EDUARDO TENÓRIO SIMÕES

**LINEAR AND NONLINEAR HIERARCHICAL PLATE MODELS AND A
POSTERIORI KINEMATICAL ERROR ESTIMATOR**

Thesis submitted to the Polytechnic School at
the University of São Paulo for award of the
Doctor of Science Degree.

**São Paulo
2015**

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Area: Structural Engineering

Advisor: Prof. Dr. Paulo de Mattos Pimenta

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Resumo

Simões, E. T. . **Modelos hierárquicos de placa lineares e não lineares e um estimador de erro a posteriori na hipótese cinemática** . 2015. 120 f. Tese (Doutorado) - Escola Politécnica da Universidade de São Paulo, São Paulo, 2015.

Este estudo explora o uso de modelos hierárquicos para representar sólidos tridimensionais de forma computacionalmente barata. Em primeiro lugar, é explorada a escolha dos espaços de elementos finitos e como isso afeta a convergência em relação ao parâmetro da espessura. Foram estudados três modelos diferentes. Mostrou-se que a menor ordem adequada do espaço de discretização cresce para todos os campos conforme a ordem do modelo é enriquecida . Isso impõe um problema, já que um maior polinômio exige maior custo computacional e modelos de alta ordem só são necessários perto do contorno. Depois, são usados estimadores de erro na discretização e na hipótese cinemática. Mostra-se que o erro implementado na discretização é capaz de capturar a camada limite de forma automatizada para qualquer modelo. Também é apresentada uma técnica de erro a posteriori na hipótese cinemática com base no erro no equilíbrio de modelos de ordem superior. No final, é apresentado um modelo hierárquico de casca geometricamente não linear e sua discretização. Mostra-se que o modelo consegue representar a solução tridimensional quando comparado com o um software comercial.

Palavras-chave: modelos hierárquicos de placa, modelo hierárquicos de casca geometricamente não linear, erro na hipótese cinemática.

Abstract

Simões, E. T. **Linear and nonlinear hierarchical plate models and a posteriori kinematical error estimator**. 2015. 120 f. Tese (Doutorado) - Escola Politécnica da Universidade de São Paulo, São Paulo, 2015.

This study explores the use of hierarchical models to represent three-dimensional solids in a computationally inexpensive way. First, it is investigated the choice of the finite element spaces and how it affects the convergence in relation to the thickness parameter. It was studied three different models. It was shown that the best lowest order suitable combination of spaces grows in all fields as the model order is enriched. After, it is presented a theory to evaluate the error in the discretization and the kinematical hypothesis. It is shown that the implemented error in discretization technique is capable of capturing the boundary layer in automated way for any model. It is also given a posteriori error procedure for kinematical hypothesis. The method is based on the equilibrium error of higher order models. Good results are shown. In the end, it is presented a geometrical nonlinear hierarchical shell model and its discretization. It is shown that the model succeeds in representing the three-dimensional solution when compared with solid elements in a commercial code.

Keywords: hierarchical plate models, geometrical nonlinear hierarchical shell model, error in the kinematical hypothesis.

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Introduction

The structural models, such as beams, plates and shells, have been popular in the engineering community for an extended time. There are two reasons for that: First, it was the first approach to representing the behaviour of a solid; Second, cause of its simplicity. Before the advent of computers, structural models results could be easily solved manually or consulted within engineering handbooks. Just a few problems had an analytical three-dimensional solution. Many complicated mechanical structures were chosen based on Reference Tables derived from real experiments what limited the possibilities of the engineering project.

Marie-Sophie Germain proposed the first plate model. The model had an error in the boundary conditions that was later correct by Kirchhoff and Poisson, see [36]. Nowadays, it is known as Kirchhoff-Love plate model. It was constructed based on the hypothesis that the orthogonal fiber to the mid-surface in the initial configuration, called within director, is still orthogonal to the mid-surface after the deformation. This hypothesis is consistent with the behaviour of a solid when its thickness goes to zero. At the beginning of the 20th century, another plate model was proposed independently by Mindlin [39] and Reissner [50]. In this model, the hypothesis was relaxed so that the original orthogonal fiber could rotate. This degree of freedom could represent well shear deformations and also represent edge effects in the border, what is not possible in the Kirchhoff-Love model. As the thickness goes to zero, the Reissner-Mindlin model tends to behave as a Kirchhoff-Love model. This property is called asymptotic consistency.

The arrival of computers and the Finite Element Method in the late 50's improved the range of solvable problems, see [23, 5]. Three-dimensional solids started to have feasible solutions, but with a precision limited by the computational capacity. Furthermore, the finite elements have limitations related to its aspect ratio and material properties. Depending on their values, the element could suffer locking. In [27], Doherty Et al. proposed a first remedy for reducing the locking. It used a reduced integration in the numerical quadrature. As the convergence strongly deteriorates when the elements have a high aspect ratio; solids with a dimension much lower than the others need an enormous number of elements. These problems and the use of structural models in the engineering practice motivated the implementation of plate elements within the framework of finite elements.

In the late 50's, the first elements for Plate Bending were proposed. It was improved later in [22]. Those elements used the Kirchhoff-Love hypothesis and they needed a continuity in the first derivative, $\mathcal{C}^1$ continuity, leading to a complicated elements like Hsieh-Clough-Tocher. Others $\mathcal{C}^1$ were also developed, see [17] page 66. Those difficulties motivated the use of plate elements based in the Reissner-Mindlin kinematics since it demands a $\mathcal{C}^0$ continuity.

Reissner-Mindlin models require less regularity than the Kirchhoff-Love model, but they can also suffer discretization error if the right choice of finite element space for the displacement and rotation is not made. This error is stronger as smaller is the effective thickness $t = d/L$. It is the

boundary layer and the locking that causes this problem.

During the text, we will use d as the plate thickness, L as the characteristic length, h as the mesh spacing in the finite element scheme and p as the degree of the elements.

The RM model has an additional boundary layer depending on the boundary conditions, see [8]. A local strategy of refinement close to the border may be needed to overcome the numerical solution deterioration. The kind of enrichment approach to treat boundary layers were studied extensively in the literature, see [53].

The tendency of the solution as the thickness d goes to 0 causes the locking. The degradation of the convergence, when the thickness goes to 0, is due to the limited number of trial spaces that satisfies the KL hypotheses. There are two ways to avoid locking. The first is the use of trial functions that are capable to approximate KL model optimally. The other possibility is to use a mixed method that enforces the constraint in a weaker sense.

The use of the expression $K(t)(h/L)^p$ simplifies the discretization error, where $K(t)$ is a locking factor. In the worst case, $K(t) \approx t^{-1}$, but there are also cases where the factor does not depend on the thickness, i.e., $K(t) \approx 1$. In the latter case, the finite element scheme solves the length scale under consideration optimally.

Suri, Babuška and Schwab studied the locking factor of the h , p and hp refinement strategy for triangular and quadrilateral meshes in [38]. They used the model proposed in [12]. They showed that for triangular meshes using an h -refinement strategy the order of the rotation trial space should be higher than 4 and the displacement one order higher than the rotation in order to avoid locking. The quadrilateral mesh can not avoid locking in an h -refinement strategy. The p -refinement strategy alleviates the need of an over-refinement to the worst case $K(t) \approx t^{-1}$, since it raises the factor p in error equation $K(t)(h/L)^p$. If it is used as point of departure of a locking free h -version, then hp -refinement can be also used.

Methods using a mixed method to cure locking were evaluated in [29]. Many of the elements are nonconforming or in displacement or rotation. An element linear nonconforming in rotations and linear conform in displacement was proposed in [43]. It did not show locking but presented a deterioration in the convergence as the thickness went higher, see [9]. In the framework of nonlinear mechanics, a close element quadratic in displacement was proposed in [18]. This element was based on the nonlinear shell formulation proposed in [44]. It showed better results than the one presented in [43], since the displacement were enriched. The deRham complex as presented in [10] gives an excellent framework to evaluate the finite element subspaces and shows that the Kirchhoff constraint can be achieved by this choice of spaces.

RM and KL models can well represent the solution of elasticity in some regions of the solid. In computational terms, they are more interesting than using a three-dimensional model since they can reduce the amount of computational cost. However, they start to lose their capability to represent the three-dimensional solution as the thickness grows, and, no matter the problem, its solution is inadequate close to the borders. The lack of a proper solution close to the edge is also known as boundary layer effect. It is shown in [13, 25] that the capability of getting the boundary effect is linked to the kinematical assumption. In [13], it was shown that close to the boundary the use of the RM model can lead to an error of around 30 % in the studied case. The kinematical hypothesis used is critical in composite materials, where the boundary is an important source of delamination [40]. In composite materials, if a simple kinematical assumption is used, e.g. Reissner-Mindlin model,

even far from the border the displacement solution is sub-optimal. So, theories that can be flexible enough to enrich just where is necessary can potentially lead to savings in computational cost.

The improvement of the kinematical hypothesis can be made a priori or a posteriori. By a priori, we mean the use of predefined basis of enrichments to the thickness. The use of a hierarchical family of plate models, e.g. [13, 25, 52], is an example of this kind of approach. The idea is to construct plate models with refinement in the thickness so that the solution is closer to the three-dimensional one. In what concerns the beams elements, a family of enrichments, warping [45] and distortion [24], were already made using as a base the finite strain beam model presented in [47]. For shells, the stretching of the thickness was included in the model [18] in [46].

Those hierarchical families suffer the same problem or discretization error as the RM. The locking of then was studied in [12] for the model with stretching in the thickness enrichment. It was shown that the choice of the same trial space for the rotations and the stretching was enough to avoid locking. In [48], it was shown that the locking just happened in the pure bending case. In [21], it was proposed a technique to evaluate a posteriori the existence of locking.

In what concerns a-posteriori method of kinematical enrichments, there are many possible ways. The idea is to optimize the kinematical assumption so that a minimum number of degrees of freedom is needed to represent the phenomenon. In composite laminates analysis, the use of post-processing techniques to get a better distribution of stresses in the thickness is common, e.g. [42, 35, 51, 19, 28].

The paper of Chaudhuri et al. [19] was one of the first to use the balance equation of three-dimensional model to improve the stress in the thickness. It was based on a triangular element that used the layer-wise constant shear angle model. The stress was post-processed in the mid-side nodes. In [32] the element was based third-order shear model of Reddy. The paper by [49] also used this technique based in a layer-wise model. In the methods above, it was needed the calculus of the second derivative of the shape functions so that no linear element could be used.

Noor et al. [41] proposed a predictor-corrector method. It is a two stages iterative technique. In the first stage, the stress in the thickness is obtained by the constitutive equation of a simplified model. In this case, it was the Reissner-Mindlin model. The correction factor is also calculated and used to improve the stiffness. The method takes into account the boundary conditions on the top and bottom faces. In the second step, the stress was used to obtain the strain by the constitutive equation. The strain in the thickness was integrated to obtain a better displacement. These displacements are used together with a Rayleigh-Ritz method to determine the amplitude of a chosen basis function. Then, the second step is redone. This process is done iteratively until convergence of the stress.

In the paper [42], it was evaluated five different approaches to modelling composite materials. Between those approaches, the majority were layer-wise models and high-order models with post-processing of stress in the thickness. The predictor-corrector method was shown to be the most efficient and it was the unique reliable for thermal problems.

Allix et al. [3] used the technique proposed in [41] analytically in the Kirchhoff-Love plate model and obtained the reduction in the other of the error in the thickness. The same paper also proposes a technique to post-process the stress taking into account the boundary layer effects.

Those techniques have the advantage of been cheaper in relation to the hierarchical models, but they fail to represent the boundary layer. It is important to have a way to improve directly the kinematical hypothesis where is necessary. This method is not too different from an evaluation of the

error in discretization, so we will start by it. There are normally two groups of techniques to evaluate the error in discretization, see [26]. The first is called recovery estimates. It works smoothing the FE stress resultants across the mesh, looking for the best possible fit of a smoothed solution in a least square sense, see [62, 63]. The computation of this kind can be carried out independently for each component of the generalized stresses.

The other big group is the residual type of estimate. They are based in the calculus of the residuum. It is possible calculate this implicitly or explicitly. The explicit method gives an error measure with a coefficient C unknown and implicit method is independent of any constant, but an internal problem must be solved. This implicit problem can be solved element by element or subdomain by subdomain.

In what concerns the error in the model, the modelling error was first studied by Vogelius and Babuška, see [59, 60, 61]. The notion of error in the kinematical assumption measured by the constitutive equation has been developed in classical papers by [33]. It was further studied by Schwab in [52] and [14] with Babuška for scalar elliptic problems. Oden and Cho developed a posteriori energy error estimates for thin plate- and shell-like structures in [20]. Stein and Ohnibus proposed another technique now based in the Posterior Equilibration Method, see [56, 55]. A close method was introduced by Bohnic and Ibrahimbegovic in [16] to calculate the error between the Reissner-Mindlin and Kirchhoff-Love. This process is done by the construction of an equilibrated stress field in the whole plate. In [35], the idea of the structural model as an internal constraint is developed, so that its reaction is used to improve the stress prediction and to verify the error in the kinematical assumption. This kind of idea has a strong link with the techniques used for a posteriori enrichments in the kinematical hypothesis.

This work is divided into three chapters. The first chapter presents the linear model for hierarchical models. In this chapter, it is evaluated the way different deformation modes are linked and how they interact with the locking. In the end, it is analyzed some elements in relation to the locking. Some conclusions are made over the limitations of each evaluated element. In the second chapter, it is presented a framework to evaluate the error in the discretization and the kinematical hypothesis. The results are shown in the end. In the third chapter, it is presented a geometrical nonlinear hierarchical shell model and its discretization. It is shown that the model succeeds in representing the three-dimensional solution when compared with Abaqus using solid elements. In the conclusion, it is presented the next direction that this research will follow.

1.1 Notation used

Throughout the text, italic Latin or Greek lower-case letters ($a, b, \dots \alpha, \beta \dots$) denotes scalar quantities. First-order tensors are denoted by bold italic Latin or Greek lower-case letters ($\mathbf{a}, \mathbf{b}, \dots \boldsymbol{\alpha}, \boldsymbol{\beta} \dots$). Bold italic Latin letters ($\mathbf{A}, \mathbf{B}, \dots$) denotes second-order tensors and script Latin capital letters ($\mathcal{A}, \mathcal{B}, \dots$) denotes third-order tensors. Fourth-order tensors are denoted by blackboard italic Latin capital letters ($\mathbb{A}, \mathbb{B}, \dots$). Vectors and matrices built of tensor components (e.g. for computational purposes) are expressed by boldface upright Latin letters ($\mathbf{A}, \mathbf{B}, \dots \mathbf{a}, \mathbf{b} \dots$). Summation convention over repeated indices is adopted, with Greek indices ranging from 1 to 2 and Latin indices from 1 to 3.

The linear thick plate model

2.1 The reference problem

Figure 2.1: *The three-dimensional domain*

We identify the initial configuration (see Figure 2.1) of the elastic plate-like solid body as a bounded domain V of material points $\boldsymbol{\xi}$. The set V is a Cartesian product of the mid-surface Ω and the thickness H so that $V = \Omega \times H$. Every material point is associated with the position vector $\boldsymbol{\xi} = \xi_\alpha \mathbf{e}_\alpha + \zeta \mathbf{e}_3$, in which $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the Cartesian frame of reference. The coordinates $\{\xi_1, \xi_2, \zeta\}$ are restricted so that $\{\xi_1, \xi_2\} \in \Omega$ and $\zeta \in H$. The boundary of V is given by $\partial V = \Omega_+ \cup \Gamma \cup \Omega_-$, where Ω_+ is the top face, Ω_- is the bottom face and Γ is the lateral surface of the shell-like solid body V .

The boundary conditions are applied over ∂V . The subscript d is used to identify the set of material point where the Dirichlet boundary conditions are applied and the subscript n where Neumann conditions are applied. This yields the relation $\partial V = \partial V_d \cup \partial V_n$. The Neumann boundary conditions are the action of body forces $\bar{\mathbf{b}}$ per unit volume and the traction forces $\bar{\mathbf{t}} = \bar{\mathbf{t}}_- \cup \bar{\mathbf{t}}_\Gamma \cup \bar{\mathbf{t}}_+$ per unit of area, see Figure 2.2. The Dirichlet boundary conditions are the imposed displacements $\bar{\mathbf{u}}$ in ∂V_d .

In the deformed configuration, the position vector, denoted as vector $\mathbf{x}$, is

$$\mathbf{x} = \boldsymbol{\xi} + \mathbf{u}, \quad (2.1)$$

where $\mathbf{u}$ is the displacement belonging to the kinematically admissible space $\mathcal{U}_k$ and $\boldsymbol{\xi}$ is the material points in the initial configuration.

Figure 2.2: *The three-dimensional domain*

The reference problem is stated as:

Find $\mathbf{u} \in \mathcal{U}_k$ such that

$$\min_{\mathbf{u} \in \mathcal{U}_k} \Pi_p(\mathbf{u}) = U_{\text{int}}(\mathbf{u}) + U_{\text{ext}}(\mathbf{u}), \quad (2.2)$$

where $U_{\text{int}}(\mathbf{u}) : \mathcal{V} \rightarrow \mathbb{R}$, $U_{\text{ext}}(\mathbf{u}) : \mathcal{V} \rightarrow \mathbb{R}$ and $\Pi_p(\mathbf{u}) : \mathcal{V} \rightarrow \mathbb{R}$ are the internal, external and total potential energy of the solid. The Dirichlet boundary conditions are imposed by the adequate choice of the kinematically admissible space in agreement with

$$\mathbf{u} \in \mathcal{U}_k := \{\mathbf{u} \in \mathcal{H}^1(\Omega) | \mathbf{u} = \bar{\mathbf{u}} \text{ on } \partial V_d\}, \quad (2.3)$$

in which $\mathcal{H}^1(\Omega^r)$ is the Hilbert Space.

In addition, the extreme value of the equation 2.2 is obtained by means of

$$\delta U_p(\mathbf{u}, \delta \mathbf{u}) = 0, \quad \forall \delta \mathbf{d}. \quad (2.4)$$

The last result is also called the weak form of the original variational problem.

The external potential energy U_{ext} , in a conservative framework, is

$$\begin{aligned} U_{\text{ext}} &= - \left(\int_{\partial V} \bar{\mathbf{t}} \cdot \mathbf{u} \, da + \int_V \bar{\mathbf{b}} \cdot \mathbf{u} \, dv \right) \\ &= - \int_{\Omega_-} (\bar{\mathbf{t}}_- \cdot \mathbf{u}_-) \, da - \int_{\Omega_+} (\bar{\mathbf{t}}_+ \cdot \mathbf{u}_+) \, da \\ &\quad - \int_{\Gamma_t} (\bar{\mathbf{t}}_\Gamma \cdot \mathbf{u}_\Gamma) \, da - \int_V (\bar{\mathbf{b}} \cdot \mathbf{u}) \, dv. \end{aligned} \quad (2.5)$$

The internal or strain potential energy U_{int} , which is given by

$$U_{\text{int}} = \int_V \varphi(\mathbf{u}) \, dv, \quad (2.6)$$

where $\varphi(\mathbf{d}) : \mathcal{U}_k \rightarrow \mathbb{R}$ is the free Helmholtz, strain energy or stored energy density function. The

first variation of the total potential energy Π_p , given by equation (2.4), can also be written as

$$\begin{aligned} \delta\Pi_p(\mathbf{u}, \delta\mathbf{u}) &= \int_V \frac{\partial\varphi(\mathbf{u})}{\partial\mathbf{E}} : \delta\mathbf{E} \, dv + \\ &- \int_{\partial V_n} \bar{\mathbf{t}} \cdot \delta\mathbf{u} \, dv - \int_V \bar{\mathbf{b}} \cdot \delta\mathbf{u} \, dv = 0, \quad \forall \delta\mathbf{u}, \end{aligned} \quad (2.7)$$

where $\delta\mathbf{E} = \frac{1}{2}(\nabla\delta\mathbf{u} + \nabla\delta\mathbf{u}^T)$ is the first variation of the symmetric part of the linear strain tensor $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$. The weak form (2.7) implies that $\mathbf{u} \in \mathcal{U}_k$ and $\delta\mathbf{u} \in \delta\mathcal{U}_k := \{\delta\mathbf{u} \in \mathcal{H}^1(\Omega^r) | \mathbf{u} = 0 \text{ on } \partial\Omega^r\}$.

An hyperelastic framework for the strain energy density φ implies

$$\mathbf{T} = \frac{\partial\varphi}{\partial\mathbf{E}}, \quad (2.8)$$

where $\mathbf{T}$ is the Cauchy stress tensor.

With the use of the divergence theorem, we can arrive at the strong form of the equation (2.7) given by

$$\operatorname{div}(\mathbf{T}) + \bar{\mathbf{b}} = \mathbf{0}, \text{ in } V \quad (2.9a)$$

$$\mathbf{T} \cdot \boldsymbol{\nu} = \bar{\mathbf{t}}, \text{ on } \partial V_n \quad (2.9b)$$

where $\boldsymbol{\nu}$ is the unit exterior normal to the boundary ∂V .

If the strain energy density function $\varphi(\mathbf{E})$ is convex then we can define the complementary energy φ^* by the use of a Legendre transformation. This yields

$$\varphi^*(\mathbf{T}) = \sup_{\mathbf{u} \in \mathcal{U}_k} (\mathbf{T} : \mathbf{E}(\mathbf{u}) - \varphi(\mathbf{E}(\mathbf{u}))), \quad \mathbf{T} \in \mathcal{S}_{Ad} \quad (2.10)$$

where $\mathcal{S}_{Ad}$ is the statical admissible space given by

$$\begin{aligned} \mathbf{T} \in \mathcal{S}_{Ad} &:= \{\mathbf{S} \in \mathcal{H}(\operatorname{div}) | \operatorname{div}(\mathbf{T}) = -\bar{\mathbf{b}} \text{ and } \text{ in } V \\ &\mathbf{T} \cdot \boldsymbol{\nu} = \bar{\mathbf{t}} \text{ on } \partial V\}. \end{aligned} \quad (2.11)$$

Substituting (2.10) in (2.2) yields the Hu-Washizu functional. Taking the divergence theorem over the term $\mathbf{T} : \mathbf{E}$, it yields the complementary problem of linear elasticity as

Find $\mathbf{T} \in \mathcal{S}_{Ad}$ such that

$$\max_{\mathbf{T} \in \mathcal{S}_{Ad}} \Pi_d(\mathbf{T}) = U_{\text{int}}^*(\mathbf{T}) + U_{\text{ext}}^*(\mathbf{T}). \quad (2.12)$$

The internal energy potential is now defined in terms of stress

$$U_{\text{int}}^* = \int_V \varphi^*(\mathbf{T}) \, dv, \quad (2.13)$$

whereas the external energy is given by

$$U_{\text{ext}}^* = - \left(\int_{\partial V_d} \bar{\mathbf{u}} \cdot (\mathbf{T} \cdot \boldsymbol{\nu}) \, da \right). \quad (2.14)$$

The corresponding strong form of the problem can finally be written as:

$$\text{Sym}(\nabla \mathbf{u}) = \frac{\partial \varphi^*}{\partial \mathbf{T}} , \text{ in } V \quad (2.15a)$$

$$\mathbf{u} = \bar{\mathbf{u}} , \text{ on } \partial V_d \quad (2.15b)$$

The two problems (2.2) and (2.12) allow us to construct the bounds upon the true solution $\mathbf{u}^*$ of the linear elasticity problem by making using of the property

$$\Pi_p(\mathbf{u}) \geq \Pi_p(\mathbf{u}^*) \geq \Pi_d(\mathbf{T}) , \forall \mathbf{u} \in \mathcal{U}_k, \forall \mathbf{T} \in \mathcal{S}_{Ad}. \quad (2.16)$$

This property will be important in later sections.

2.2 The hierarchical family models

In the case in which the thickness parameter H is much smaller than a characteristic dimension of the mid-surface Ω , it is possible to substitute the original three-dimensional problem by a simpler two-dimensional one without losing the overall behaviour. The justification to this kind of approach is the asymptotic consistency of the two models when the thickness H tends to 0. For instance, if there is a displacement restriction $\mathbf{d}^0$ that works on $\mathbf{u}$ such that

$$\lim_{H \rightarrow 0} \frac{\|\mathbf{u} - \mathbf{u}(\mathbf{d}^0)\|_E}{\|\mathbf{u}\|_E} = 0. \quad (2.17)$$

then we say that the restriction $\mathbf{u}(\mathbf{d}^0)$ is asymptotic consistent and is a limit model of the solid. In 2.17, the $\|\bullet\|_E$ is the energy norm.

The Kirchhoff-Love plate model is the lowest limit model for thin plates. It has only one unknown, the displacement w perpendicular to the mid-surface. The displacement $\mathbf{u}$ in this model is denoted as

$$\mathbf{u}(\mathbf{d}^0) = w \mathbf{e}_3 + \boldsymbol{\theta} \times \zeta \mathbf{e}_3 \quad (2.18)$$

where $\boldsymbol{\theta} = [w_{,2}, w_{,1}]$ and ζ measures the distance from the plate mid-surface.

As this model is a kinematic restriction of the original solution space $\mathcal{U}_K$ it implies that $\mathbf{u}(\mathbf{d}^0) \in \mathcal{U}_{K_0} \subset \mathcal{U}_K$, where $\mathcal{U}_{K_0}$ is given by

$$\mathcal{U}_{k_0} := \{w \in \mathcal{H}_2(\Omega) | w = \bar{w} \text{ on } \partial\Omega_d\}. \quad (2.19)$$

The inequality (2.16) implies that

$$\Pi_p(\mathbf{u}(\mathbf{d}^0)) \geq \Pi_p(\mathbf{u}^*) , \forall \mathbf{u} \in \mathcal{U}_k, \forall \mathbf{d} \in \mathcal{U}_{K_0}.$$

The superscript 0, in equations 2.17 and 2.18, represents the fact that there are n models that are asymptotic consistent. This allows a construction of a family of n hierarchical models where all models are asymptotic consistent. To take advantage of this, we can define a hierarchical family such that the condition

$$\lim_{n \rightarrow \infty} \|\mathbf{u} - \mathbf{u}(\mathbf{d}^n)\|_E = 0 \quad (2.20)$$

is respected for fixed H . This is known as approximability condition.

Another plate model that is asymptotic consistent and better approaches the three-dimensional field is the Reissner-Mindlin model. Its kinematic restriction is given by

$$\mathbf{u}(\mathbf{d}^1) = w \mathbf{e}_3 + \boldsymbol{\theta} \times \zeta \mathbf{e}_3 \quad (2.21)$$

where $\mathbf{d} = [w, \boldsymbol{\theta}^T]^T \in \mathcal{U}_{K_1}$ and

$$\mathcal{U}_{k_1} := \{w, \boldsymbol{\theta} \in \mathcal{H}_1(\Omega) \times \mathcal{H}_1^2(\Omega) | w = \bar{w} \text{ on } \partial\Omega_d\}. \quad (2.22)$$

One important remark is that $\mathcal{U}_{k_0} \subset \mathcal{U}_{k_1}$, so that

$$U_T(\mathbf{u}(\mathbf{d}^0)) \geq U_T(\mathbf{u}(\mathbf{d}^1)) \geq U_T(\mathbf{u}^*), \quad \forall \mathbf{d}^1 \in \mathcal{U}_{k_1}, \forall \mathbf{d}^0 \in \mathcal{U}_{K_0}. \quad (2.23)$$

Following the concepts presented in [25], a hierarchy of models is given by a subsequence of spaces $\mathcal{U}_{K_0} \subset \dots \subset \mathcal{U}_{K_n} \subset \mathcal{U}_k$, in which K_n stands for the order of the model.

As stated in [11], posing more than the conditions given in (2.17) and (2.20), any hierarchical model will lead to an improvement in the convergence rate γ_n given by

$$\|\mathbf{u} - \mathbf{u}(\mathbf{d}^n)\|_E \leq Ch^{\gamma_n} \|\mathbf{u}\|_E \quad (2.24)$$

if there is no boundary layer.

2.3 An a priori hierarchical family of models

Ideally the hierarchical family of models should be obtained from an asymptotic consistent method, like homogenization or multiple scales. Herein, we take a simpler approach. We construct a family of hierarchical models by an additive composition of polynomials, so that the asymptotic consistency and approximability properties are respected.

As the Kirchhoff-Love model requires $\mathcal{C}^1$ continuity, it was decided to construct a hierarchical model starting from the Reissner-Mindlin kinematic hypothesis. The corresponding kinematic restriction can thus be written as

$$\mathbf{u}(\mathbf{d}^n) = w \mathbf{e}_3 + \boldsymbol{\theta} \times \mathbf{a} + \Delta \mathbf{a}, \quad (2.25)$$

in which w is the displacement of the mid-surface, $\mathbf{a} = \zeta \mathbf{e}_3$ is the director vector, $\boldsymbol{\theta}$ is the infinitesimal rotation of the director and $\Delta \mathbf{a}$ is the director enrichment vector.

2.3.1 The choice of director enrichment

The enrichment of the director vector is defined

$$\Delta \mathbf{a} = \sum_{i=1}^n \Delta \mathbf{a}^n = \sum_{i=1}^n \Delta a_i^n \mathbf{e}_i, \quad (2.26)$$

where each component Δa_i^n is further described in terms of an additive decomposition by

$$\Delta a_\alpha^n = \phi_\alpha^n \left(\frac{\zeta}{h} \right) p_\alpha^n(\xi_\alpha) \quad \text{and} \quad (2.27)$$

$$\Delta a_3^n = \phi_3^n \left(\frac{\zeta}{h} \right) p_3^n(\xi_\alpha). \quad (2.28)$$

In 2.27, $\phi_i^j \left(\frac{\zeta}{h} \right)$ are the hierarchic basis or director functions, $p_i(\xi_\alpha)$ are the generalized deformations and q_i are the order in hierarchic basis.

We will consider the classical asymptotic expansion $(\frac{\zeta}{h})^n$ for the hierarchical base, where n is the order of the polynomial.

Table 2.1: *Adopted hierarchical family*

$\Delta \mathbf{a}$	$\Delta \mathbf{a}_1$	$\Delta \mathbf{a}_2$	$\Delta \mathbf{a}_3$
$\Delta \mathbf{a}^1$	0	0	$(\frac{\zeta}{h})^2$.
$\Delta \mathbf{a}^2$	$(\frac{\zeta}{h})^3$	$(\frac{\zeta}{h})^3$	$(\frac{\zeta}{h})^4$.
$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\Delta \mathbf{a}^n$	$(\frac{\zeta}{h})^{2n-1}$	$(\frac{\zeta}{h})^{2n-1}$	$(\frac{\zeta}{h})^{2n}$.

The hierarchical basis are gathered in the column matrix $\phi_i \left(\frac{\zeta}{h} \right)$ and its associated parameters in $\mathbf{p}_i(\xi_\alpha)$; we can thus write the corresponding expression in matrix notation:

$$\Delta \mathbf{a} = (\phi_i \cdot \mathbf{p}_i) \mathbf{e}_i = (\mathbf{e}_i \otimes \phi_i) \cdot \mathbf{p}_i = \mathbf{\Psi} \cdot \mathbf{p}, \quad (2.29)$$

where

$$\mathbf{\Psi} = \begin{bmatrix} \phi_1^T & \mathbf{0}_{1 \times np_2} & \mathbf{0}_{1 \times np_3} \\ \mathbf{0}_{1 \times np_1} & \phi_2^T & \mathbf{0}_{1 \times np_3} \\ \mathbf{0}_{1 \times np_1} & \mathbf{0}_{1 \times np_2} & \phi_3^T \end{bmatrix} \quad (2.30)$$

is the hierarchical basis matrix and

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}^T \quad (2.31)$$

is the column matrix that gathers all the generalized deformations.

2.4 Generalized deformations

The corresponding expression for displacement gradient is given by

$$\begin{aligned} \mathbf{L} = \frac{\partial \mathbf{u}}{\partial \boldsymbol{\xi}} = & (\boldsymbol{\eta}_\alpha + \boldsymbol{\kappa}_\alpha \times \mathbf{a} + \Delta \mathbf{a}_{,\alpha}) \otimes \mathbf{e}_\alpha \\ & + \Delta \mathbf{a}_{,3} \otimes \mathbf{e}_3 + \boldsymbol{\Theta} \end{aligned} \quad (2.32)$$

where

$$\boldsymbol{\eta}_\alpha = w_{,\alpha} \mathbf{e}_3 - \boldsymbol{\theta} \times \mathbf{e}_\alpha, \quad (2.33)$$

$$\boldsymbol{\kappa}_\alpha = \boldsymbol{\theta}_{,\alpha}, \quad (2.34)$$

$$\Delta \mathbf{a}_{,\alpha} = \begin{bmatrix} \boldsymbol{\phi}_1 \cdot \mathbf{p}_{1,\alpha} \\ \boldsymbol{\phi}_2 \cdot \mathbf{p}_{2,\alpha} \\ \boldsymbol{\phi}_3 \cdot \mathbf{p}_{3,\alpha} \end{bmatrix} \text{ and } \Delta \mathbf{a}_{,3} = \begin{bmatrix} \boldsymbol{\phi}_{1,3} \cdot \mathbf{p}_1 \\ \boldsymbol{\phi}_{2,3} \cdot \mathbf{p}_2 \\ \boldsymbol{\phi}_{3,3} \cdot \mathbf{p}_3 \end{bmatrix}. \quad (2.35)$$

We note that the vectors $\boldsymbol{\eta}_\alpha$ are given by

$$\boldsymbol{\eta}_1 = \begin{bmatrix} 0 \\ 0 \\ w_{,1} + \theta_2 \end{bmatrix} \text{ and } \boldsymbol{\eta}_2 = \begin{bmatrix} 0 \\ 0 \\ w_{,2} - \theta_1 \end{bmatrix}, \quad (2.36)$$

which simply recovers the shear strains $\gamma_\alpha = \boldsymbol{\eta}_\alpha \cdot \mathbf{e}_3$.

The linear strain tensor can also be recast in matrix notation by

$$\mathbf{E} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \quad (2.37)$$

where its components are given by :

$$\begin{aligned} E_{11} &= \zeta \theta_{2,1} + \boldsymbol{\phi}_1 \cdot \mathbf{p}_{1,1}; \\ E_{22} &= -\zeta \theta_{1,2} + \boldsymbol{\phi}_2 \cdot \mathbf{p}_{2,2}; \\ E_{33} &= \boldsymbol{\phi}_{3,3} \cdot \mathbf{p}_3; \\ E_{21} &= \frac{1}{2}(\zeta(\theta_{2,2} - \theta_{1,1}) + \boldsymbol{\phi}_2 \cdot \mathbf{p}_{2,1} + \boldsymbol{\phi}_1 \cdot \mathbf{p}_{1,2}); \\ E_{31} &= \frac{1}{2}(\gamma_1 + \boldsymbol{\phi}_3 \cdot \mathbf{p}_{3,1} + \boldsymbol{\phi}_{1,3} \cdot \mathbf{p}_1) \text{ and} \\ E_{32} &= \frac{1}{2}(\gamma_2 + \boldsymbol{\phi}_3 \cdot \mathbf{p}_{3,2} + \boldsymbol{\phi}_{2,3} \cdot \mathbf{p}_2). \end{aligned}$$

All the generalized strain components can be gathered in a single vector by using again the matrix notation:

$$\boldsymbol{\epsilon} = \begin{bmatrix} \gamma_1 & \gamma_2 & \boldsymbol{\kappa}_1^T & \boldsymbol{\kappa}_2^T & \mathbf{p}^T & \mathbf{p}_{,1}^T & \mathbf{p}_{,2}^T \end{bmatrix}^T = \mathcal{T}_c(\mathbf{d}). \quad (2.38)$$

The operator $\mathcal{T}_c$ above is the one who takes $\mathbf{d}$ to the $\boldsymbol{\epsilon}$.

2.5 Constitutive equation

For simplicity, we will just choose the generalized Hooke's law. The strain energy density function of this constitutive model is given by

$$\varphi^0(I_1, I_2) = \frac{\lambda}{2}(I_1)^2 + \mu(I_2), \quad (2.39)$$

where $I_1 = \mathbf{I} : \mathbf{E}$ and $I_2 = \mathbf{I} : \mathbf{E}^2$ are the corresponding strain invariants. The stress tensor is given by

$$\mathbf{T} = \frac{\partial \psi}{\partial \mathbf{E}} = \lambda I_1 \mathbf{I} + 2\mu \mathbf{E} \quad (2.40)$$

where $\psi(\mathbf{E})$ is the Helmholtz free energy.

The constitutive law can also be written in matrix notation:

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} (\mu + \lambda) & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & (\mu + \lambda) & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & (\mu + \lambda) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix}. \quad (2.41)$$

The corresponding constitutive equations can also be written in the form:

$$t_i = \frac{\partial \psi}{\partial \gamma_j} = C_{ij} \quad (2.42)$$

where $t_i = \mathbf{T} \cdot \mathbf{e}_i$ and $\gamma_i = \mathbf{u}_{,i}$. In this way, the constitutive equations can again be written in the matrix notation as

$$\underbrace{\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}}_{\mathbf{t}} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}}_{\boldsymbol{\gamma}}. \quad (2.43)$$

We can relate the generalized strains $\boldsymbol{\epsilon}$ with $\boldsymbol{\gamma}$ by using

$$\boldsymbol{\gamma} = \boldsymbol{\Lambda}(\zeta) \boldsymbol{\epsilon} \quad (2.44)$$

where

$$\boldsymbol{\Lambda}(\zeta) = \begin{bmatrix} e_3 & \mathbf{O} & (-A) & \mathbf{O} & \mathbf{O} & \boldsymbol{\Psi} & \mathbf{O} \\ \mathbf{O} & e_3 & \mathbf{O} & (-A) & \mathbf{O} & \mathbf{O} & \boldsymbol{\Psi} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \boldsymbol{\Psi}_{,3} & \mathbf{O} & \mathbf{O} \end{bmatrix}. \quad (2.45)$$

The use of a fully 3D constitutive model requires the adequate enrichment of the strain through-the-thickness, what is not need for the Kirchhoff-Love and Reissner-Mindlin models.

2.6 Through-the-thickness enrichments

To keep the asymptotic consistency of all the models, some precautions should be take. It could be proved, by taking the limit of each model, that they converge to Kirchhoff-Love model as thickness tends to zero. Herein, we took a simpler approach. It is known the the limit model has two properties in mechanical terms. First, it has zero shear strain and second it has $T_{33} = 0$, also

known as "plane stress" condition. Our approach is to see if the choice of enrichments can represent these characteristics.

First, the zero transverse shear strain problem is more important in the discretization. It occurs when there is an incapacity of the element to represent a state pure bending. This is normally evaluated by the transverse shear energy term that must be able to go to zero. In our case, this condition implies:

$$\begin{aligned} \mu E_{13}^2 + \mu E_{23}^2 &= 2\mu(\gamma_1 + \phi_3 \cdot \mathbf{p}_{3,1} + \phi_{1,3} \cdot \mathbf{p}_1)^2 \\ &\quad + 2\mu(\gamma_2 + \phi_3 \cdot \mathbf{p}_{3,2} + \phi_{2,3} \cdot \mathbf{p}_2)^2 \end{aligned} \quad (2.46)$$

equal to 0 as the thickness goes to 0. The most important terms here are the pure shears γ_1 . They will be evaluated in the discretization section.

Second, the "plane stress" condition to be enforced is $T_{33} = 0$. Given that we have

$$\begin{aligned} T_{33} &= \lambda E_{11} + \lambda E_{22} + (\lambda + \mu) E_{33} \\ &= \lambda(\zeta \theta_{2,1} + \phi_1 \cdot \mathbf{p}_{1,1}) \\ &\quad + \lambda(-\zeta \theta_{1,2} + \phi_2 \cdot \mathbf{p}_{2,2}) \\ &\quad + (\lambda + \mu) \phi_{3,3} \cdot \mathbf{p}_3 \end{aligned} \quad (2.47)$$

we will require that ϕ_3 must start with ζ^2 and that $\phi_{3,3} = \phi_\alpha$.

2.7 Internal power

In this section, the internal power of the enriched plate model is deduced. This is important because the notion of power is more general than that of work when there is some interest in numerical methods. It allows to derive the virtual work directly for the linear and nonlinear case. Also, since the time derivative is a linearization then it allows to directly substitute for any tensor $\dot{\mathbf{v}}$ the equivalent version $\delta \mathbf{v}$. The virtual work will be needed after to the finite element discretization.

The internal power of the enriched model is given by

$$P_{int} = \int_V \mathbf{T} : \dot{\mathbf{E}} \, dv. \quad (2.48)$$

The symmetry of $\mathbf{T}$ allows us to rewrite the internal power as

$$P_{int} = \int_V \mathbf{T} : \dot{\mathbf{L}} \, dv. \quad (2.49)$$

Substituting (2.32) in (2.49) and integrating through the thickness leads to

$$P_{int} = \int_\Omega (V_\alpha \gamma_\alpha + \mathbf{m}_\alpha \cdot \boldsymbol{\kappa}_\alpha + \boldsymbol{\beta}_i \cdot \mathbf{p}_i + \boldsymbol{\rho}_{i\alpha} \cdot \mathbf{p}_{i,\alpha}) \, da \quad (2.50)$$

where we can write explicitly

$$V_\alpha = \int_H \mathbf{t}_\alpha \cdot \mathbf{e}_3 \, dx, \quad (2.51)$$

$$\mathbf{m}_\alpha = \mathbf{e}_3 \times \int_H \zeta \mathbf{t}_\alpha \, dx, \quad (2.52)$$

$$\beta_i = \int_H (\mathbf{e}_i \cdot \mathbf{t}_3) \phi_{i,3} \, dx \text{ and} \quad (2.53)$$

$$\rho_{i\alpha} = \int_H (\mathbf{e}_i \cdot \mathbf{t}_\alpha) \phi_i \, dx. \quad (2.54)$$

Here, V_α is the generalized shear stress, $\mathbf{m}_\alpha$ is the bending moment, β_α and $\rho_{3\beta}$ are the bi-shear, $\rho_{\alpha\beta}$ is the bi-moment and β_3 is the bi-normal stress.

Those generalized stresses can be gathered in a column matrix by using the following matrix notation:

$$\boldsymbol{\sigma} = \begin{bmatrix} V_1 & V_2 & \mathbf{m}_1^T & \mathbf{m}_2^T & \beta^T & \rho_1^T & \rho_2^T \end{bmatrix}^T, \quad (2.55)$$

where

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \text{ and } \boldsymbol{\rho}_\alpha = \begin{bmatrix} \rho_{1\alpha} \\ \rho_{2\alpha} \\ \rho_{3\alpha} \end{bmatrix}. \quad (2.56)$$

The internal power can then simply be written as:

$$P_{int} = \int_\Omega \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\epsilon}} \, da \quad (2.57)$$

For a linear elastic constitutive model, the internal work is further given by

$$W_{int} = \frac{1}{2} \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \, da = \frac{1}{2} \int_\Omega \boldsymbol{\epsilon} \cdot \mathbf{D} \cdot \boldsymbol{\epsilon} \, da. \quad (2.58)$$

where

$$\mathbf{D} = \int_H \boldsymbol{\Lambda}(\zeta)^T \mathbf{C} \boldsymbol{\Lambda}(\zeta) \, dx. \quad (2.59)$$

2.8 External Power

The external power of the enriched model in the domain Ω can be expressed as

$$\begin{aligned} P_{ext} &= \int_{\Omega_-} (\bar{\mathbf{t}}_- \cdot \dot{\mathbf{u}}_-) \, da + \int_{\Omega_+} (\bar{\mathbf{t}}_+ \cdot \dot{\mathbf{u}}_+) \, da \\ &\quad + \int_{\Gamma_t} (\bar{\mathbf{t}}_\Gamma \cdot \dot{\mathbf{u}}_\Gamma) \, da + \int_V (\mathbf{b} \cdot \dot{\mathbf{u}}) \, dv \end{aligned} \quad (2.60)$$

where $\bar{\mathbf{t}}_+$, $\bar{\mathbf{t}}_-$ and $\bar{\mathbf{t}}_\Gamma$ are the external surface tractions on the top, bottom and the lateral surfaces of the shell per unit of reference area, and $\mathbf{b} = b\mathbf{e}_3$ is the vector of body forces per unit of reference

volume. By time differentiation of (4.4), we can further obtain:

$$\dot{\mathbf{u}} = \dot{w} \mathbf{e}_3 + \dot{\boldsymbol{\theta}} \times \mathbf{a} + (\boldsymbol{\phi}_i \cdot \dot{\mathbf{p}}_i) \mathbf{e}_i = \boldsymbol{\Sigma} \dot{\mathbf{d}} \quad (2.61)$$

where

$$\boldsymbol{\Sigma}(\zeta) = \begin{bmatrix} \mathbf{e}_3 & -\mathbf{A} & \boldsymbol{\Psi} \end{bmatrix}. \quad (2.62)$$

Substituting the expression 2.62 in 2.60 we can arrive at

$$P_{ext} = \int_{\Omega} \bar{\mathbf{q}} \cdot \dot{\mathbf{d}} \, da + \int_{\Gamma} \bar{\mathbf{q}}^{\Gamma} \cdot \dot{\mathbf{d}} \, dx \quad (2.63)$$

where

$$\bar{\mathbf{q}} = \int_H \boldsymbol{\Sigma}^T \mathbf{b} \, dx + (\boldsymbol{\Sigma}|_{-})^T \mathbf{t}_{-} + (\boldsymbol{\Sigma}|_{+})^T \mathbf{t}_{+} \quad (2.64)$$

and

$$\mathbf{q}^{\Gamma} = \int_H \boldsymbol{\Sigma}^T (\mathbf{t}|_{\Gamma}) \, dx. \quad (2.65)$$

The use of the matrix $\boldsymbol{\Sigma}$ simplifies the deduction but hide how each part of the vector $\mathbf{q}$ is energetically conjugated with those of the $\dot{\mathbf{d}}$. This can be seen by expanding the terms in 2.60 with the aid of 2.61 what yields

$$P_{ext} = \int_{\Omega} \bar{\mathbf{V}} \cdot \dot{w} + \bar{\mathbf{m}} \cdot \dot{\boldsymbol{\theta}} + \bar{\boldsymbol{\beta}} \cdot \dot{\mathbf{p}} \, da + \int_{\Gamma} \bar{\mathbf{V}}^{\Gamma} \cdot \dot{w} + \bar{\mathbf{m}}^{\Gamma} \cdot \dot{\boldsymbol{\theta}} + \bar{\boldsymbol{\beta}}^{\Gamma} \cdot \dot{\mathbf{p}} \, dx, \quad (2.66)$$

where

$$\bar{\mathbf{q}} = \begin{bmatrix} \bar{\mathbf{V}} \\ \bar{\mathbf{m}} \\ \bar{\boldsymbol{\beta}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{t}}_{+} \cdot \mathbf{e}_3 + \bar{\mathbf{t}}_{-} \cdot \mathbf{e}_3 + \int_H b \, dx \\ h \mathbf{e}_3 \times \bar{\mathbf{t}}_{+} - h \mathbf{e}_3 \times \bar{\mathbf{t}}_{-} + \int_H \mathbf{a} \times \mathbf{b} \, dx \\ \boldsymbol{\Psi}_{+}^T \bar{\mathbf{t}}_{+} + \boldsymbol{\Psi}_{-}^T \bar{\mathbf{t}}_{-} + \int_H \boldsymbol{\Psi}^T \mathbf{b} \, dx \end{bmatrix} \quad (2.67)$$

and

$$\bar{\mathbf{q}}^{\Gamma} = \begin{bmatrix} \bar{\mathbf{V}}^{\Gamma} \\ \bar{\mathbf{m}}^{\Gamma} \\ \bar{\boldsymbol{\beta}}^{\Gamma} \end{bmatrix} = \begin{bmatrix} \int_H \mathbf{t}^{\Gamma} \, dx \\ \int_H \mathbf{a} \times \mathbf{t}^{\Gamma} \, dx \\ \int_H \boldsymbol{\Psi}^T \mathbf{t}^{\Gamma} \, dx \end{bmatrix}. \quad (2.68)$$

2.9 Balance equations and weak form

Evoking the the internal power in eq. 2.57, the external power in eq. 4.67 and the property that the time derivative $\dot{\mathbf{d}}$ is equivalent to the first variation $\delta \mathbf{d}$ then the weak form of the enriched model can be written as

$$\begin{aligned} \delta W &= \int_{\Omega} \boldsymbol{\sigma}(\mathbf{d}) \cdot \delta \boldsymbol{\epsilon} \, da - \int_{\Omega^r} \bar{\mathbf{q}} \cdot \delta \mathbf{d} \, da \\ &\quad - \int_{\Gamma^r} \bar{\mathbf{q}}^{\Gamma} \cdot \delta \mathbf{d} \, dx^r, \forall \delta \mathbf{d}. \end{aligned} \quad (2.69)$$

The use of the divergence theorem allows us to obtain the corresponding strong form of balance

equations

$$\mathcal{T}_e \boldsymbol{\sigma} + \mathbf{q} = \mathbf{0}, \text{ in } \Omega \quad (2.70)$$

$$\mathbf{n} \mathbf{H} \boldsymbol{\sigma} = \mathbf{q}^\Gamma \text{ in } \partial\Omega \quad (2.71)$$

where $\mathcal{T}_e$ is the adjoint operator of $\mathcal{T}_c$. The same expression in components is

$$V_{\alpha,\alpha} + \bar{V} = 0, \quad (2.72)$$

$$\mathbf{m}_{\alpha,\alpha} + (\mathbf{e}_\alpha \times \mathbf{e}_3) V_\alpha + \bar{\mathbf{m}} = \mathbf{0} \text{ and} \quad (2.73)$$

$$\rho_{i\alpha,\alpha} - \beta_i + \bar{\beta}_i = 0. \quad (2.74)$$

Denoting by $\partial\Omega$ the boundary of Ω and by $\boldsymbol{\nu}$ its unit exterior normal vector, the Neumann boundary conditions are stated as

$$V_\alpha \nu_\alpha = \bar{V}^\Gamma, \quad (2.75)$$

$$\mathbf{m}_\alpha \nu_\alpha = \bar{\mathbf{m}}^\Gamma \text{ and} \quad (2.76)$$

$$\rho_{i\alpha} \nu_\alpha = \bar{\beta}_i^\Gamma \quad (2.77)$$

over $\partial\Omega_n$ and the Dirichlet boundary conditions

$$w = \bar{w}, \quad (2.78)$$

$$\boldsymbol{\theta} = \bar{\boldsymbol{\theta}} \text{ and} \quad (2.79)$$

$$\boldsymbol{\beta} = \bar{\boldsymbol{\beta}} \quad (2.80)$$

over $\partial\Omega_d$ and the essential boundary conditions.

2.10 Finite element discretization

The approximate solution of the problem defined in the eq. 2.69 is sought by the FEM discretization. The domain Ω is represented as a partition $\mathcal{P}$ of triangular elements $\mathcal{K}$, where $\partial\mathcal{P}$ are the element edges, $\mathcal{N}$ are the set of element vertices and $\mathbf{m}_\gamma \in \partial\mathcal{P}$ are the mid-points of the edges.

Figure 2.3: *T63i finite element*

In order to avoid shear locking, the fields are interpolated by different types of FE discretization. The displacement field $\mathbf{u}$ is interpolated by the triangular element with quadratic Lagrangian shape functions, given by

$$\mathcal{M}^2 := \{\mathbf{v} \in \mathcal{C}^0; \mathbf{v}|_{\mathcal{K}} \in \mathbb{P}_2, \forall \mathcal{K} \in \mathcal{P}\}, \quad (2.81)$$

where $\mathbb{P}_n$ denotes the space of polynomials of degree up to n . This element is further referred to as T6. Its shape function are given by

$$N_1 = (2L_3 - 1)L_3, \quad (2.82a)$$

$$N_2 = (2L_1 - 1)L_1, \quad (2.82b)$$

$$N_3 = (2L_2 - 1)L_2, \quad (2.82c)$$

$$N_4 = 4L_1L_3, \quad (2.82d)$$

$$N_5 = 4L_1L_2 \text{ and} \quad (2.82e)$$

$$N_6 = 4L_2L_3, \quad (2.82f)$$

where $L_3 = 1 - L_1 - L_2$. This implies that $w \approx w^h \in \mathcal{M}^2 \subset \mathcal{H}_1$.

The other fields $\boldsymbol{\theta}$, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ are interpolated by the non-conforming finite element space

$$\begin{aligned} \mathcal{M}_{\text{nc}}^1 &:= \{\mathbf{v} \in \mathcal{L}^2; \mathbf{v}|_{\mathcal{K}} \in \mathbb{P}_1, \\ &\mathbf{v} \text{ is continuous at } \mathbf{m}_\gamma \forall \mathcal{K} \in \mathcal{P}\}. \end{aligned} \quad (2.83)$$

This element is called T3i. Its shape functions are given by

$$\bar{N}_4 = 1 - 2L_2, \quad (2.84a)$$

$$\bar{N}_5 = 1 - 2L_3 \text{ and} \quad (2.84b)$$

$$\bar{N}_6 = 1 - 2L_1. \quad (2.84c)$$

Hence, the displacement vector $\mathbf{d}$ in the finite element approximation is given by

$$\mathbf{d} \approx \mathbf{d}^h = \sum_i^{n_{\text{nodes}}} \mathbf{N}_i \mathbf{p}_i \quad (2.85)$$

where for $i = 1, 2, 3$

$$\begin{bmatrix} N_i \\ \mathbf{0}_{\mathbf{np}^* \times 1} \end{bmatrix}, \quad (2.86)$$

for $i = 4, 5, 6$

$$\begin{bmatrix} N_i & \mathbf{0}_{1 \times \mathbf{np}^*} \\ \mathbf{0}_{\mathbf{np}^* \times 1} & \bar{N}_i \mathbf{I}_{\mathbf{np}^* \times \mathbf{np}^*} \end{bmatrix} \quad (2.87)$$

, $\mathbf{np}^* = 2 + \mathbf{np}$ and $\mathbf{np}$ is the total number of parameters added by the enrichment in the thickness. This can be expressed in matrix notation with $\mathbf{d}^h = \mathbf{N}\mathbf{p}$, where the matrix $\mathbf{N}$ is equal to

$$\begin{bmatrix} \mathbf{N}_1 & \cdots & \mathbf{N}_6 \end{bmatrix}. \quad (2.88)$$

We can define the matrix $\mathbf{B} = \mathcal{T}_{\mathbf{c}}\mathbf{N}$, so that

$$\boldsymbol{\epsilon} \approx \boldsymbol{\epsilon}^h = \mathbf{B}\mathbf{p}. \quad (2.89)$$

And The stiffness matrix is given by

$$\mathbf{K}^e = \int_{\Omega_e} \mathbf{B}^T \mathbf{C} \mathbf{B} \, da. \quad (2.90)$$

This computation is carried out by numerical integration, where the 3 Gauss points in the mid-side points are used.

2.10.1 Closed-form for locking cure

With a special choice of an 3-node geometry mapping, we can easily obtained the closed-form expressions for strains, showing that the proposed approach can avoid the usual locking phenomena and provide an excellent performance.

The locking is given by an deterioration of the convergence rate as the thickness H goes to 0. The problem is linked to the incapacity of the model to represent the limite model condtion. In the case of plates this means that shear γ_α must be capable of being zero and the normal stress T_{33} also must be capable of been zero.

The shear strain is given by

$$\boldsymbol{\gamma} = \nabla w - \boldsymbol{\alpha}, \quad (2.91)$$

, where $\boldsymbol{\alpha} = [\theta_2 - \theta_1]^T$. The condition $\boldsymbol{\gamma} = \mathbf{0}$ implies $\nabla w = \boldsymbol{\alpha}$. It's known from [7] that a condition for stability of the discretization is that

$$\nabla w_h \subseteq \boldsymbol{\alpha}_h. \quad (2.92)$$

what implies the discontinuity of the space $\boldsymbol{\theta}_h$.

In the literature there are two popular ways of solving this. The first is to use the Enhanced Assumed Strain so that the ∇w is enhanced with a deformation $\tilde{\epsilon}$ by some non-conforming deformation mode. The second is the Assumed Strain Method that uses a projection $R(\bullet)$ over $\boldsymbol{\theta}$, so that $\nabla w_h \subseteq R(\boldsymbol{\theta})$.

The deRham complex as presented in [10] gives a good framework to evaluate the finite element subspaces. The deRham complex show that $\nabla \mathcal{M}^2 = \mathcal{RT}_1 \subseteq \mathcal{M}_{nc}^1$, where $\mathcal{RT}_1$ is the Raviart-Thomas element of first-order.

The element has one order higher in the displacement w from the element [43], this avoids the problem evaluated in [9].

It is interesting to note that no special assumed strain modification are needed for this element.

2.11 Results

This section presents some numerical results that evaluate the asymptotic consistency and approximabilty of the proposed finite element. It was chosen a plate clamped like a cantilever as presented in 3.6.

The clamped condition allows the rise of a strong boundary layer. Since this result doesn't have an analytical solution it is necessary a numerical solution to have some comparison. It was not found a three-dimensional solution with strong boundary layer. All analytical solutions found were without any boundary layer.

Figure 2.4: *First benchmark*

The results were evaluated for a fixed thickness h and varying enrichment model and for fixed enriched model and varying thickness.

The data is presented in the Table 4.2.

Table 2.2: *Data of the first benchmark*

Data name	Value
(force) $b =$	0.001
(width) $w =$	1
(Length) $L =$	1
$E =$	20000
$\nu =$	0.25

2.11.1 d^1 model

1st discretization

Table 2.3: 1st discretization

Field	Finite element
w	Lagrangean of 1st order
θ_1, θ_2	Lagrangean of 1st order
p_{31}	Lagrangean of 1st order

Table 2.4: 1st discretization, model d^1 , $h = 0.1$

Field	Convergence rate
u_1	2.6325668815513588
θ_1	2.625016886018986
θ_2	2.6314072764442247
p_{31}	1.844656393970748

Table 2.5: 1st discretization, model d^1 , $h = 0.01$

Field	Convergence rate
u_1	1.8126781731775772
θ_1	0.9856445914610696
θ_2	1.8091655913668927
p_{31}	1.7134519713040564

Table 2.6: 1st discretization, model d^1 , $h = 0.001$

Field	Convergence rate
u_1	0.7216972155205952
θ_1	0.7605172384799745
θ_2	0.7232437726791978
p_{31}	0.7227790298711568

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ **Figure 2.5:** 1st discretization, model d^1 (a) w (b) θ_1 (c) θ_2 (d) p_{31} **Figure 2.6:** 1st discretization, model d^1 , $h = 0.1$

Figure 2.7: 1st discretization, model d^1 , $h = 0.01$

Figure 2.8: 1st discretization, model d^1 , $h = 0.001$

2nd discretization

Table 2.7: 2nd discretization

Field	Finite element
w	Lagrangean of 2nd order
θ_1, θ_2	Crouzeix-Raviart of 1st order
p_{31}	Crouzeix-Raviart of 1st order

Table 2.8: 2nd discretization, model d^1 , $h = 0.1$

Field	Convergence rate
u_1	2.3505664857085873
θ_1	-0.20097463089840584
θ_2	-0.202408689385016
p_{31}	1.1658415038739396

Table 2.9: 2nd discretization, model d^1 , $h = 0.01$

Field	Convergence rate
u_1	2.0927126169269896
θ_1	1.179404483106979
θ_2	2.059171446525681
p_{31}	0.6285592226689287

Table 2.10: 2nd discretization, model d^1 , $h = 0.001$

Field	Convergence rate
u_1	2.1026236671176375
θ_1	2.1792889187611397
θ_2	2.1189875266929294
p_{31}	0.5820054434505044

Table 2.11: 2nd discretization, model d^1 , $h = 0.0001$

Field	Convergence rate
u_1	-0.013816169436407212
θ_1	0.8022068479406821
θ_2	-0.39904757805243013
p_{31}	0.5767559539627974

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$ **Figure 2.9:** 2nd discretization, model d^1

(a) w (b) θ_1 (c) θ_2 (d) p_{31} **Figure 2.10:** 2nd discretization, model d^1 , $h = 0.1$

Figure 2.11: 2nd discretization, model d^1 , $h = 0.01$

Figure 2.12: 2nd discretization, model d^1 , $h = 0.001$

3rd discretization

Table 2.12: 3rd discretization

Field	Finite element
w	Lagrangean of 2nd order
θ_1, θ_2	Crouzeix-Raviart of 1st order
p_{31}	Lagrangean of 1st order

Table 2.13: 3rd discretization, model d^1 , $h = 0.1$

Field	Convergence rate
u_1	1.841678267015977
θ_1	-0.20034164247399355
θ_2	-0.16790857106872575
p_{31}	1.8376278415949558

Table 2.14: 3rd discretization, model d^1 , $h = 0.01$

Field	Convergence rate
u_1	2.0503853897294673
θ_1	1.3389180186342156
θ_2	2.0366002516069157
p_{31}	1.1991093264901356

Table 2.15: 3rd discretization, model d^1 , $h = 0.001$

Field	Convergence rate
u_1	2.055198160004889
θ_1	2.1397417801826752
θ_2	2.062899138317646
p_{31}	1.1583439197511576

Table 2.16: 3rd discretization, model d^1 , $h = 0.0001$

Field	Convergence rate
u_1	2.832429397549282
θ_1	1.0815697470941912
θ_2	2.70706191453826
p_{31}	1.1679032450800653

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$ **Figure 2.13:** 3rd discretization, model d^1

(a) w (b) θ_1 (c) θ_2 (d) p_{31} **Figure 2.14:** 3rd discretization, model d^1 , $h = 0.1$

Figure 2.15: 3rd discretization, model d^1 , $h = 0.01$

Figure 2.16: 3rd discretization, model d^1 , $h = 0.001$

4th discretization

Table 2.17: 4th discretization

Field	Finite element
w	Lagrangean of 2st order
θ_1, θ_2	Crouzeix-Raviart of 1st order
p_{31}	Lagrangean of 2 order

Table 2.18: 4th discretization, model d^1 , $h = 0.1$

Field	Convergence rate
u_1	1.9088840130599447
θ_1	-0.20148840747277988
θ_2	-0.20539193065572475
p_{31}	2.8874963468523127

Table 2.19: 4th discretization, model d^1 , $h = 0.01$

Field	Convergence rate
u_1	2.1541096337523555
θ_1	1.0792011319160826
θ_2	2.0831523051671232
p_{31}	1.1965246849612687

Table 2.20: 4th discretization, model d^1 , $h = 0.001$

Field	Convergence rate
u_1	2.132534241452662
θ_1	2.1803639363744454
θ_2	2.15317153344634
p_{31}	0.9375732589015802

Table 2.21: 4th discretization, model d^1 , $h = 0.0001$

Field	Convergence rate
u_1	3.0075350652820316
θ_1	0.4539649924153726
θ_2	2.650514042519329
p_{31}	0.9225106068024219

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$ **Figure 2.17:** 4th discretization, model d^1

(a) w (b) θ_1 (c) θ_2 (d) p_{31} **Figure 2.18:** 4th discretization, model d^1 , $h = 0.1$

Figure 2.19: 4th discretization, model d^1 , $h = 0.01$

Figure 2.20: 4th discretization, model d^1 , $h = 0.001$

5th discretization

Table 2.22: 5th discretization

Field	Finite element
w	Lagrangean of 2nd order
θ_1, θ_2	Lagrangean of 2nd order
p_{31}	Lagrangean of 2nd order

Table 2.23: 5th discretization, model d^1 , $h = 0.1$

Field	Convergence rate
u_1	2.039031398183664
θ_1	1.6369482307886696
θ_2	2.6705490872168296
p_{31}	3.0371054311508465

Table 2.24: 5th discretization, model d^1 , $h = 0.01$

Field	Convergence rate
u_1	2.2342115279721755
θ_1	2.72855182859306
θ_2	2.238862870061412
p_{31}	1.3001034166431031

Table 2.25: 5th discretization, model d^1 , $h = 0.001$

Field	Convergence rate
u_1	2.6221279551980152
θ_1	3.5084972008996824
θ_2	2.719794979235333
p_{31}	0.9818652301155654

Table 2.26: 5th discretization, model d^1 , $h = 0.0001$

Field	Convergence rate
u_1	2.6221279551980152
θ_1	3.5084972008996824
θ_2	2.719794979235333
p_{31}	0.9818652301155654

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$ **Figure 2.21:** 5th discretization, model d^1

Figure 2.22: 5th discretization, model d^1 , $h = 0.1$

Figure 2.23: 5th discretization, model d^1 , $h = 0.01$

Figure 2.24: 5th discretization, model d^1 , $h = 0.001$

2.11.2 d^2 model

1st discretization

Table 2.27: 1st discretization

Field	Finite element
w	Crouzeix-Raviart of 1st order
θ_1, θ_2	Lagrangean of 2nd order
p_{11}, p_{21}	Lagrangean of 2nd order
p_{31}	Crouzeix-Raviart of 1st order

Table 2.28: 1st discretization, model d^2 , $h = 0.1$

Field	Convergence rate
u_1	2.750883488232455
θ_1	2.658063996622881
θ_2	2.7308891182746864
p_{11}	2.600887289580462
p_{21}	1.7937820037946115
p_{31}	0.843698507186774

Table 2.29: 1st discretization, model d^2 , $h = 0.01$

Field	Convergence rate
u_1	1.9950837805246031
θ_1	1.5017954413942896
θ_2	1.9905438750240594
p_{11}	1.050805870850197
p_{21}	0.8340017544322151
p_{31}	1.799287186414251

Table 2.30: 1st discretization, model d^2 , $h = 0.001$

Field	Convergence rate
u_1	0.729956300914217
θ_1	0.7936078748627902
θ_2	0.7307941333661789
p_{11}	-0.06438471262733007
p_{21}	0.2899818646663301
p_{31}	0.7349610443994302

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ **Figure 2.25:** 1st discretization, model d^2 (a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{21} (f) p_{31} **Figure 2.26:** 1st discretization, model d^2 , $h = 0.1$

Figure 2.27: 1st discretization, model d^2 , $h = 0.01$

Figure 2.28: 1st discretization, model d^2 , $h = 0.001$

2nd discretization

Table 2.31: 2nd discretization

Field	Finite element
w	Lagrangean of 2nd order
θ_1, θ_2	Crouzeix-Raviart of 1st order
p_{11}, p_{21}	Crouzeix-Raviart of 1st order
p_{31}	Crouzeix-Raviart of 1st order

Table 2.32: 2nd discretization, model d^2 , $h = 0.1$

Field	Convergence rate
u_1	2.7006689174104985
θ_1	-0.19506075895336702
θ_2	-0.2050251126908255
p_{11}	-0.16622527211150664
p_{21}	-0.16804637277714377
p_{31}	1.203187101821119

Table 2.33: 2nd discretization, model d^2 , $h = 0.01$

Field	Convergence rate
u_1	2.077597925500127
θ_1	0.8846460023360879
θ_2	2.006281805345041
p_{11}	0.2686612568664468
p_{21}	-0.015224778580392664
p_{31}	0.6457713450012081

Table 2.34: 2nd discretization, model d^2 , $h = 0.001$

Field	Convergence rate
u_1	2.1068683399923103
θ_1	2.179774249439427
θ_2	2.124138280373205
p_{11}	-0.05807183796649546
p_{21}	-0.30117172552050503
p_{31}	0.5825948873166689

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ **Figure 2.29:** 2nd discretization, model d^2 (a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{21} (f) p_{31} **Figure 2.30:** 2nd discretization, model d^2 , $h = 0.1$

Figure 2.31: 2nd discretization, model d^2 , $h = 0.01$

Figure 2.32: 2nd discretization, model d^2 , $h = 0.001$

3rd discretization

Table 2.35: 3rd discretization

Field	Finite element
w	Lagrangean of 2nd order
θ_1, θ_2	Crouzeix-Raviart of 1st order
p_{11}, p_{21}	Crouzeix-Raviart of 1st order
p_{31}	Lagrangean of 2nd order

Table 2.36: 3rd discretization, model d^2 , $h = 0.1$

Field	Convergence rate
u_1	2.485162656245949
θ_1	-0.1961222857601936
θ_2	-0.20713404272427852
p_{11}	-0.17107521594701974
p_{21}	-0.17011637348559286
p_{31}	2.3998151793408704

Table 2.37: 3rd discretization, model d^2 , $h = 0.01$

Field	Convergence rate
u_1	2.1376802750603274
θ_1	0.7886257128144509
θ_2	1.9969148262976382
p_{11}	0.13332361112828392
p_{21}	-0.009769977233092779
p_{31}	1.2578678018847973

Table 2.38: 3rd discretization, model d^2 , $h = 0.001$

Field	Convergence rate
u_1	2.131375463931121
θ_1	2.183520495556466
θ_2	2.1520082159316254
p_{11}	-0.1219824334148893
p_{21}	-0.3021505953482983
p_{31}	0.9398741236840497

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ **Figure 2.33:** 3rd discretization, model d^2 (a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{21} (f) p_{31} **Figure 2.34:** 3rd discretization, model d^2 , $h = 0.1$

Figure 2.35: 3rd discretization, model d^2 , $h = 0.01$

Figure 2.36: 3rd discretization, model d^2 , $h = 0.001$

5th discretization

Table 2.39: 5th discretization

Field	Finite element
w	Lagrangean of 2nd order
θ_1, θ_2	Lagrangean of 2nd order
p_{11}, p_{21}	Lagrangean of 2nd order
p_{31}	Lagrangean of 2nd order

Table 2.40: 5th discretization, model d^2 , $h = 0.1$

Field	Convergence rate
u_1	2.0978526059093343
θ_1	1.6406905192312409
θ_2	2.6321659075624186
p_{11}	3.2101516714056815
p_{21}	2.708242119623654
p_{31}	3.124830343468199

Table 2.41: 5th discretization, model d^2 , $h = 0.01$

Field	Convergence rate
u_1	2.232606965655369
θ_1	2.765095067115047
θ_2	2.228023354781486
p_{11}	0.6793102322949869
p_{21}	1.9299366314723865
p_{31}	1.3699344711033663

Table 2.42: 5th discretization, model d^2 , $h = 0.001$

Field	Convergence rate
u_1	2.614748221236825
θ_1	3.5074510083945394
θ_2	2.7106740273587886
p_{11}	-0.09100818650529868
p_{21}	-0.08290390775611192
p_{31}	0.9851393948720041

Table 2.43: 5th discretization, model d^2 , $h = 0.0001$

Field	Convergence rate
u_1	2.614748221236825
θ_1	3.5074510083945394
θ_2	2.7106740273587886
p_{11}	-0.09100818650529868
p_{21}	-0.08290390775611192
p_{31}	0.9851393948720041

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$ **Figure 2.37:** 5th discretization, model d^1

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{21} (f) p_{31} **Figure 2.38:** 5th discretization, model d^1 , $h = 0.1$

Figure 2.39: 5th discretization, model d^1 , $h = 0.01$

Figure 2.40: 5th discretization, model d^1 , $h = 0.001$

5th discretization

Table 2.44: 5th discretization

Field	Finite element
w	Lagrangean of 3rd order
θ_1, θ_2	Lagrangean of 3rd order
p_{11}, p_{21}	Lagrangean of 3rd order
p_{31}	Lagrangean of 3rd order

Table 2.45: 5th discretization, model d^2 , $h = 0.1$

Field	Convergence rate
u_1	—
θ_1	1.5505755850769738
θ_2	2.6332902079142304
p_{11}	3.5746550056451407
p_{21}	2.7179100973705657
p_{31}	3.388513093039135

Table 2.46: 5th discretization, model d^2 , $h = 0.01$

Field	Convergence rate
u_1	2.4290338187131972
θ_1	2.631726793527394
θ_2	2.4005343160976578
p_{11}	1.0765911347638883
p_{21}	1.1151113666141672
p_{31}	1.5071285853102818

Table 2.47: 5th discretization, model d^2 , $h = 0.001$

Field	Convergence rate
u_1	2.754589979797355
θ_1	2.2366744969967103
θ_2	2.8364638569910525
p_{11}	0.34116459328104626
p_{21}	0.5944192916486888
p_{31}	1.0899819413661982

Table 2.48: 5th discretization, model d^2 , $h = 0.0001$

Field	Convergence rate
u_1	2.754589979797355
θ_1	2.2366744969967103
θ_2	2.8364638569910525
p_{11}	0.34116459328104626
p_{21}	0.5944192916486888
p_{31}	1.0899819413661982

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$ **Figure 2.41:** 5th discretization, model d^1

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{21} (f) p_{31} **Figure 2.42:** 5th discretization, model d^1 , $h = 0.1$

Figure 2.43: 5th discretization, model d^1 , $h = 0.01$

Figure 2.44: 5th discretization, model d^1 , $h = 0.001$

2.11.3 d^3 model

1st discretization

Table 2.49: 1st discretization

Field	Finite element
w	Lagrangean of 3rd order
θ_1, θ_2	Lagrangean of 3rd order
$p_{11}, p_{12}, p_{21}, p_{22}$	Lagrangean of 3rd order
p_{31}, p_{32}	Lagrangean of 3rd order

Table 2.50: 1st discretization, model d^3 , $h = 0.1$

Field	Convergence rate
u_1	—
θ_1	1.564232613688242
θ_2	2.0021182658787895
p_{11}	2.9931252782382947
p_{12}	3.0053015643734864
p_{21}	2.678607004753345
p_{22}	2.8038710456394647
p_{31}	2.757620409759178
p_{32}	2.7731106474428513

Table 2.51: 1st discretization, model d^3 , $h = 0.01$

Field	Convergence rate
u_1	2.3055843629703543
θ_1	2.5363633094582028
θ_2	2.305156216803144
p_{11}	0.17888789844254604
p_{12}	0.3702157476409058
p_{21}	0.36984913995805435
p_{22}	0.415607480631427
p_{31}	1.8453985897165812
p_{32}	0.3966348329232215

Table 2.52: 1st discretization, model d^3 , $h = 0.001$

Field	Convergence rate
u_1	2.8598750152867973
θ_1	2.0431532296772335
θ_2	2.9198264803648177
p_{11}	0.38564904751113854
p_{12}	-0.03745722507865332
p_{21}	0.71667052664572
p_{22}	-0.052630957835812876
p_{31}	1.0962035147566114
p_{32}	-0.1685810337295464

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ **Figure 2.45:** 1st discretization, model d^3

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{12} (f) p_{21} (g) p_{22} (h) p_{31} (i) p_{32} **Figure 2.46:** 1st discretization, model d^3 , $h = 0.1$

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{12} (f) p_{21} (g) p_{22} (h) p_{31} (i) p_{32} **Figure 2.47:** 1st discretization, model d^3 , $h = 0.01$

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{12} (f) p_{21} (g) p_{22} (h) p_{31} (i) p_{32} Figure 2.48: 1st discretization, model d^3 , $h = 0.001$

2nd discretization

Table 2.53: 2nd discretization

Field	Finite element
w	Lagrangean of 3rd order
θ_1, θ_2	Lagrangean of 3rd order
$p_{11}, p_{12}, p_{21}, p_{22}$	Lagrangean of 3rd order
p_{31}, p_{32}	Lagrangean of 3rd order

Table 2.54: 2nd discretization, model d^3 , $h = 0.1$

Field	Convergence rate
u_1	—
θ_1	—
θ_2	2.225189515107465
p_{11}	3.8953128435733464
p_{12}	3.9127621797455565
p_{21}	2.802399000016283
p_{22}	2.9184463499235003
p_{31}	3.7343190171209844
p_{32}	3.7174814845395097

Table 2.55: 2nd discretization, model d^3 , $h = 0.01$

Field	Convergence rate
u_1	2.5080672113248768
θ_1	2.409437150131387
θ_2	2.505260629303098
p_{11}	0.08790738668119083
p_{12}	0.389883884271111
p_{21}	-0.1019278331817581
p_{22}	0.1819778171786652
p_{31}	0.7923011548302755
p_{32}	0.4831839077536301

Table 2.56: 2nd discretization, model d^3 , $h = 0.001$

Field	Convergence rate
u_1	3.400685298571223
θ_1	0.8009110674072085
θ_2	1.9334905989393067
p_{11}	0.07320870609081455
p_{12}	-0.13734075794197426
p_{21}	-0.13558006255118374
p_{22}	-0.21920791328194775
p_{31}	0.9518841992479731
p_{32}	-0.3126835562321617

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ **Figure 2.49:** 2nd discretization, model d^3

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{12} (f) p_{21} (g) p_{22} (h) p_{31} (i) p_{32} **Figure 2.50:** 2nd discretization, model d^3 , $h = 0.1$

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{12} (f) p_{21} (g) p_{22} (h) p_{31} (i) p_{32} **Figure 2.51:** 2nd discretization, model d^3 , $h = 0.01$

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{12} (f) p_{21} (g) p_{22} (h) p_{31} (i) p_{32} **Figure 2.52:** 2nd discretization, model d^3 , $h = 0.001$

2nd discretization

Table 2.57: 2nd discretization

Field	Finite element
w	Lagrangean of 3rd order
θ_1, θ_2	Lagrangean of 3rd order
$p_{11}, p_{12}, p_{21}, p_{22}$	Lagrangean of 3rd order
p_{31}, p_{32}	Lagrangean of 3rd order

Table 2.58: 2nd discretization, model d^3 , $h = 0.1$

Field	Convergence rate
u_1	1.8113995088163686
θ_1	1.566056018721568
θ_2	1.8130663034437862
p_{11}	2.397092344485686
p_{12}	2.5055891012278324
p_{21}	1.7010522063896012
p_{22}	1.8017366179078886
p_{31}	2.7568685541023643
p_{32}	2.7381810377643645

Table 2.59: 2nd discretization, model d^3 , $h = 0.01$

Field	Convergence rate
u_1	2.2382477818055078
θ_1	2.51518747075009
θ_2	2.240216413514585
p_{11}	1.5668720473373894
p_{12}	0.02057141538713522
p_{21}	1.1486524145410844
p_{22}	0.3946598467341649
p_{31}	2.6557400741660495
p_{32}	0.47837606619197237

Table 2.60: 2nd discretization, model d^3 , $h = 0.001$

Field	Convergence rate
u_1	2.3884379873106782
θ_1	1.8460087518849868
θ_2	2.4388924185624163
p_{11}	0.5001423547370613
p_{12}	0.06153032139980976
p_{21}	0.6910450999811929
p_{22}	-0.07141440589849449
p_{31}	1.120920846890829
p_{32}	0.19808733475917423

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ **Figure 2.53:** 2nd discretization, model d^3

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{12} (f) p_{21} (g) p_{22} (h) p_{31} (i) p_{32} **Figure 2.54:** 2nd discretization, model d^3 , $h = 0.1$

 (a) w

 (b) θ_1

 (c) θ_2

 (d) p_{11}

 (e) p_{12}

 (f) p_{21}

 (g) p_{22}

 (h) p_{31}

 (i) p_{32}

 Figure 2.55: 2nd discretization, model d^3 , $h = 0.01$

(a) w (b) θ_1 (c) θ_2 (d) p_{11} (e) p_{12} (f) p_{21} (g) p_{22} (h) p_{31} (i) p_{32} Figure 2.56: 2nd discretization, model d^3 , $h = 0.001$

Linear Error

The analytical solution of the original problem 2.2 is unavailable in the majority of the practical cases. This imposes the use of a numerical procedure. The Finite Element Method uses the Ritz-Galerkin method to find a solution that minimize $\Pi_p(\mathbf{u})$ in a subspace $\mathcal{U}_K^h$ of the original solution space $\mathcal{U}_K$. The use of a posteriori error techniques allows the analyst to have a strong confidence level in the final result.

This chapter starts with the classical error in the discretization. The next section compares the classical error equation to the reaction imposed by the finite element shape function restriction. The error in hierarchical models is presented after that, followed by the error in the kinematical restriction. Those sections show a general view of the error. After that, it is presented techniques to estimate the error. It is shown ZZ super patch recovery, after explicit error and equilibrated implicit error. It is presented only the theory of implicit error.

The notation used here is the same of the section 2.10 , where the domain Ω is represented as a partition $\mathcal{P}$ of triangular elements $\mathcal{K}$, where $\partial\mathcal{P}$ are the element edges, $\mathcal{N}$ are the set of element vertices and $\mathbf{m}_\gamma \in \partial\mathcal{P}$ are the mid-points of the edges.

3.1 Error in discretization

The relation in equation 2.16 implies a energy gap between the solution, so that $\Pi_p(\mathbf{u}_h) - \Pi_p(\mathbf{u}) > 0$. Furthermore, defining the error to be $\mathbf{e} = \mathbf{u} - \mathbf{u}_h \in \mathcal{U}_K$ and evoking the first variation of the problem in 2.7, we arrive at the equation

$$\int_V \mathbf{T}(\mathbf{e}) : \delta \mathbf{E} \, dv = \int_{\partial V_n} \bar{\mathbf{t}} \cdot \delta \mathbf{u} \, dv + \int_V \bar{\mathbf{b}} \cdot \delta \mathbf{u} \, dv - \int_V \mathbf{T}(\mathbf{u}_h) : \delta \mathbf{E} \, dv, \quad \forall \delta \mathbf{u} \in \delta \mathcal{U}_K, \quad (3.1)$$

where the linearity of the stress tensor $\mathbf{T}$ in relation to $\mathbf{u}$ was used. One of the properties of the Ritz-Galerkin methods is its orthogonality with relation to the error given by

$$\int_V \mathbf{T}(\mathbf{u}_h) : \delta \mathbf{E}_h \, dv - \int_{\partial V_n} \bar{\mathbf{t}} \cdot \delta \mathbf{u}_h \, dv - \int_V \bar{\mathbf{b}} \cdot \delta \mathbf{u}_h \, dv = 0, \quad \forall \delta \mathbf{u}_h \in \delta \mathcal{U}_K^h. \quad (3.2)$$

We can decompose the solution space in $\mathcal{U}_K = \mathcal{U}_K^h \oplus \mathcal{U}_K^+$ and use it to subtract the equation 3.2 of 3.1. This yields

$$\int_V \mathbf{T}(\mathbf{e}) : \delta \mathbf{E}_+ \, dv = \int_{\partial V_n} \bar{\mathbf{t}} \cdot \delta \mathbf{u}_+ \, dv + \int_V \bar{\mathbf{b}} \cdot \delta \mathbf{u}_+ \, dv - \int_V \mathbf{T}(\mathbf{u}_h) : \delta \mathbf{E}_+ \, dv, \quad \forall \delta \mathbf{u}_+ \in \delta \mathcal{U}_K^+, \quad (3.3)$$

where $\delta \mathbf{E}_+ = \text{Sym}(\nabla \delta \mathbf{u}_+)$. The eq. 3.3 is the error equation.

The energy norm of the error is given by

$$\eta^2 = \|e\|_E^2 = \int_V \mathbf{T}(e) : \delta \mathbf{E}(e) \, dv \quad (3.4)$$

and the strong form of the error is obtained by the use of the divergence theorem and is given by

$$\operatorname{div}(\mathbf{T}(e) + \mathbf{T}(\mathbf{u}_h)) + \mathbf{b} = \mathbf{0}, \text{ in } \mathcal{K} \quad (3.5a)$$

$$\llbracket \mathbf{T}(e) \cdot \boldsymbol{\nu} \rrbracket + \llbracket \mathbf{T}(\mathbf{u}_h) \cdot \boldsymbol{\nu} \rrbracket = \mathbf{0}, \text{ on } \partial \mathcal{K} \quad (3.5b)$$

$$\mathbf{T}(e) \cdot \boldsymbol{\nu} = \bar{\mathbf{t}} - \mathbf{T}(\mathbf{u}_h) \cdot \boldsymbol{\nu}, \text{ on } \partial \mathcal{K} \cap \partial \Omega_n \quad (3.5c)$$

where $\boldsymbol{\nu}$ is the normal to the edge $\partial \mathcal{K}$ of the element $\mathcal{K}$ and $\llbracket \bullet \rrbracket$ is the jump operator described by the eq. 3.6.

$$\llbracket \mathbf{T} \cdot \boldsymbol{\nu} \rrbracket = \begin{cases} \mathbf{T} \cdot \boldsymbol{\nu}_1, & \text{on } \partial \mathcal{K}_1 \cap \partial \Omega \\ \mathbf{T} \cdot \boldsymbol{\nu}_1 + \mathbf{T} \cdot \boldsymbol{\nu}_2, & \text{on } \partial \mathcal{K}_1 \cap \partial \mathcal{K}_2 \end{cases} \quad (3.6)$$

Another way to write the strong form of error is given by

$$\sum_{\mathcal{K} \in \mathcal{P}} \int_{\mathcal{K}} \mathbf{T}(e) : \delta \mathbf{E}_+ \, dv = \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_V \mathbf{r} \cdot \delta \mathbf{u}_+ \, dv + \int_{\partial \mathcal{K}} \mathbf{j} \cdot \delta \mathbf{u}_+ \, da \right\}, \quad (3.7)$$

where

$$\mathbf{j} = \begin{cases} \bar{\mathbf{t}} - \mathbf{T}(\mathbf{u}_h) \cdot \boldsymbol{\nu}, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket \mathbf{T} \cdot \boldsymbol{\nu} \rrbracket), & \text{on } \partial \mathcal{K} \end{cases} \quad (3.8)$$

and

$$\mathbf{r} = \operatorname{div}(\mathbf{T}(\mathbf{u}_h)) + \mathbf{b}. \quad (3.9)$$

The strong form of the error in the hierarchical model can be easily found by the evocation of the strong form presented in section 2.9.

$$\sum_{\mathcal{K} \in \mathcal{P}} \int_{\mathcal{K}} \boldsymbol{\sigma}(e) \cdot \delta \boldsymbol{\epsilon}(\mathbf{d}_+) \, dv = \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_{\mathcal{K}} \mathbf{r} \cdot \delta \mathbf{d}_+ \, da + \int_{\partial \mathcal{K}} \mathbf{j} \cdot \delta \mathbf{d}_+ \, dx \right\}, \quad (3.10)$$

where

$$\mathbf{j} = \begin{cases} \mathbf{n} \mathbf{H} \boldsymbol{\sigma}(\mathbf{d}^h) - \mathbf{q}^\Gamma, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket \mathbf{n} \mathbf{H} \boldsymbol{\sigma} \rrbracket), & \text{on } \partial \mathcal{K} \end{cases} \quad (3.11)$$

and

$$\mathbf{r} = \mathcal{T}_e(\boldsymbol{\sigma}(\mathbf{d}^h)) + \mathbf{q}. \quad (3.12)$$

Another form to write 3.10 is by using the the strong form in the generalized stresses as presented

in section 2.9.

$$\begin{aligned}
\sum_{\mathcal{K} \in \mathcal{P}} \int_{\mathcal{K}} \boldsymbol{\sigma}(\mathbf{e}) \cdot \delta \boldsymbol{\epsilon}(\mathbf{d}_+) \, dv &= \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_{\mathcal{K}} \mathbf{r}_V \cdot \delta w_+ \, da + \int_{\partial \mathcal{K}} \mathbf{j}_V \cdot \delta w_+ \, dx \right\} \\
&+ \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_{\mathcal{K}} \mathbf{r}_M \cdot \delta \boldsymbol{\theta}_+ \, da + \int_{\partial \mathcal{K}} \mathbf{j}_M \cdot \delta \boldsymbol{\theta}_+ \, dx \right\} \\
&+ \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_{\mathcal{K}} \mathbf{r}_{p_i} \cdot \delta \mathbf{p}_{i+} \, da + \int_{\partial \mathcal{K}} \mathbf{j}_{p_i} \cdot \delta \mathbf{p}_{i+} \, dx \right\},
\end{aligned} \tag{3.13}$$

where

$$\mathbf{j}_V = \begin{cases} V_\alpha \nu_\alpha - \bar{V}^\Gamma, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket V_\alpha \nu_\alpha \rrbracket), & \text{on } \partial \mathcal{K} \end{cases} \tag{3.14}$$

$$\mathbf{j}_M = \begin{cases} \mathbf{m}_\alpha \nu_\alpha - \bar{\mathbf{m}}^\Gamma, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket \mathbf{m}_\alpha \nu_\alpha \rrbracket), & \text{on } \partial \mathcal{K} \end{cases} \tag{3.15}$$

$$\mathbf{j}_{p_i} = \begin{cases} \boldsymbol{\rho}_{i\alpha} \nu_\alpha - \bar{\boldsymbol{\beta}}_i^\Gamma, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket \boldsymbol{\rho}_{i\alpha} \nu_\alpha \rrbracket), & \text{on } \partial \mathcal{K} \end{cases} \tag{3.16}$$

and

$$\begin{aligned}
\mathbf{r}_V &= V_{\alpha,\alpha} + \bar{V}, \\
\mathbf{r}_M &= \mathbf{m}_{\alpha,\alpha} + (\mathbf{e}_\alpha \times \mathbf{e}_3) V_\alpha + \bar{\mathbf{m}}, \\
\mathbf{r}_{p_i} &= \boldsymbol{\rho}_{i\alpha,\alpha} - \boldsymbol{\beta}_i + \bar{\boldsymbol{\beta}}_i.
\end{aligned} \tag{3.17}$$

3.1.1 Error as reaction

Another approach to error is to see it as a restriction to original problem 2.2 so that the functional

$$\Pi_M(\mathbf{u}, \mathbf{u}_h, \boldsymbol{\lambda}) = U_{\text{int}}(\mathbf{u}) + U_{\text{ext}}(\mathbf{u}) - \int_V \boldsymbol{\lambda} \cdot (\mathbf{u} - \mathbf{u}_h) \, dv + \int_{\partial V} \boldsymbol{\lambda} \cdot (\mathbf{u} - \mathbf{u}_h) \, da, \tag{3.18}$$

represents the same problem presented before, where $\boldsymbol{\lambda}$ is a Lagrange multiplier. An important remark is that the Lagrange multiplier will be different of zero when the restriction $(\mathbf{u} - \mathbf{u}_h)$ is activated. A possible interpretation is that the lagrange multiplier is the reaction to the finite element restriction. The solution of eq. 3.18 is found by taking the first variation, given by

$$\delta \Pi_M \cdot \delta \mathbf{u} = \int_V \mathbf{T} : \delta \mathbf{E} \, dv - \int_V \bar{\mathbf{b}} \cdot \delta \mathbf{u} \, dv - \int_{\partial V} \bar{\mathbf{t}} \cdot \delta \mathbf{u} \, da - \int_V \boldsymbol{\lambda} \cdot \delta \mathbf{u} \, dv + \int_{\partial V} \boldsymbol{\lambda} \cdot \delta \mathbf{u} \, da = 0, \tag{3.19a}$$

$$\delta \Pi_M \cdot \delta \boldsymbol{\lambda} = \int_V \delta \boldsymbol{\lambda} \cdot (\mathbf{u} - \mathbf{u}_h) \, dv - \int_{\partial V} \delta \boldsymbol{\lambda} \cdot (\mathbf{u} - \mathbf{u}_h) \, da = 0 \text{ and} \tag{3.19b}$$

$$\delta \Pi_M \cdot \delta \mathbf{u}_h = \int_V \boldsymbol{\lambda} \cdot \delta \mathbf{u}_h \, dv - \int_{\partial V} \boldsymbol{\lambda} \cdot \delta \mathbf{u}_h \, da = 0, \tag{3.19c}$$

what results in the KKT conditions. If we take

$$\boldsymbol{\lambda} = \text{div}(\mathbf{T}^R) \text{ in } V \text{ and } \mathbf{T}^R \cdot \boldsymbol{\nu} = \boldsymbol{\lambda} \text{ on } \partial V. \tag{3.20}$$

and evoke the Green theorem in the equation 3.19c, it leads to

$$\int_V \mathbf{T}^R : \nabla \mathbf{u}_h \, dv = \int_V \mathbf{T}^R : \mathbf{E}_h \, dv = 0. \quad (3.21)$$

Therefore, the reaction stress do no work. This is a mechanically coherent condition that follows directly from the KKT conditions and with the Galerkin orthogonality.

If one impose the Finite Element restriction, that is $\mathbf{u} = \mathbf{u}_h$, in the equation 3.19a yields

$$\int_V \mathbf{T}^R : \delta \mathbf{E} \, dv = \int_V \bar{\mathbf{b}} \cdot \delta \mathbf{u} \, dv + \int_{\partial V} \bar{\mathbf{t}} \cdot \delta \mathbf{u} \, da - \int_V \mathbf{T}(\mathbf{u}_h) : \delta \mathbf{E} \, dv \quad (3.22)$$

that is equivalent to the equation 3.1. The equation 3.19c implies the orthogonality of the reaction in relation to the finite element space. Evoking the green theorem over eq. 3.19a yields

$$\operatorname{div}(\mathbf{T}(\mathbf{u}_h) + \mathbf{T}^R) + \bar{\mathbf{b}} = 0 \text{ in } V \text{ and} \quad (3.23a)$$

$$\bar{\mathbf{t}} = (\mathbf{T}(\mathbf{u}_h) + \mathbf{T}^R) \cdot \boldsymbol{\nu} \text{ on } \partial V_n \quad (3.23b)$$

that is the strong form of the error equation. Calling $\bar{\mathbf{T}} = \mathbf{T}(\mathbf{u}_h) + \mathbf{T}^R$, those $\bar{\mathbf{T}} \in \mathcal{S}_{Ad}$ are known to be an upper bound of error by the relation show in eq. 2.16.

It is known that functionals with Lagrange multipliers have their solution in a saddle point what imposes it to be a minimum in one direction and a maximum in the other. This would implies that any $\mathbf{T}^R$ that is orthogonal to restriction is also an upper bound of the true error. To compare the error in a common measure it is important to have a dissipation energy function, so that

$$\eta^2 = \|\mathbf{T}^R\|_E^2 = \varphi^*(\mathbf{T}^R). \quad (3.24)$$

3.2 Error in the model discretization

The model presented in chapter 1.1 is used in a discretized version, therefore it has an error associated. For the n -th model, the error is given by

$$\mathbf{e} = \mathbf{u} - \mathbf{u}(\mathbf{d}^h) = \mathbf{u} - \mathbf{u}(\mathbf{d}) + \mathbf{u}(\mathbf{d}) - \mathbf{u}(\mathbf{d}^h) = \mathbf{e}_m + \mathbf{e}_h, \quad (3.25)$$

where $\mathbf{e}_m$ is the modelling error and $\mathbf{e}_h$ is the model discretization error. This allows a decomposition of the solution space in $\mathcal{U}_K = \mathcal{U}_K^+ \oplus \mathcal{U}_{Kn}^{h+} \oplus \mathcal{U}_{Kn}^h$, where $\mathbf{e}_m \in \mathcal{U}_K^+$, $\mathbf{e}_h \in \mathcal{U}_{Kn}^{h+}$ and $\mathbf{d}_h \in \mathcal{U}_{Kn}^h$.

The kinematical restriction herein used is based in a series construction. The series is based in the Legendre polynomials for the linear case. As they are a complete polynomial between $[-1, 1]$, we can decompose the displacement $\mathbf{u}$ by

$$\mathbf{u} \approx \sum_{n=1}^{\infty} \Delta \mathbf{u}^n = \mathbf{u}(\mathbf{d}^n) + \sum_{m=n+1}^{\infty} \Delta \mathbf{u}^m \quad (3.26)$$

where $\Delta \mathbf{u}^m = \left(\frac{2m+1}{2} \int_H L_n(\frac{\zeta}{h}) u_i \, dx \right) L_n(\frac{\zeta}{h}) \mathbf{e}_i$. If we approximate the displacement by

$$\mathbf{u} \approx w \mathbf{e}_3 + \boldsymbol{\theta} \times \mathbf{a} + \boldsymbol{\Psi} \cdot \mathbf{p} + \boldsymbol{\Psi}^+ \cdot \mathbf{d}_+, \quad (3.27)$$

where

$$\Psi^+ \cdot \mathbf{d}_+ \in \mathcal{U}_K^+, \quad (3.28)$$

Ψ^+ are extra terms that are not in Ψ and $\mathbf{d}_+$ are the extra enrichments modes, then we can calculate approximately the error in the model. This is a pragmatical approach since we can evaluate how much error reduction does a new mode in relation to the ones used. This allows us to calculate the error in modelling approximately by $\mathbf{d}_+$ and the error in discretization by $\mathbf{d}_{h+} = \mathbf{d} - \mathbf{d}_h$.

We can calculate the generalized strains related to the errors in eq. 3.25 as it was for the generalized displacements $\mathbf{d}$ in 2.44. The generalized displacement of the discretization error is given by $\epsilon(\mathbf{d}_{h+})$, of the model error is $\epsilon(\mathbf{d}_+)$, so that the eq. 2.44 is now decomposed in

$$\gamma = \Lambda(\zeta)\epsilon(\mathbf{d}_h) + \Lambda(\zeta)\epsilon(\mathbf{d}_{h+}) + \Lambda_+(\zeta)\epsilon(\mathbf{d}_+) \quad (3.29)$$

where

$$\Lambda_+(\zeta) = \begin{bmatrix} \mathbf{O} & \Psi_+ & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \Psi_+ \\ \Psi_{+,3} & \mathbf{O} & \mathbf{O} \end{bmatrix}. \quad (3.30)$$

The calculation of the error uses the same technique of the finite element methods. It uses a discrete subspace of the original problem to solve it numerically. The space used here will be the same used for the error in the discretization, that is $\mathcal{U}^{h+}$. So, The modelling error is discretized by $\mathbf{d}_+ \approx \mathbf{N}_+ \mathbf{e}_m$, where the orthogonal basis shape function $\mathbf{N}_+ \in \mathcal{U}^{h+}$. The same is true for the discretization error given by $\mathbf{d}_{h+} \approx \mathbf{N}_+ \mathbf{e}_h$. Applying the relation 2.89 implies $\epsilon(\mathbf{d}_+) \approx \mathbf{B}_+ \mathbf{e}_m$ and $\epsilon(\mathbf{d}_{h+}) \approx \mathbf{B}_+ \mathbf{e}_h$. The variation of the generalized displacement vector has also an orthogonal form in the space $\mathcal{U}^{h+}$ so that $\delta\epsilon(\mathbf{d}_h) \approx \mathbf{B}\delta\mathbf{d}_h + \mathbf{B}_+\delta\mathbf{d}_h$ and its shape function $\delta\mathbf{d}_h \approx \mathbf{N}\delta\mathbf{d}_h + \mathbf{N}_+\delta\mathbf{d}_h$.

The error equation is derived from the weak form 2.69 by evoking the error decomposition in eq. 3.25, what implies

$$\int_{\Omega} \boldsymbol{\sigma}(\mathbf{e}_m) \cdot \delta\epsilon \, da + \int_{\Omega} \boldsymbol{\sigma}(\mathbf{e}_h) \cdot \delta\epsilon \, da = \int_{\Omega} \bar{\mathbf{q}} \cdot \delta\mathbf{d} \, da + \int_{\partial\Omega_n} \bar{\mathbf{q}}^\Gamma \cdot \delta\mathbf{d} \, dx^r - \int_{\Omega} \boldsymbol{\sigma}(\mathbf{d}^h) \cdot \delta\epsilon \, da. \quad (3.31)$$

Its discretized form is given by

$$\mathbf{K}_{hh}\mathbf{e}_h + \mathbf{K}_{hm}\mathbf{e}_m = \mathbf{r}_h \quad (3.32)$$

where

$$\mathbf{K}_{hm} = \int_{\Omega} \mathbf{B}_+^T \left(\int_H \Lambda^T \mathbf{D} \Lambda_+ \, dx \right) \mathbf{B}_+ \, da, \quad (3.33)$$

$$\mathbf{K}_{hh} = \int_{\Omega} \mathbf{B}_+^T \left(\int_H \Lambda^T \mathbf{D} \Lambda \, dx \right) \mathbf{B}_+ \, da \quad (3.34)$$

and

$$\mathbf{r}_h = \int_{\Omega} \mathbf{N}_+^T \bar{\mathbf{q}} \, da + \int_{\partial\Omega_n} \mathbf{N}_+^T \bar{\mathbf{q}}^\Gamma \, dx^r - \int_{\Omega} \mathbf{B}_+^T \boldsymbol{\sigma}(\mathbf{d}^h) \, da. \quad (3.35)$$

3.3 Error in the kinematical restriction

The analogy between error and reaction to a finite element restriction gives a mechanical analogy to interpret the error in the kinematical restriction. The functional presented in eq. 3.18 for the finite element restriction takes the form

$$\Pi_M(\mathbf{u}, \mathbf{u}_h, \boldsymbol{\lambda}) = U_{\text{int}}(\mathbf{u}) + U_{\text{ext}}(\mathbf{u}) - \int_V \boldsymbol{\lambda} \cdot (\mathbf{u} - \mathbf{u}(\mathbf{d}^n)) \, dv - \int_{\partial V} \boldsymbol{\lambda} \cdot (\mathbf{u} - \mathbf{u}(\mathbf{d}^n)) \, da, \quad (3.36)$$

when applied to kinematical restriction problem. The $\mathbf{u}(\mathbf{d}^n) \in \mathcal{U}_{K^n}$ is the kinematical restriction and n the order of the model. By imposing the relation $\mathcal{U}_K = \mathcal{U}_{K^n} \oplus \mathcal{U}_K^+$ and $\mathbf{e}_+ = \mathbf{u} - \mathbf{u}(\mathbf{d}^n)$, then it yields the kinematical error equation as

$$\int_V \mathbf{T}^R : \delta \mathbf{E}_+ \, dv = \int_{\partial V_n} \bar{\mathbf{t}} \cdot \delta \mathbf{e}_+ \, dv + \int_V \bar{\mathbf{b}} \cdot \delta \mathbf{e}_+ \, dv - \int_V \mathbf{T}(\mathbf{u}(\mathbf{d}_n)) : \delta \mathbf{E}_+ \, dv, \quad \forall \delta \mathbf{e}_+ \in \delta \mathcal{U}_K^+, \quad (3.37)$$

where $\mathbf{T}^R$ is the reaction of the kinematical restriction, $\mathbf{E}_+$ is the linear strain tensor of $\mathbf{e}_+$. If we use the relation in eq. 3.27 for the rest of the kinematical enrichment modes in $\mathbf{e}_+$ to find the error of the model, then the equation takes the form

$$\begin{aligned} \int_{\Omega} \boldsymbol{\beta}_i^+ \cdot \delta \mathbf{p}_{i+} + \boldsymbol{\rho}_{i\alpha}^+ \cdot \delta \mathbf{p}_{i+,\alpha} \, da &= \int_{\partial \Omega} \tilde{\boldsymbol{\beta}}_i^\Gamma \cdot \delta \mathbf{p}_{i+} \, dx + \int_{\Omega} \tilde{\boldsymbol{\beta}}_i \cdot \delta \mathbf{p}_{i+} \, da \\ &\quad - \int_{\Omega} \tilde{\boldsymbol{\beta}}_i \cdot \delta \mathbf{p}_{i+} + \tilde{\boldsymbol{\rho}}_{i\alpha} \cdot \delta \mathbf{p}_{i+,\alpha} \, da, \quad \forall \delta \mathbf{p}_{i+} \in \delta \mathcal{U}_K^+, \end{aligned} \quad (3.38)$$

where

$$\tilde{\boldsymbol{\beta}}_i = \int_H (\mathbf{e}_i \cdot \mathbf{T}(\mathbf{u}_h) \cdot \mathbf{e}_3) \boldsymbol{\phi}_{i,3}^+ \, dx, \quad (3.39)$$

$$\boldsymbol{\beta}_i^+ = \int_H (\mathbf{e}_i \cdot \mathbf{T}^R \cdot \mathbf{e}_3) \boldsymbol{\phi}_{i,3}^+ \, dx, \quad (3.40)$$

$$\tilde{\boldsymbol{\rho}}_{i\alpha} = \int_H (\mathbf{e}_i \cdot \mathbf{T}(\mathbf{u}_h) \cdot \mathbf{e}_\alpha) \boldsymbol{\phi}_i^+ \, dx, \quad (3.41)$$

$$\boldsymbol{\rho}_{i\alpha}^+ = \int_H (\mathbf{e}_i \cdot \mathbf{T}^R \cdot \mathbf{e}_\alpha) \boldsymbol{\phi}_i^+ \, dx, \quad (3.42)$$

$$\tilde{\boldsymbol{\beta}}^+ = (\boldsymbol{\Psi}_+(h+))^T \bar{\mathbf{t}}_+ + (\boldsymbol{\Psi}_+(h-))^T \bar{\mathbf{t}}_- + \int_H (\boldsymbol{\Psi}_+)^T \mathbf{b} \, dx \text{ and} \quad (3.43)$$

$$\tilde{\boldsymbol{\beta}}^{\Gamma+} = \int_H \boldsymbol{\Psi}_+^T \mathbf{t}^\Gamma \, dx. \quad (3.44)$$

The strong form of the error is given by

$$(\boldsymbol{\rho}_\alpha^+ + \tilde{\boldsymbol{\rho}}_\alpha^+)_\alpha - (\boldsymbol{\beta}^+ + \tilde{\boldsymbol{\beta}}^+) + \bar{\boldsymbol{\beta}}^+ = \mathbf{0} \text{ in } \Omega \text{ and} \quad (3.45)$$

$$(\boldsymbol{\rho}_\alpha^+ + \tilde{\boldsymbol{\rho}}_\alpha^+) \nu_\alpha = \bar{\boldsymbol{\beta}}^{\Gamma+} \text{ on } \partial \Omega. \quad (3.46)$$

This equation complements the original strong form given in the eq. 2.72. This equations shows the importance of the boundary condition to activate the high order modes. The discretized form of the equation 3.38 is given by

$$\mathbf{K}_{mh} \mathbf{e}_h + \mathbf{K}_{mm} \mathbf{e}_m = \mathbf{r}_m \quad (3.47)$$

where

$$\mathbf{K}_{mh} = \int_{\Omega} \mathbf{B}_+^T \tilde{\boldsymbol{\sigma}}^+(e_h) da = \int_{\Omega} \mathbf{B}_+^T \left(\int_H \boldsymbol{\Lambda}_+^T D \boldsymbol{\Lambda} dx \right) \mathbf{B}_+ da, \quad (3.48)$$

$$\mathbf{K}_{hh} = \int_{\Omega} \mathbf{B}_+^T \boldsymbol{\sigma}^+(e_m) da = \int_{\Omega} \mathbf{B}_+^T \left(\int_H \boldsymbol{\Lambda}_+^T D \boldsymbol{\Lambda}_+ dx \right) \mathbf{B}_+ da, \quad (3.49)$$

$$\mathbf{r}_m = \int_{\Omega} \mathbf{N}_+^T \bar{\boldsymbol{\beta}}^+ da + \int_{\partial\Omega_n} \mathbf{N}_+^T \bar{\boldsymbol{\beta}}^{\Gamma^+} dx^r - \int_{\Omega} \mathbf{B}_+^T \tilde{\boldsymbol{\sigma}}^+(d_h) da. \quad (3.50)$$

and

$$\tilde{\boldsymbol{\sigma}}^+(d_h) = \left(\int_H \boldsymbol{\Lambda}_+^T D \boldsymbol{\Lambda} dx \right) \mathbf{B}_p. \quad (3.51)$$

The equations 3.47 and 3.32 shows the dependency of the solution of the error in modelling of the error in discretization. Both have to be calculated in the same time. We can then define an global formula to solve the error in the model and in the discretization by

$$\begin{bmatrix} \mathbf{K}_{hm} & \mathbf{K}_{hh} \\ \mathbf{K}_{mh} & \mathbf{K}_{mm} \end{bmatrix} \begin{bmatrix} \mathbf{e}_h \\ \mathbf{e}_m \end{bmatrix} = \begin{bmatrix} \mathbf{r}_h \\ \mathbf{r}_m \end{bmatrix}. \quad (3.52)$$

This equation is equivalent to the equation of error in a $m+1$ -th order higher then the model n -th order. Therefore, along the text we don't separate the model error from simple error in kinematics. The second will be calculated by the increase in the order of the model.

The strong form of the error model and the discretization error together follows as:

$$\begin{aligned} \sum_{\mathcal{K} \in \mathcal{P}} \int_{\mathcal{K}} \boldsymbol{\sigma}(e_h) \cdot \delta \boldsymbol{\epsilon}(d_+) dv + \sum_{\mathcal{K} \in \mathcal{P}} \int_{\mathcal{K}} \boldsymbol{\sigma}(e_m) \cdot \delta \boldsymbol{\epsilon}(d_+) dv = & \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_{\mathcal{K}} r_V \cdot \delta w_+ da + \int_{\partial \mathcal{K}} \mathbf{j}_V \cdot \delta w_+ dx \right\} \\ & + \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_{\mathcal{K}} \mathbf{r}_M \cdot \delta \boldsymbol{\theta}_+ da + \int_{\partial \mathcal{K}} \mathbf{j}_M \cdot \delta \boldsymbol{\theta}_+ dx \right\} \\ & + \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_{\mathcal{K}} \mathbf{r}_{p_i} \cdot \delta \mathbf{p}_{i+} da + \int_{\partial \mathcal{K}} \mathbf{j}_{p_i} \cdot \delta \mathbf{p}_{i+} dx \right\} \\ & + \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \int_{\mathcal{K}} \mathbf{r}_{\tilde{p}_i} \cdot \delta \tilde{\mathbf{p}}_{i+} da + \int_{\partial \mathcal{K}} \mathbf{j}_{\tilde{p}_i} \cdot \delta \tilde{\mathbf{p}}_{i+} dx \right\}, \end{aligned} \quad (3.53)$$

where

$$j_V = \begin{cases} V_{\alpha} \nu_{\alpha} - \bar{V}^{\Gamma}, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket V_{\alpha} \nu_{\alpha} \rrbracket), & \text{on } \partial \mathcal{K} \end{cases}, \quad (3.54)$$

$$j_M = \begin{cases} \mathbf{m}_{\alpha} \nu_{\alpha} - \bar{\mathbf{m}}^{\Gamma}, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket \mathbf{m}_{\alpha} \nu_{\alpha} \rrbracket), & \text{on } \partial \mathcal{K} \end{cases}, \quad (3.55)$$

$$j_{p_i} = \begin{cases} \boldsymbol{\rho}_{i\alpha} \nu_{\alpha} - \bar{\boldsymbol{\beta}}_i^{\Gamma}, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket \boldsymbol{\rho}_{i\alpha} \nu_{\alpha} \rrbracket), & \text{on } \partial \mathcal{K} \end{cases}, \quad (3.56)$$

$$j_{\tilde{p}_i} = \begin{cases} \tilde{\boldsymbol{\rho}}_{i\alpha} \nu_{\alpha} - \tilde{\bar{\boldsymbol{\beta}}}_i^{\Gamma}, & \text{on } \partial \mathcal{K} \cap \partial \Omega_n \\ \frac{1}{2}(\llbracket \tilde{\boldsymbol{\rho}}_{i\alpha} \nu_{\alpha} \rrbracket), & \text{on } \partial \mathcal{K} \end{cases}, \quad (3.57)$$

$$\begin{aligned}
r_V &= V_{\alpha,\alpha} + \bar{V}, \\
r_M &= \mathbf{m}_{\alpha,\alpha} + (\mathbf{e}_\alpha \times \mathbf{e}_3)V_\alpha + \bar{\mathbf{m}}, \\
r_{p_i} &= \rho_{i\alpha,\alpha} - \beta_i + \bar{\beta}_i \text{ and} \\
r_{\tilde{p}_i} &= \tilde{\rho}_{i\alpha,\alpha} - \tilde{\beta}_i + \tilde{\bar{\beta}}_i.
\end{aligned} \tag{3.58}$$

3.4 Error localization

Until this moment, the error presented was based in global equation calculation. This demands a costly solution. To reduce it, there are many methods that localizes the problem to a specific region.

A possible way to classify then is in two big groups. The first is called recovery estimates. They work smoothing the FE stress resultants across the whole mesh, so that the best possible fit to a smooth solution in the least square sense is achieved. The computation of this kind can be carried out independently for each stress resultant component.

The other group is the residual type of estimate. They are based in the calculus of the residual 3.3, where it can be calculated implicitly or explicitly. The explicitly gives a error measure with a coefficient C unknown and implicit is independent of any constant but a internal problem must be solved. To solve the implicit method, one has to apply a technique to localize the problem so that the problem can be solved element by element or subdomain by subdomain.

3.4.1 Recovery type of estimate

The recovery type of the estimate is the simplest kind of posteriori error method to implement. Because of that, they are really popular. There are many different techniques to implement then. The most popular of then is ZZ super patch recovery, see [62, 63]. The technique is based in the interpolation of the stress field around the vertex using the displacement shape functions. The stress nodal value is calculated, so that, it minimizes the least square error in energy of the stresses at the super-convergent points.

Given the component j of the enhanced stress field $\hat{\sigma}$ around the node i by

$$\hat{\sigma}_j^i(\bar{\mathbf{x}}) = \mathbf{S}_{ij}(\bar{\mathbf{x}})\alpha_{ij} = \begin{bmatrix} 1 & \bar{x}_1 & \bar{x}_2 \end{bmatrix} \begin{bmatrix} \alpha_{ij1} \\ \alpha_{ij2} \\ \alpha_{ij3} \end{bmatrix}, \tag{3.59}$$

where $\mathbf{S}_{ij}$ is a column matrix that gathers the polynomial terms, $(\bar{\mathbf{x}}) = \mathbf{x} - \mathbf{x}_i$ is the local coordinate system and $\mathbf{x}_i$ is the nodal global coordinates of the node i . The problem is given by the minimization of

$$J(\alpha_i) = \sum_{k=1}^{n_{\text{dim}}} \sum_{j=1}^{n_{\text{Bp},i}} (\hat{\sigma}_k^i - \sigma_k(\bar{\mathbf{x}}_j))^2 \tag{3.60}$$

, where $\mathbf{x}_j$ are the barlow points, $n_{\text{Bp},I}$ is the number of Barlow points around the node i and n_{dim} is dimension of the stress vector $\boldsymbol{\sigma}$. The Barlow point used was the centroid of the element. The

solution is obtained by

$$\frac{\partial J(\alpha_{ij})}{\partial \alpha_{ij}} = \sum_{k=1}^{n_{\text{BP},i}} (\mathbf{S}_{ij}^T \mathbf{S}_{ij} \alpha_i - \mathbf{S}_{ij}^T \sigma_j^i(\bar{\mathbf{x}}_k)) = \mathbf{0}. \quad (3.61)$$

The new stresses nodal value are given by

$$\hat{\sigma}_i = \mathbf{S}_i(\mathbf{0}) \alpha_i. \quad (3.62)$$

This technique has some limitations. The eq. 3.61 is ill-conditioned for nodes inside the element edge and in the geometry border. The way to solve this is to interpolate the value of the ill-conditioned nodes from the value of the closest nodes. The methods that use equilibrated interpolation around the node like [58] does not have any ill-conditioned node. Unfortunately, the employed shape functions do not allow this kind of approach.

The results of the Recovery type of estimate are presented in the next section.

3.4.2 Recovery estimate results

Figure 3.1: *First benchmark*

The first benchmark is presented in 3.1. The data is presented in the table 3.1. This a simple example that exploit the classical kinematical hypothesis of a plate model but at the same time it has a localized solution, better captured by a three dimensional model.

Table 3.1: *Data of the first benchmark*

Data name	Value
(force) $q =$	1
(wide) $w =$	10
(Length) $L =$	100
(height) $h =$	2
$E =$	20000
$\nu =$	0.25

The following three figures show the discretization error in energy for three different meshes. The forth figure shows how they vary through the length. The error shows how important is the boundary layer in this case.

Figure 3.2: *Error in discretization mesh 1*

Figure 3.3: *Error in discretization mesh 2*

Figure 3.4: *Error in discretization mesh 3*

Figure 3.5: Error through the length in the three different meshes

3.4.3 Residual type of estimate

The residual type of estimates solves the error equation 3.3 in a local way. This localization can be using implicit or explicit methods. The explicit methods does not solve any local system of equations. So they are less expensive, but they are also less sharp estimators. They are based in the use of the Cauchy-Schwarz inequality in the equation 3.7.

The implicit methods solve local problems what gives sharper estimate but also with a more expensive solution. Herein, we will focus on implicit estimators.

Explicit error estimator

The use of explicit type of estimators simplifies the error calculus. They are computationally less expensive then other methods, but they are less sharp and they depend of an unknown constant. They are based in the Cauchy-Schwarz inequality in the Hilbert space:

$$\sum_{\mathcal{K} \in \mathcal{P}} \int_{\mathcal{K}} \boldsymbol{\sigma}(\mathbf{e}) : \delta \boldsymbol{\epsilon}_+ \, dv \leq \sum_{\mathcal{K} \in \mathcal{P}} \left\{ \|\mathbf{r}\|_{\mathcal{L}^2(\mathcal{K})} \|\delta \mathbf{u}_+\|_{\mathcal{L}^2(\mathcal{K})} + \|\mathbf{j}\|_{\mathcal{L}^2(\partial \mathcal{K})} \|\delta \mathbf{u}_+\|_{\mathcal{L}^2(\partial \mathcal{K})} \right\}. \quad (3.63)$$

Using the properties

$$\begin{aligned} \|\delta \mathbf{u}_+\|_{\mathcal{L}^2(\mathcal{K})} &= \|\delta \mathbf{u} - \delta \mathbf{u}_h\|_{\mathcal{L}^2(\mathcal{K})} \leq Ch_{\mathcal{K}} \|\delta \mathbf{u}\|_{\mathcal{H}^1(\mathcal{K})} \text{ and} \\ \|\delta \mathbf{u}_+\|_{\mathcal{L}^2(\partial \mathcal{K})} &= \|\delta \mathbf{u} - \delta \mathbf{u}_h\|_{\mathcal{L}^2(\partial \mathcal{K})} \leq Ch_{\mathcal{K}}^{1/2} \|\delta \mathbf{u}\|_{\mathcal{H}^1(\partial \mathcal{K})}, \end{aligned} \quad (3.64)$$

where C is a constant that is independent of $h_{\mathcal{K}}$, in eq. 3.63 yields

$$\sum_{\mathcal{K} \in \mathcal{P}} \int_{\mathcal{K}} \boldsymbol{\sigma}(\mathbf{e}) : \delta \boldsymbol{\epsilon}_+ \, dv \leq C \|\delta \mathbf{u}\|_{\mathcal{H}^1(\partial \mathcal{K})} \left\{ \sum_{\mathcal{K} \in \mathcal{P}} \left\{ h_{\mathcal{K}}^2 \|\mathbf{r}\|_{\mathcal{L}^2(\mathcal{K})}^2 + h_{\mathcal{K}} \|\mathbf{j}\|_{\mathcal{L}^2(\partial \mathcal{K})}^2 \right\} \right\}^{1/2}. \quad (3.65)$$

Evoking the Korn's inequality given by $\|\delta \mathbf{u}\|_{\mathcal{H}^1(\partial \mathcal{K})} \leq C \sum_{\mathcal{K} \in \mathcal{P}} \int_{\mathcal{K}} \boldsymbol{\sigma}(\delta \mathbf{u}) : \delta \boldsymbol{\epsilon}_+(\delta \mathbf{u}) \, dv = C \|\delta \mathbf{u}\|$ and substituting it in eq. 3.63 yields

$$\|\mathbf{e}_h\|^2 \leq C \sum_{\mathcal{K} \in \mathcal{P}} \eta_{\mathcal{K}}^2, \quad (3.66)$$

where $\eta_{\mathcal{K}}^2 = h_{\mathcal{K}}^2 \|\mathbf{r}\|_{\mathcal{L}^2(\mathcal{K})}^2 + h_{\mathcal{K}} \|\mathbf{j}\|_{\mathcal{L}^2(\partial \mathcal{K})}^2$.

We can use also the error equation presented in 3.53 to calculate the model error. Applying the Cauchy-Schwarz inequality to it yields

$$\|\mathbf{e}_h\|^2 + \|\mathbf{e}_m\|^2 \leq C \sum_{\mathcal{K} \in \mathcal{P}} (\eta_{h\mathcal{K}}^2 + \eta_{m\mathcal{K}}^2), \quad (3.67)$$

where

$$\begin{aligned} \eta_{h\mathcal{K}}^2 &= \sum_{\mathcal{K} \in \mathcal{P}} (h_{\mathcal{K}}^2 \|r_V\|_{\mathcal{L}^2(\mathcal{K})}^2 + h_{\mathcal{K}} \|j_V\|_{\mathcal{L}^2(\partial \mathcal{K})}^2) \\ &\quad + \sum_{\mathcal{K} \in \mathcal{P}} (h_{\mathcal{K}}^2 \|\mathbf{r}_M\|_{\mathcal{L}^2(\mathcal{K})}^2 + h_{\mathcal{K}} \|\mathbf{j}_M\|_{\mathcal{L}^2(\partial \mathcal{K})}^2) \\ &\quad + \sum_{\mathcal{K} \in \mathcal{P}} (h_{\mathcal{K}}^2 \|\mathbf{r}_{p_i}\|_{\mathcal{L}^2(\mathcal{K})}^2 + h_{\mathcal{K}} \|\mathbf{j}_{p_i}\|_{\mathcal{L}^2(\partial \mathcal{K})}^2) \\ \eta_{m\mathcal{K}}^2 &= \sum_{\mathcal{K} \in \mathcal{P}} (h_{\mathcal{K}}^2 \|\mathbf{r}_{\tilde{p}_i}\|_{\mathcal{L}^2(\mathcal{K})}^2 + h_{\mathcal{K}} \|\mathbf{j}_{\tilde{p}_i}\|_{\mathcal{L}^2(\partial \mathcal{K})}^2). \end{aligned} \quad (3.68)$$

The error in discretization is given by

$$\|e_h\| \left(1 + \frac{\|e_m\|^2}{\|e_h\|^2} \right)^{1/2} \leq C \left(\sum_{\mathcal{K} \in \mathcal{P}} \eta_{h\mathcal{K}}^2 \right)^{1/2} \left(1 + \frac{\sum_{\mathcal{K} \in \mathcal{P}} \eta_{m\mathcal{K}}^2}{\sum_{\mathcal{K} \in \mathcal{P}} \eta_{h\mathcal{K}}^2} \right)^{1/2}. \quad (3.69)$$

Discretization adaptativity

Hierarchical models have a strong boundary layer. This imposes the use of error adaptativity strategy to increase the method reliability. We use bisection of the element $\mathcal{K}$ that have an local error $\eta_{h\mathcal{K}} > 0.3 \max(\eta_{h\mathcal{K}})$. This is done until the tolerance of $\|e_h\| < 0.03$ is achieved.

Results

This section presents some numerical results that evaluate the discretization error and model error. It was chosen the same example as in section 2.11.

Figure 3.6: *First benchmark*

The results were evaluated for a fixed thickness h and varying enrichment model and for fixed enriched model and varying thickness.

The data is presented in the table 4.2.

Table 3.2: *Data of the first benchmark*

Data name	Value
(force) $b =$	0.001
(width) $w =$	1
(Length) $L =$	1
$E =$	20000
$\nu =$	0.25

d^1 Model

Table 3.3: 5th discretization

Field	Finite element
w	Lagrangean of 2nd order
θ_1, θ_2	Lagrangean of 2nd order
p_{31}	Lagrangean of 2nd order

Figure 3.7: Explicit error in discretization for the d^1 model, $h = 0.1$

Figure 3.8: Mesh adaptativity for the d^1 model, $h = 0.1$

Figure 3.9: *Explicit error in the model for the d^1 model, $h = 0.1$*

Figure 3.10: *model d^1 , $h = 0.1$*

Implicit residual type of estimator

The implicit methods localize the problem by two different techniques. One is by the partition of unity.

$$\sum_{i \in \mathcal{N}} N_i = 1. \quad (3.70)$$

By multiplying the error equation 3.3 by the partition of unity in eq. 3.70 then the original problem will be localized to patches of elements around the nodes i . The problem with this technique is that the same element is calculated many times depending on the number of patches around the element.

Another technique is to localize the problem around the element $\mathcal{K} \in \mathcal{P}$. This is done by the use of the strong form in equation 3.5. The equilibration equation 3.5b represents the equilibrium in the edge $\partial\mathcal{K}$ of the element $\mathcal{K}$. It can be reformulated as

$$\mathbf{T}(e|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu}_1 + \mathbf{T}(e|_{\mathcal{K}_2}) \cdot \boldsymbol{\nu}_2 + \mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu}_1 + \mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_2}) \cdot \boldsymbol{\nu}_2 = \mathbf{0}, \quad (3.71)$$

$$\mathbf{T}(e|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu} - \mathbf{T}(e|_{\mathcal{K}_2}) \cdot \boldsymbol{\nu} + \mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu} - \mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_2}) \cdot \boldsymbol{\nu} = \mathbf{0}, \quad (3.72)$$

$$\mathbf{T}(e|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu} - \mathbf{T}(\mathbf{u}) \cdot \boldsymbol{\nu} + \mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu} = \mathbf{0}, \quad (3.73)$$

$$\mathbf{T}(e|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu} = \mathbf{T}(\mathbf{u}) \cdot \boldsymbol{\nu} - \mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu}, \quad (3.74)$$

$$\mathbf{T}(e|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu} = \mathbf{g}_{\mathcal{K}} - \mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu}, \quad (3.75)$$

where $\boldsymbol{\nu}_2 = -\boldsymbol{\nu}_1 = \boldsymbol{\nu}$, $\mathbf{u} = \mathbf{e} + \mathbf{u}_h$, $\mathbf{u}$ is the exact solution and $\mathbf{g}_{\mathcal{K}} = \mathbf{T}(\mathbf{u}) \cdot \boldsymbol{\nu}$. Therefore, the strong form eq. 3.5 allows the localization of the problem to the element level. The problem is that we don't have the quantity $\mathbf{g}_{\mathcal{K}}$. There is two techniques to approximate this quantity, the first is

$$\mathbf{g}_{\mathcal{K}} \approx \frac{1}{2} \left(\mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_1}) \cdot \boldsymbol{\nu} + \mathbf{T}(\mathbf{u}_h|_{\mathcal{K}_2}) \cdot \boldsymbol{\nu} \right), \quad (3.76)$$

where an average value of the jump is calculated.

The other is to solve a local equilibrium equation, so that $\mathbf{g}_{\mathcal{K}}$ is equilibrated. This is essential since the new problem is a Neumann one that just have a solution if the loads are equilibrated. The equilibrated condition is given by

$$\int_{\partial\mathcal{K}} \mathbf{g}_{\mathcal{K}} \cdot \mathbf{1} \, da + \int_{\mathcal{K}} \bar{\mathbf{b}} \cdot \mathbf{1} \, dv - \int_{\mathcal{K}} \operatorname{div}(\mathbf{T}(\mathbf{u}_h)) \cdot \mathbf{1} \, dv = 0, \quad (3.77a)$$

$$\mathbf{g}_{\mathcal{K}} + \mathbf{g}_{\mathcal{K}'} = 0 \text{ on } \partial\mathcal{K} \cap \partial\mathcal{K}' \text{ and} \quad (3.77b)$$

$$\mathbf{g}_{\mathcal{K}} = \bar{\mathbf{t}} \text{ on } \partial\mathcal{K} \cap \partial\Omega_n. \quad (3.77c)$$

This equilibration flux condition is called strong prolongation condition in [34] or zeroth-order equilibration condition in [2].

Equilibrated flux

This section is based in [2] and is herein described to have a more self-contained text.

To simplify the calculus of the equilibrated flux around the element, it is simpler to use an equilibrium condition called the weak prolongation condition. It is given by the use of the partition

of unit to the error equation 3.96. This implies

$$\int_{\mathcal{K}} \mathbf{N}_i^T \bar{\mathbf{q}} \, da - \int_{\mathcal{K}} \mathbf{B}_i^T \boldsymbol{\sigma}(\mathbf{d}^h) \, da + \int_{\partial \mathcal{K}} \mathbf{N}_i^T \mathbf{g}_{\mathcal{K}} \, dx = 0, \forall i \in \mathcal{N}(\mathcal{K}) \quad (3.78a)$$

$$\mathbf{g}_{\mathcal{K}} + \mathbf{g}_{\mathcal{K}'} = 0 \text{ on } \partial \mathcal{K} \cap \partial \mathcal{K}' \text{ and} \quad (3.78b)$$

$$\mathbf{g}_{\mathcal{K}} = \bar{\mathbf{t}} \text{ on } \partial \mathcal{K} \cap \partial \Omega_n. \quad (3.78c)$$

The equilibrated flux is calculated in terms of flux moments given by

$$\boldsymbol{\mu}_{\mathcal{K},i}^{\gamma} = \int_{\gamma} \mathbf{N}_i^T \mathbf{g}_{\mathcal{K}} \, dx, \quad (3.79)$$

where γ is an edge and i a node of the element $\mathcal{K}$.

The equilibrium condition in terms of flux moments is given by

$$\sum_{\gamma \in \partial \mathcal{K}} \boldsymbol{\mu}_{\mathcal{K},i}^{\gamma} = \mathbf{r}_{\mathcal{K},i} \quad \forall i \in \mathcal{N}(\mathcal{K}), \quad (3.80a)$$

$$\boldsymbol{\mu}_{\mathcal{K},i}^{\gamma} + \boldsymbol{\mu}_{\mathcal{K}',i}^{\gamma} = 0 \text{ on } \partial \mathcal{K} \cap \partial \mathcal{K}' \text{ and} \quad (3.80b)$$

$$\boldsymbol{\mu}_{\mathcal{K},i}^{\gamma} = \int_{\gamma} \mathbf{N}_i^T \bar{\mathbf{q}}^{\Gamma} \, dx \text{ on } \partial \mathcal{K} \cap \partial \Omega_n, \quad (3.80c)$$

where

$$\mathbf{r}_{\mathcal{K},i} = \int_{\mathcal{K}} \mathbf{B}_i^T \boldsymbol{\sigma}(\mathbf{d}^h) \, da - \int_{\mathcal{K}} \mathbf{N}_i^T \bar{\mathbf{q}} \, da. \quad (3.81)$$

The equilibrated flux $\mathbf{g}_{\mathcal{K}}$ is approximated by

$$\mathbf{g}_{\mathcal{K}}|_{\gamma} = \mathbf{N}_l \mathbf{q}_l + \mathbf{N}_r \mathbf{q}_r, \quad (3.82)$$

where N_i is the shape function of the edge γ and $\mathcal{N}(\gamma) = \{l, r\}$ are the nodes in the edge γ . By the flux moment definition, see eq. 3.79, we can derive the system of equation

$$\begin{bmatrix} \int_{\gamma} \mathbf{N}_l^T \mathbf{N}_l \, dx & \int_{\gamma} \mathbf{N}_l^T \mathbf{N}_r \, dx \\ \int_{\gamma} \mathbf{N}_r^T \mathbf{N}_l \, dx & \int_{\gamma} \mathbf{N}_r^T \mathbf{N}_r \, dx \end{bmatrix} \begin{bmatrix} \mathbf{q}_l \\ \mathbf{q}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_{\mathcal{K},l}^{\gamma} \\ \boldsymbol{\mu}_{\mathcal{K},r}^{\gamma} \end{bmatrix}, \quad (3.83)$$

that gives the nodal values of the flux stress.

The equilibrium of the flux moment is solved in a node by node bases. The type of equations that arises depends on the node position. There is four possibilities. The first one is an internal node. The other three are boundary nodes, where the difference are if they have a Neumann-Neumann, a Dirichlet-Neumann or a Dirichlet-Dirichlet boundary conditions.

The interior node problem is given by two set of equations. The first is related to the equilibrium in each element, see eq. 3.80a. So, for the notation presented in fig. 3.11, the system of equations is given by:

$$\begin{cases} \boldsymbol{\mu}_{e_1,i}^{\gamma_1} + \boldsymbol{\mu}_{e_1,i}^{\gamma_2} = \mathbf{r}_{e_1,i} \\ \boldsymbol{\mu}_{e_2,i}^{\gamma_2} + \boldsymbol{\mu}_{e_2,i}^{\gamma_3} = \mathbf{r}_{e_2,i} \\ \vdots \\ \boldsymbol{\mu}_{e_n,i}^{\gamma_n} + \boldsymbol{\mu}_{e_n,i}^{\gamma_1} = \mathbf{r}_{e_n,i} \end{cases}. \quad (3.84)$$

The second system of equations is related to the equilibrium of the flux moments in eq. 3.80b.

Figure 3.11: A star patch $\mathcal{P}_i$ surrounding the interior node i

So, for the specific case of the interior nodes it is given by

$$\begin{cases} \mu_{e_1,i}^{\gamma_1} + \mu_{e_n,i}^{\gamma_1} = 0 \\ \mu_{e_1,i}^{\gamma_2} + \mu_{e_2,i}^{\gamma_2} = 0 \\ \vdots \\ \mu_{e_n,i}^{\gamma_n} + \mu_{e_{n-1},i}^{\gamma_n} = 0 \end{cases} . \quad (3.85)$$

The other case is the boundary node, see fig. 3.12. The equation related to the equilibrium in each element is given by

$$\begin{cases} \mu_{e_1,i}^{\gamma_1} + \mu_{e_1,i}^{\gamma_2} = \mathbf{r}_{e_1,i} \\ \mu_{e_2,i}^{\gamma_2} + \mu_{e_2,i}^{\gamma_3} = \mathbf{r}_{e_2,i} \\ \vdots \\ \mu_{e_n,i}^{\gamma_n} + \mu_{e_n,i}^{\gamma_{n+1}} = \mathbf{r}_{e_n,i} \end{cases} . \quad (3.86)$$

The continuity of the flux moments depends on the type of boundary node. Its is equal to the interior node case as in eq. 3.85, but the edges are in the boundary. The edges in the boundary can be in a Neumann or a Dirichlet boundary condition. As for each node there is always two edges in the boundary, then there are only three possibilities. They are

- Neumann-Neumann boundary conditions: if $\gamma_1 \in \partial\Omega_n$ and $\gamma_n \in \partial\Omega_n$, then

$$\begin{cases} \mu_{e_1,i}^{\gamma_1} = \int_{\gamma_1} \mathbf{N}_i^T \bar{\mathbf{q}}^\Gamma dx \\ \mu_{e_n,i}^{\gamma_{n+1}} = \int_{\gamma_n} \mathbf{N}_i^T \bar{\mathbf{q}}^\Gamma dx \end{cases} ; \quad (3.87)$$

- Neumann-Dirichlet boundary conditions: if $\gamma_1 \in \partial\Omega_n$ and $\gamma_n \in \partial\Omega_d$, then

$$\begin{cases} \mu_{e_1,i}^{\gamma_1} = \int_{\gamma_1} \mathbf{N}_i^T \bar{\mathbf{q}}^\Gamma dx \\ \mu_{e_n,i}^{\gamma_{n+1}} = \text{unconstrained} \end{cases} ; \quad (3.88)$$

- Dirichlet-Dirichlet boundary conditions: if $\gamma_1 \in \partial\Omega_n$ and $\gamma_n \in \partial\Omega_d$, then

$$\begin{cases} \mu_{e_1,i}^{\gamma_1} = \text{unconstrained} \\ \mu_{e_n,i}^{\gamma_{n+1}} = \text{unconstrained} \end{cases} . \quad (3.89)$$

Figure 3.12: A star patch $\mathcal{P}_i$ surrounding the exterior node i

Regularization of the residual type of estimate

Unfortunately, the equation to find the equilibrated flux is ill-posed and need some kind of regularization. We use here the technique presented in [16]. The idea is that $\mathbf{r}_{I,\Gamma}^e$ must be as close as possible to the stresses obtained in the recovered type of estimate. To do that, we construct the following constrained patch-wise minimization problem:

$$\begin{aligned} \mathcal{L} = \frac{1}{2} \left(\mathbf{r}_{\Gamma,i}^0 - \mathbf{r}_{\Gamma,i} \right)^T \left(\mathbf{r}_{\Gamma,i}^0 - \mathbf{r}_{\Gamma,i} \right) \\ + \lambda^T (\mathbf{r}_i^e - \mathbf{A} \mathbf{r}_{\Gamma,i}) \end{aligned} \quad (3.90)$$

where the constraint 3.93 are also introduced by means of the corresponding Lagrange multiplier vector λ .

The recovered stresses in the star patch ω_i can be used to obtain a new residuo $\mathbf{r}_{\Gamma_j,k}^{e_k,0}$ given by

$$\mathbf{r}_{\Gamma_j,i}^{e_k,0} = \int_{\Omega_e} \mathbf{N}_i^T (\nu_{\Gamma_j} \mathbf{H} \boldsymbol{\sigma}_{0,i}|_{\Gamma_j}) da \quad (3.91)$$

The minimization of the lagrangean 3.90 is given by

$$\begin{bmatrix} \mathbf{r}_{\Gamma,i}^0 \\ \mathbf{r}_i^e \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{r}_{\Gamma,i} \\ \lambda \end{bmatrix}, \quad (3.92)$$

by taking the Shur-complement of the above equation we can simplify the problem to

$$\begin{bmatrix} \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \mathbf{A}^T \end{bmatrix} \begin{bmatrix} \lambda \end{bmatrix}, \quad (3.93)$$

where

$$\mathbf{b} = \mathbf{A} \mathbf{r}_{\mathbf{T},i}^0 - \mathbf{r}_i^e \quad (3.94)$$

and the value of $\mathbf{r}_{\mathbf{T},i}$ can be found by the relation

$$\mathbf{r}_{\mathbf{T},i} = \mathbf{r}_{\mathbf{T},i}^0 - \mathbf{A}^T \lambda. \quad (3.95)$$

3.4.4 Local Neuman problem

With the stress resultant equilibrated loads $\mathbf{g}_K$ applied at the edges γ we have to solve the local Neumann problem given by

$$\int_{\Omega_e} \boldsymbol{\sigma}(\mathbf{e}) \cdot \delta \boldsymbol{\epsilon} \, da = \int_{\Omega_e} \bar{\mathbf{q}} \cdot \delta \mathbf{d} \, da - \int_{\partial\Omega_e} \mathbf{g}_K \cdot \delta \mathbf{d} \, dx, \quad (3.96)$$

where we want to find the error $\mathbf{e}$. Numerical test presented in [15] showed that the primal problem is an upper bound from error. It is known from [1] that this is always the case.

An element with only Neumann boundary conditions is essentially a floating structure, and one thus ought to eliminate the rigid body modes to get a unique result. A rigid body displacement $\mathbf{d}_R^e$ of an element is a linear combination of all its rigid body modes

$$\mathbf{d}_R^e = \mathbf{D}^e \boldsymbol{\alpha} \quad (3.97)$$

where $\boldsymbol{\alpha}$ is a vector of amplitudes that correspond to rigid body modes, and $\mathbf{D}^e$ is a matrix containing the rigid body modes of the element (arranged column-wise). Since the element stiffness matrix $\mathbf{K}^e$ is singular due to the rigid body modes contribution, one can form a modified nonsingular stiffness matrix $\mathbf{K}^{e'}$ by adding a product $\mathbf{D}^e \mathbf{D}^{eT}$ to $\mathbf{K}^e$, so that

$$\mathbf{K}^{e'} = \mathbf{K}^e + \mathbf{D}^e \mathbf{D}^{eT}.$$

The inversion of $\mathbf{K}^{e'}$ is possible and solution $\mathbf{d}^{e'}$ ('rigid-body-polluted' element nodal displacements/rotations) can be obtained. Since the matrix $\mathbf{D}^e \mathbf{D}^{eT}$ projects onto the space spanned by rigid body modes, we have

$$\mathbf{d}^{e'} = \mathbf{d}^e + \mathbf{d}_R^e$$

where $\mathbf{d}^e$ are real nodal displacements/rotations. As we are interested only in stress resultants, such a 'pollution' is not critical due to the property $\mathbf{K}^e \mathbf{d}_R^e = \mathbf{0}$.

As the error is orthogonal to actual discretization space, the actual space must be enriched. The enrichment can be an h -enrichment or a p -enrichment. In this case the analysis was decided to be made in relation to a h -enrichment. This is done by solving a enriched problem locally. The solution of this problem gives the local error.

The nonlinear thick shell model

The main purpose of this work is to present a different geometrically-exact hierarchical shell model. Herein a pure displacement approach was applied. We account for an enrichment of director kinematics along the thickness, based on addition of a warping and stretching vector to the shell director. The shell is regarded as a Cosserat surface with the additional degrees of freedom described as a vector. The use of this vector enables the shell model to approach the solid one. This kind of approach can benefit from all the features shell elements have in treating locking and from its convergence rate. The model constitutes an extension of an earlier work in Campello et al. [18].

Figure 4.1: *Deformation from the initial configuration to the deformed one*

Finite rotations are exactly treated by the Euler-Rodrigues formula in a total lagrangean framework. A plane reference configuration is assumed for the shell mid-surface, but initially curved shells can also be considered if regarded as a stress-free deformed state from the plane position. The use of convective non-Cartesian coordinate systems is avoided here so that only components related to orthogonal frames are employed.

A complete set of high-order shell local equilibrium equations and the related boundary conditions are consistently derived in terms of stress resultants, starting from the weak form of equilibrium.

4.1 The shell model

In this section, we describe the notation used during this chapter. We focus here in the definition of the geometry, since this is the main difference between a solid and a shell. In Shells, clearly there exists a top and bottom surface (Ω_+ and Ω_-) and a lateral boundary (Γ), what is not clear in the

Figure 4.2: *Reference configuration*

case of a general solid. Those are the boundary of the volume V in the deformed configuration, see fig. 4.1. In the reference configuration, we add the superscript $(\bullet)^r$. Any material point is described by a orthonormal system of coordinates $\{e_1^r, e_2^r, e_3^r\}$. The materials points ξ in the reference configuration are mapped to points $\mathbf{x}$ in deformed configuration by the map $\mathbf{x} = \chi(\xi)$.

Another geometrical entity is the middle surface in the reference configuration $\Omega^r \subset V^r$ defined by $\zeta = \xi_\alpha e_\alpha^r$. So, any material point ξ is given by

$$\xi = \zeta + \mathbf{a}^r, \quad (4.1)$$

where $\mathbf{a}^r = \zeta e_3^r$ is the director of the shell, see fig. 4.2. The thickness coordinate $\zeta \in H$ defines the shell thickness in the reference configuration (observe that $\{\xi_\alpha, \zeta\}$ sets a three-dimensional Cartesian frame).

In the current configuration the position $\mathbf{x}$ of the shell material points is given by

$$\mathbf{x} = \mathbf{z} + \mathbf{a}, \quad (4.2)$$

where $\mathbf{z}$ represents the current position of a point in the middle surface and $\mathbf{a}$ the current director, see fig. 4.3.

Figure 4.3: *Deformed configuration*

The displacement of any point in the mid-surface is given by

$$\mathbf{u} = \mathbf{z} - \zeta. \quad (4.3)$$

In the deformed configuration, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a local orthogonal system depending on the planar coordinates $\{\xi_1, \xi_2\}$. This approach is different from the classical literature that uses covariant $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ and contra-variant basis $\{\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3\}$ to construct a invariant (or covariant) metric and derivative measure. The deformed configuration can becomes plane by the use of the map $\mathbf{Q} = \mathbf{e}_i \otimes \mathbf{e}_i^r$ so that, for any vector $\mathbf{b}$ in the deformed configuration we have $\mathbf{Q}^T \mathbf{b} = \mathbf{b}^r \Leftrightarrow \mathbf{Q} \mathbf{b}^r = \mathbf{b}$. The vector $\mathbf{b}^r$ is then said to be the back-rotated counterpart of $\mathbf{b}$ and is not affected by superimposed rigid body motions. Herein, differently from Campello et al. [18], the director $\mathbf{a}$ is allowed to stretch and warp.

4.2 The hierarchical model kinematical restriction

For the same reason presented in the section 2.3, it was decided to construct a hierarchical model starting from the Reissner-Mindlin kinematic hypothesis. The corresponding kinematic restriction can thus be written as

$$\mathbf{u}(\mathbf{d}^n) = \mathbf{u} + \mathbf{Q}(\boldsymbol{\theta})(\zeta \mathbf{e}_3^r + \Delta \mathbf{a}^r), \quad (4.4)$$

where $\Delta \mathbf{a}^r$ is the additional vector in relation to Reissner-Mindlin model.

In the nonlinear framework the rotation has important role. It has many peculiarities that should be mentioned before the obtaining the weak form and its linearization. In the next section, we discuss the properties of the rotation operator.

4.2.1 The choice of director enrichment

The enrichment of the director vector is defined

$$\Delta \mathbf{a}^r = \sum_{i=1}^n \Delta \mathbf{a}^{rn} = \sum_{i=1}^n \Delta a_i^{rn} \mathbf{e}_i^r, \quad (4.5)$$

where each component Δa_i^{rn} is further described in terms of a additive decomposition by

$$\Delta a_\alpha^{rn} = \phi_\alpha^n \left(\frac{\zeta}{h} \right) p_\alpha^n(\xi_\alpha) \quad \text{and} \quad (4.6)$$

$$\Delta a_3^{rn} = \phi_3^n \left(\frac{\zeta}{h} \right) p_3^n(\xi_\alpha). \quad (4.7)$$

In 4.6, $\phi_i^j \left(\frac{\zeta}{h} \right)$ are the hierarchic basis or director functions, $p_i(\xi_\alpha)$ are the generalized deformations and q_i are the order in hierarchic basis.

We will consider the classical asymptotic expansion $\left(\frac{\zeta}{h} \right)^n$ for the hierarchical base, where n is the order of the polynomial.

The same asymptotic consistency is needed to the the classical asymptotic expansion $\left(\frac{\zeta}{h} \right)^n$, see table 4.1.

The extra terms in table 4.1 in relation to the table 2.1 are due to membrane modes of deformation that are absent in the plate model. They are necessary to guarantee the "plane stress" condition needed to the family be asymptotic consistent, see section 2.6.

The hierarchical basis are gathered in the column matrix $\boldsymbol{\phi}_i \left(\frac{\zeta}{h} \right)$ and its associated parameters

Table 4.1: *Adopted hierarchical family for the nonlinear framework (classical asymptotic expansion)*

$\Delta \mathbf{a}$	$\Delta \mathbf{a}_1$	$\Delta \mathbf{a}_2$	$\Delta \mathbf{a}_3$
$\Delta \mathbf{a}^1$	0	0	$(\frac{\xi}{h})^1, (\frac{\xi}{h})^2$.
$\Delta \mathbf{a}^2$	$(\frac{\xi}{h})^2, (\frac{\xi}{h})^3$	$(\frac{\xi}{h})^2, (\frac{\xi}{h})^3$	$(\frac{\xi}{h})^3, (\frac{\xi}{h})^4$.
$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\Delta \mathbf{a}^n$	$(\frac{\xi}{h})^{2n-2}, (\frac{\xi}{h})^{2n-1}$	$(\frac{\xi}{h})^{2n-2}, (\frac{\xi}{h})^{2n-1}$	$(\frac{\xi}{h})^{2n-1}, (\frac{\xi}{h})^{2n}$.

in $\mathbf{p}_i(\xi_\alpha)$; we can thus write the corresponding expression in the matrix notation:

$$\Delta \mathbf{a} = (\phi_i \cdot \mathbf{p}_i) \mathbf{e}_i = (\mathbf{e}_i \otimes \phi_i) \cdot \mathbf{p}_i = \Psi \cdot \mathbf{p}, \quad (4.8)$$

where

$$\Psi = \begin{bmatrix} \phi_1^T & \mathbf{0}_{1 \times np_2} & \mathbf{0}_{1 \times np_3} \\ \mathbf{0}_{1 \times np_1} & \phi_2^T & \mathbf{0}_{1 \times np_3} \\ \mathbf{0}_{1 \times np_1} & \mathbf{0}_{1 \times np_2} & \phi_3^T \end{bmatrix} \quad (4.9)$$

is the hierarchical basis matrix and

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}^T \quad (4.10)$$

is the column matrix that gathers all the generalized deformations.

4.3 Rotation parametrization

Rotation tensor can be represented minimally by three parameters. It is well known that there exists no single three parameters parametrization [57]. The singularity changes depending of what parametrization was chosen. A simpler and more useful parametrization can be obtained by the use of a rotation vector parametrization [4]

$$\mathbf{Q} = \mathbf{I} + h_1(\theta) \Theta + h_2(\theta) \Theta^2, \quad (4.11)$$

where Θ is the skew tensor of the vector θ and $\theta = |\theta|$. This formula is also known as Euler-Rodrigues formula. Its singularity is in $\theta = 2\pi$.

As presented in [6], expanding the trigonometric formulae in 4.11 in series it yields

$$\mathbf{Q} = \exp(\Theta). \quad (4.12)$$

The skew tensor Θ belongs to the tangent space of the rotation manifold, see [6, 37, 31], the notation $\Theta \in \mathcal{TSO}(3)$, where the identity $\mathbf{I} \in \mathcal{SO}(3)$ represents the base point.

The skew-symmetric tensors form a Lie-algebra with Lie-brackets defined as

$$[\mathbf{A}, \mathbf{B}] := \mathbf{AB} - \mathbf{BA}, \forall \mathbf{A}, \mathbf{B} \in \mathfrak{so}(3), \quad (4.13)$$

where the set of the skew-symmetric tensors $\mathfrak{so}(3)$ are defined as

$$\mathfrak{so}(3) = \{\Theta : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \mid \Theta^T = -\Theta\}. \quad (4.14)$$

4.3.1 Compound rotations

A compound rotation can be defined in two-different way: by the material description, and by the spatial description [37]. The material is given by

$$Q Q' = Q \exp(\Psi), \quad (4.15)$$

where Ψ is the incremental material rotation and the spatial is given by

$$Q' Q = \exp(\Omega) Q, \quad (4.16)$$

where Ω is the incremental spatial rotation.

4.3.2 Tangent space

Figure 4.4: Rotation manifold, inspired by the figure in [54]

The tangent space of a compound rotation is given by

$$\begin{aligned} \mathcal{T}_Q \mathcal{SO}(3) &= \{Q\Psi = \Omega Q \mid \Psi \in \mathcal{T}_I \mathcal{SO}(3) \\ &\text{and } \Omega = Q\Psi Q^T \in \mathcal{T}_Q \mathcal{SO}(3)\} \end{aligned} \quad (4.17)$$

and both spaces can be seen in the figure 4.4. The connection between the intrinsic parametrization rotation θ and the incremental material rotation is given by

$$\exp(\Theta + t\delta\Theta) = Q \exp(t\delta\Psi), \quad (4.18)$$

taking the Frechet derivative in respect to t and after taking the skew of each side we arrive at

$$\Gamma(\theta)\delta\theta = Q\delta\psi = \delta\omega, \quad (4.19)$$

where $\mathbf{\Gamma}(\boldsymbol{\theta})$ is the material tangential transformation. A important property is that $\mathbf{Q}^T \mathbf{\Gamma} = \mathbf{\Gamma}^T$.

4.3.3 Material tangential transformation and its derivative

The Frenchet derivative of $\mathbf{\Gamma}(\boldsymbol{\theta} + t\Delta\boldsymbol{\theta})$ is given by

$$\dot{\mathbf{\Gamma}}|_{t=0} = \mathbf{U}(\boldsymbol{\theta}, \Delta\boldsymbol{\theta}), \quad (4.20)$$

this result is also valid for the derivation over the paramter ξ_α in way that

$$\mathbf{\Gamma}_{,\alpha} = \mathbf{U}(\boldsymbol{\theta}, \boldsymbol{\theta}_{,\alpha}). \quad (4.21)$$

An important property of the tensor $\mathbf{U}$ is

$$\mathbf{U}^T(\boldsymbol{\theta}, \Delta\boldsymbol{\theta})t = \mathbf{Q}^T \mathbf{U}(\boldsymbol{\theta}, t)\Delta\boldsymbol{\theta}, \quad (4.22)$$

and we can define a new tensor $\mathbf{V}(\boldsymbol{\theta}, t) = \mathbf{Q}^T \mathbf{U}(\boldsymbol{\theta}, t)$ that has the property

$$\mathbf{V}^T(\boldsymbol{\theta}, t) = \mathbf{V}(\boldsymbol{\theta}, t) - \mathbf{\Gamma}^T \mathbf{T} \mathbf{\Gamma}. \quad (4.23)$$

The differentiation of $\mathbf{U}(\boldsymbol{\theta} + \Delta\boldsymbol{\theta}, \boldsymbol{\theta}_{,\alpha})$ is given by

$$\dot{\mathbf{U}}|_{t=0} = \mathbf{U}'(\boldsymbol{\theta}, \boldsymbol{\theta}_{,\alpha}, \Delta\boldsymbol{\theta}), \quad (4.24)$$

with the property

$$\begin{aligned} \mathbf{U}'^T(\boldsymbol{\theta}, \boldsymbol{\theta}_{,\alpha}, \Delta\boldsymbol{\theta})t &= \mathbf{Q}^T \mathbf{U}'(\boldsymbol{\theta}, \boldsymbol{\theta}_{,\alpha}, t)\Delta\boldsymbol{\theta} \\ &= \mathbf{V}'(\boldsymbol{\theta}, \boldsymbol{\theta}_{,\alpha}, t)\Delta\boldsymbol{\theta}. \end{aligned} \quad (4.25)$$

Details of those formulas and deductions can be found in [18].

4.4 Deformation Gradient

Differentiating the eq. 4.2 with respect to $\boldsymbol{\xi}$ yields the following deformation gradient

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} = \mathbf{x}_{,i} \otimes \mathbf{e}_i^r = \mathbf{x}_{,\alpha} \otimes \mathbf{e}_\alpha^r + \mathbf{x}_{,3} \otimes \mathbf{e}_3^r, \quad (4.26)$$

where $(\bullet)_{,\alpha} = \frac{\partial(\bullet)}{\partial \xi_\alpha}$ and $(\bullet)_{,3} = \frac{\partial(\bullet)}{\partial \zeta}$.

Evaluating each derivative in 4.26, we arrive at

$$\mathbf{z}_{,\alpha} = \mathbf{u}_{,\alpha}, \quad (4.27a)$$

$$\begin{aligned} \mathbf{a}_{,\alpha} &= (\mathbf{Q} \mathbf{a}^r)_{,\alpha} = \mathbf{Q}_{,\alpha} \mathbf{a}^r + \mathbf{Q} \mathbf{a}_{,\alpha}^r \\ &= \mathbf{Q}_{,\alpha} \mathbf{Q}^T \mathbf{a} + \mathbf{Q}(\phi_i \cdot \mathbf{p}_{i,\alpha}) \mathbf{e}_i^r \\ &= \boldsymbol{\kappa}_\alpha \times \mathbf{a} + \mathbf{Q}(\phi_i \cdot \mathbf{p}_{i,\alpha}) \mathbf{e}_i^r \end{aligned} \quad (4.27b)$$

$$\begin{aligned} \mathbf{a}_{,3} &= (\mathbf{Q} \mathbf{a}^r)_{,3} = \mathbf{Q} \mathbf{a}_{,3}^r \\ &= \mathbf{Q}(\phi_{i,3} \cdot \mathbf{p}_i) \mathbf{e}_i^r. \end{aligned} \quad (4.27c)$$

Substituting 4.27a, 4.27b and 4.27c in 4.26 yields

$$\begin{aligned} \mathbf{F} &= \{\mathbf{u}_{,\alpha} + \boldsymbol{\kappa}_\alpha \times \mathbf{a} + (\phi_i \cdot \mathbf{p}_{i,\alpha}) \mathbf{e}_i\} \otimes \mathbf{e}_\alpha^r \\ &\quad + \{\mathbf{e}_3 + (\phi_{i,3} \cdot \mathbf{p}_i) \mathbf{e}_i\} \otimes \mathbf{e}_3^r \\ &= \{\boldsymbol{\eta}_\alpha + \boldsymbol{\kappa}_\alpha \times \mathbf{a} + (\phi_i \cdot \mathbf{p}_{i,\alpha}) \mathbf{e}_i\} \otimes \mathbf{e}_\alpha^r \\ &\quad + \{(\phi_{i,3} \cdot \mathbf{p}_i) \mathbf{e}_i\} \otimes \mathbf{e}_3^r + \mathbf{Q}, \end{aligned} \quad (4.28)$$

where $\boldsymbol{\eta}_\alpha = \mathbf{u}_{,\alpha} - \mathbf{e}_\alpha$. The gradient of deformation can be described as

$$\mathbf{F} = \mathbf{Q} [I + \boldsymbol{\gamma}_\alpha^r \otimes \mathbf{e}_\alpha^r + \boldsymbol{\gamma}_3^r \otimes \mathbf{e}_3^r] = \mathbf{Q} \mathbf{F}^r, \quad (4.29)$$

where $\boldsymbol{\gamma}_i^r$ are the back-rotated strain vector and $\mathbf{F}^r$ is the back-rotated gradient of deformation. The back-rotated strains are given by

$$\boldsymbol{\gamma}_\alpha^r = \{\boldsymbol{\eta}_\alpha^r + \boldsymbol{\kappa}_\alpha^r \times \mathbf{a}^r + (\phi_i \cdot \mathbf{p}_{i,\alpha}) \mathbf{e}_i^r\} \text{ and} \quad (4.30)$$

$$\boldsymbol{\gamma}_3^r = \{(\phi_{i,3} \cdot \mathbf{p}_i) \mathbf{e}_i^r\}. \quad (4.31)$$

The generalized strains are gather in a column vector by

$$\boldsymbol{\epsilon}^r = \begin{bmatrix} \boldsymbol{\eta}_1^r & \boldsymbol{\eta}_2^r & \boldsymbol{\kappa}_1^r & \boldsymbol{\kappa}_2^r & \mathbf{p} & \mathbf{p}_{,1} & \mathbf{p}_{,2} \end{bmatrix}. \quad (4.32)$$

4.5 Velocity Gradient

The material velocity gradient is given by time differentiation of 4.29 (denoted by a superposed dot) as follows

$$\dot{\mathbf{F}} = \boldsymbol{\Omega} \mathbf{F} + \mathbf{Q} [\dot{\boldsymbol{\gamma}}_\alpha^r \otimes \mathbf{e}_\alpha^r + \dot{\boldsymbol{\gamma}}_3^r \otimes \mathbf{e}_3^r], \quad (4.33)$$

where $\boldsymbol{\Omega} = \dot{\mathbf{Q}} \mathbf{Q}^T$ represents the director spin, whose axial vector is $\boldsymbol{\omega} = \text{axial}(\boldsymbol{\Omega}) = \boldsymbol{\Gamma} \dot{\boldsymbol{\theta}}$. The derivatives $\dot{\boldsymbol{\gamma}}_\alpha^r$ and $\dot{\boldsymbol{\gamma}}_3^r$ in 4.33 are obtained directly from 4.30 and 4.31, and yields :

$$\dot{\boldsymbol{\gamma}}_\alpha^r = \{\dot{\boldsymbol{\eta}}_\alpha^r + \dot{\boldsymbol{\kappa}}_\alpha^r \times \mathbf{a}^r + \boldsymbol{\kappa}_\alpha^r \times \dot{\mathbf{a}}^r + \dot{\mathbf{a}}_{,\alpha}^r\} \text{ and} \quad (4.34)$$

$$\dot{\boldsymbol{\gamma}}_3^r = \dot{\mathbf{a}}_{,3}^r. \quad (4.35)$$

In order to fully evaluate expressions 4.34 and 4.35, the time derivatives $\dot{\boldsymbol{\eta}}_\alpha^r$ and $\dot{\boldsymbol{\kappa}}_\alpha^r$ are needed.

From 4.28, it is possible to arrive at

$$\dot{\eta}_\alpha^r = Q^T(\dot{u}_{,\alpha} + Z_{,\alpha}\omega), \quad (4.36)$$

$$\dot{\kappa}_\alpha^r = Q^T\omega_{,\alpha}, \quad (4.37)$$

where $Z_{,\alpha} = \text{skew}(z_{,\alpha})$. The time derivatives related to the director are given by

$$\dot{a}^r = \Psi \cdot \dot{p}, \quad (4.38a)$$

$$\dot{a}_{,\alpha}^r = \Psi \cdot \dot{p}_{,\alpha} \text{ and} \quad (4.38b)$$

$$\dot{a}_{,3}^r = \Psi_{,3} \cdot \dot{p}. \quad (4.38c)$$

In matricial form we can define a column matrix $\dot{\epsilon}^r = \Phi(d)\dot{\bar{d}}'$ where

$$\Phi = \begin{bmatrix} \Phi_{15 \times 15}^* & O_{15 \times np} \\ O_{np \times 15} & I_{np \times np} \end{bmatrix}, \quad (4.39)$$

$$\Phi^* = \underbrace{\begin{bmatrix} Q & O & O & O \\ O & Q & O & O \\ O & O & Q & O \\ O & O & O & Q \end{bmatrix}}_{\Lambda}^T \underbrace{\begin{bmatrix} I & O & Z_{,1} & O & O \\ O & I & Z_{,2} & O & O \\ O & O & O & I & O \\ O & O & O & O & I \end{bmatrix}}_{\Phi} \quad (4.40)$$

and

$$\bar{d}' = \begin{bmatrix} u_{,1} & u_{,2} & \omega & \omega_{,1} & \omega_{,2} & p & p_{,1} & p_{,2} \end{bmatrix}^T. \quad (4.41)$$

Using the matrix

$$Y = \begin{bmatrix} I_{6 \times 6} & O & O_{3np \times 3np} \\ O_{6 \times 6} & \bar{Y}(\theta) & O_{3np \times 3np} \\ O_{6 \times 6} & O & I_{3np \times 3np} \end{bmatrix} \quad (4.42)$$

and

$$\bar{Y}(\theta) = \begin{bmatrix} \Gamma & O & O \\ \Gamma_{,1} & \Gamma & O \\ \Gamma_{,2} & O & \Gamma \end{bmatrix}, \quad (4.43)$$

we can relate $\dot{\bar{d}}' = Y(d)\dot{d}'$. The relation between d and d' is given by $d' = \Delta d$ where Δ is matrix that gathers the ∂_α in a suitable way. The terms that are not dependent on the thickness are given by

$$\dot{\epsilon}^r = \Phi Y \Delta \dot{d} = \mathcal{T}_c^r(d)\dot{d}. \quad (4.44)$$

The terms that depends on the thickness are related by

$$\dot{\gamma}^r = \Xi(\zeta, d)\dot{\epsilon}^r, \quad (4.45)$$

where

$$\Xi(\zeta, d) = \begin{bmatrix} I & O & (-A^r) & O & K_1^r Y & Y & O \\ O & I & O & (-A^r) & K_2^r Y & O & Y \\ O & O & O & O & Y_{,3} & O & O \end{bmatrix} \quad (4.46)$$

and

$$\dot{\gamma}^r = \begin{bmatrix} \dot{\gamma}_1^r \\ \dot{\gamma}_2^r \\ \dot{\gamma}_3^r \end{bmatrix}. \quad (4.47)$$

4.6 Internal Power

The first Piola-Kirchhoff stress tensor, when the material is hyperelastic, is defined by

$$\mathbf{P} = \frac{\partial \varphi(\mathbf{F})}{\partial \mathbf{F}}, \quad (4.48)$$

where $\varphi(\mathbf{F})$ is the strain energy density function.

The first Piola-Kirchhoff stress tensor can be decomposed in

$$\mathbf{P} = \mathbf{Q}\mathbf{P}^r = \mathbf{Q}\boldsymbol{\tau}_i^r \otimes \mathbf{e}_i^r, \quad (4.49)$$

where $\mathbf{P}^r = \boldsymbol{\tau}_i^r \otimes \mathbf{e}_i^r$ is called the back-rotated first Piola-Kirchhoff stress tensor. The quantities $\boldsymbol{\tau}_i^r$ are back-rotated stress vectors and act on cross-sectional planes whose normals on the reference configuration are $\mathbf{e}_i^r$.

The intern power is given by

$$P_{int} = \int_{V^r} \mathbf{P} : \dot{\mathbf{F}} \, dv^r. \quad (4.50)$$

Substituting 4.33 in 4.50 yields

$$\begin{aligned} \mathbf{P} : \dot{\mathbf{F}} &= \mathbf{P} : \boldsymbol{\Omega}\mathbf{F} + \mathbf{P} : \mathbf{Q} [\dot{\gamma}_\alpha^r \otimes \mathbf{e}_\alpha^r + \dot{\gamma}_3^r \otimes \mathbf{e}_3^r] \\ &= \mathbf{P}\mathbf{F}^T : \boldsymbol{\Omega} + \mathbf{P}^r : [\dot{\gamma}_\alpha^r \otimes \mathbf{e}_\alpha^r + \dot{\gamma}_3^r \otimes \mathbf{e}_3^r] \\ &= \boldsymbol{\tau}_i^r \cdot \dot{\gamma}_i^r, \end{aligned} \quad (4.51)$$

where $\mathbf{P}\mathbf{F}^T : \boldsymbol{\Omega} = 0$ cause of the rotation invariance. Substituting 4.34 and 4.35 in 4.51 yields

$$\mathbf{P} : \dot{\mathbf{F}} = \boldsymbol{\tau}_\alpha^r \cdot (\dot{\boldsymbol{\eta}}_\alpha^r + \dot{\boldsymbol{\kappa}}_\alpha^r \times \mathbf{a}^r + \boldsymbol{\kappa}_\alpha^r \times \dot{\mathbf{a}}^r + \dot{\mathbf{a}}_{,\alpha}^r) + \boldsymbol{\tau}_3^r \cdot \dot{\mathbf{a}}_{,3}^r. \quad (4.52)$$

Then, imposing the following relations

$$\mathbf{P} : \dot{\mathbf{F}} = \boldsymbol{\tau}_\alpha^r \cdot \dot{\boldsymbol{\eta}}_\alpha^r + (\mathbf{a}^r \times \boldsymbol{\tau}_\alpha^r) \cdot \dot{\boldsymbol{\kappa}}_\alpha^r + (\boldsymbol{\tau}_\alpha^r \times \boldsymbol{\kappa}_\alpha^r) \cdot \dot{\mathbf{a}}^r + \boldsymbol{\tau}_\alpha^r \cdot \dot{\mathbf{a}}_{,\alpha}^r + \boldsymbol{\tau}_3^r \cdot \dot{\mathbf{a}}_{,3}^r \quad (4.53)$$

and integrating in the thickness, we can derive

$$\begin{aligned} \int_H \mathbf{P} : \dot{\mathbf{F}} \, dx^r &= \underbrace{\int_{H^r} \boldsymbol{\tau}_\alpha^r \, dx^r}_{\mathbf{n}_\alpha^r} \cdot \dot{\boldsymbol{\eta}}_\alpha^r + \underbrace{\int_{H^r} (\mathbf{a}^r \times \boldsymbol{\tau}_\alpha^r) \, dx^r}_{\mathbf{m}_\alpha^r} \cdot \dot{\boldsymbol{\kappa}}_\alpha^r \\ &\quad + \int_{H^r} (\boldsymbol{\tau}_\alpha^r \times \boldsymbol{\kappa}_\alpha^r) \cdot \dot{\mathbf{a}}^r \, dx^r + \int_{H^r} \boldsymbol{\tau}_\alpha^r \cdot \dot{\mathbf{a}}_{,\alpha}^r \, dx^r + \int_{H^r} \boldsymbol{\tau}_3^r \cdot \dot{\mathbf{a}}_{,3}^r \, dx^r, \end{aligned} \quad (4.54)$$

where $\mathbf{n}_\alpha^r$ is the generalized stress energetically conjugated with $\dot{\boldsymbol{\eta}}_\alpha^r$ and $\mathbf{m}_\alpha^r$ is the generalized moment energetically conjugated with $\dot{\boldsymbol{\kappa}}_\alpha^r$. If we use the hypothesis 4.4, it yields

$$\begin{aligned} \int_H \mathbf{P} : \dot{\mathbf{F}} \, dx^r &= \mathbf{n}_\alpha^r \cdot \dot{\boldsymbol{\eta}}_\alpha^r + \mathbf{m}_\alpha^r \cdot \dot{\boldsymbol{\kappa}}_\alpha^r \\ &\quad + \int_{H^r} (\boldsymbol{\tau}_\alpha^r \times \boldsymbol{\kappa}_\alpha^r) \mathbf{e}_i \phi_i \, dx^r \cdot \dot{\mathbf{p}}_i + \int_{H^r} \boldsymbol{\tau}_\alpha^r \mathbf{e}_i \phi_i \, dx^r \cdot \dot{\mathbf{p}}_{i,\alpha} + \int_{H^r} \boldsymbol{\tau}_3^r \mathbf{e}_i \phi_{i,3} \, dx^r \cdot \dot{\mathbf{p}}_i \\ &= \mathbf{n}_\alpha^r \cdot \dot{\boldsymbol{\eta}}_\alpha^r + \mathbf{m}_\alpha^r \cdot \dot{\boldsymbol{\kappa}}_\alpha^r \\ &\quad + \underbrace{\int_{H^r} ((\boldsymbol{\tau}_\alpha^r \times \boldsymbol{\kappa}_\alpha^r) \cdot \mathbf{e}_i \phi_i + \boldsymbol{\tau}_3^r \cdot \mathbf{e}_i \phi_{i,3}) \, dx^r}_{\boldsymbol{\beta}_i} \cdot \dot{\mathbf{p}}_i + \underbrace{\int_{H^r} \boldsymbol{\tau}_\alpha^r \cdot \mathbf{e}_i \phi_i \, dx^r}_{\boldsymbol{\rho}_{i\alpha}} \cdot \dot{\mathbf{p}}_{i,\alpha}, \end{aligned} \quad (4.55)$$

where $\boldsymbol{\beta}_i$ is the bi-shear energetically conjugated with $\dot{\mathbf{p}}_i$ and $\boldsymbol{\rho}_{i\alpha}$ is the bi-moment energetically conjugated with $\dot{\mathbf{p}}_{i,\alpha}$. An important remark is that $\mathbf{p}_i$ is already a deformation, and so there are energetically conjugated stresses related to it.

And we can finally obtain the final expression for the internal power in the body V^r given by

$$P_{int} = \int_{\Omega^r} (\mathbf{n}_\alpha^r \cdot \dot{\boldsymbol{\eta}}_\alpha^r + \mathbf{m}_\alpha^r \cdot \dot{\boldsymbol{\kappa}}_\alpha^r + \boldsymbol{\beta}_i \cdot \dot{\mathbf{p}}_i + \boldsymbol{\rho}_{i\alpha} \cdot \dot{\mathbf{p}}_{i,\alpha}) \, d\Omega^r. \quad (4.56)$$

The generalized stresses and strain can be gathered in a single column matrix as

$$P_{int} = \int_{\Omega^r} \boldsymbol{\sigma}^r \cdot \dot{\boldsymbol{\epsilon}}^r \, da^r, \quad (4.57)$$

where

$$\boldsymbol{\sigma}^r = \int_{H^r} \boldsymbol{\Xi}^T \boldsymbol{\tau}^r \, dx^r, \quad (4.58)$$

$$\boldsymbol{\sigma}^r = \begin{bmatrix} \mathbf{n}_1^r & \mathbf{n}_2^r & \mathbf{m}_1^r & \mathbf{m}_2^r & \boldsymbol{\beta} & \boldsymbol{\rho}_1 & \boldsymbol{\rho}_2 \end{bmatrix}^T \quad (4.59)$$

and

$$\boldsymbol{\tau}^r = \begin{bmatrix} \boldsymbol{\tau}_1^r & \boldsymbol{\tau}_2^r & \boldsymbol{\tau}_3^r \end{bmatrix}^T. \quad (4.60)$$

4.7 External Power

The external power on the same domain Ω^r can be expressed by

$$P_{ext} = \int_{\Omega_-^r} (\bar{\mathbf{t}}_-^r \cdot \dot{\mathbf{x}}_-) \, da^r + \int_{\Omega_+^r} (\bar{\mathbf{t}}_+^r \cdot \dot{\mathbf{x}}_+) \, da^r + \int_{\Gamma_t^r} (\bar{\mathbf{t}}_\Gamma^r \cdot \dot{\mathbf{x}}_\Gamma) \, da^r + \int_{V^r} (\mathbf{b}^r \cdot \dot{\mathbf{x}}) \, dV^r, \quad (4.61)$$

Figure 4.5: Boundary Conditions

where $\bar{\mathbf{t}}_+$, $\bar{\mathbf{t}}_-$ and $\bar{\mathbf{t}}_\Gamma$ are the external surface traction on the top, bottom and the lateral surfaces of the shell per unit area of reference and $\mathbf{b}^r$ is the vector of body forces per unit volume of reference, see fig. 4.5. The time differentiation of 4.2 yields

$$\begin{aligned}\dot{\mathbf{x}} &= \dot{\mathbf{u}} + \dot{\mathbf{a}} = \dot{\mathbf{z}} + \dot{\mathbf{Q}}\mathbf{a}^r + \mathbf{Q}\dot{\mathbf{a}}^r \\ &= \dot{\mathbf{z}} + \boldsymbol{\Omega}\mathbf{a} + \mathbf{Q}\dot{\mathbf{a}}^r.\end{aligned}\quad (4.62)$$

From the hypothesis made in 4.4, 4.5 and 4.6 stems

$$\dot{\mathbf{x}} = \dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{a} + (\phi_i \cdot \dot{\mathbf{p}}_i)\mathbf{e}_i. \quad (4.63)$$

The analyzes of each integral in equation 4.61 yields

$$\begin{aligned}\int_{V^r} (\mathbf{b}^r \cdot \dot{\mathbf{x}}) dV^r &= \int_{V^r} (\mathbf{b}^r \cdot (\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{a} + (\phi_i \cdot \dot{\mathbf{p}}_i)\mathbf{e}_i)) dV^r \\ &= \int_{V^r} (\mathbf{b}^r \cdot \dot{\mathbf{u}}) dV^r + \int_{V^r} (\mathbf{b}^r \cdot \boldsymbol{\omega} \times \mathbf{a}) dV^r + \int_{V^r} (\mathbf{b}^r \cdot (\phi_i \cdot \dot{\mathbf{p}}_i)\mathbf{e}_i) dV^r \\ &= \int_{\Omega^r} \left(\int_{H^r} \mathbf{b}^r dx^r \right) \cdot \dot{\mathbf{u}} da^r + \int_{\Omega^r} \left(\int_{H^r} \mathbf{a} \times \mathbf{b}^r dx^r \right) \cdot \boldsymbol{\omega} da^r \\ &\quad + \int_{\Omega^r} \left(\int_{H^r} \mathbf{b}^r \cdot \mathbf{e}_i \phi_i dx^r \right) \cdot \dot{\mathbf{p}}_i da^r,\end{aligned}\quad (4.64)$$

and

$$\begin{aligned}\int_{\Omega_-^r} (\bar{\mathbf{t}}_- \cdot \dot{\mathbf{x}}_-) da^r &= \int_{\Omega^r} (\bar{\mathbf{t}}_- \cdot (\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{a}_- + (\phi_{i-} \cdot \dot{\mathbf{p}}_i)\mathbf{e}_i)) da^r \\ &= \int_{\Omega^r} \bar{\mathbf{t}}_- \cdot \dot{\mathbf{u}} da^r + \int_{\Omega^r} \bar{\mathbf{t}}_- \cdot (\boldsymbol{\omega} \times \mathbf{a}_-) da^r + \int_{\Omega^r} (\bar{\mathbf{t}}_- \cdot \mathbf{e}_i (\phi_{i-} \cdot \dot{\mathbf{p}}_i)) da^r \\ &= \int_{\Omega^r} \bar{\mathbf{t}}_- \cdot \dot{\mathbf{u}} da^r + \int_{\Omega^r} (\mathbf{a}_- \times \bar{\mathbf{t}}_-) \cdot \boldsymbol{\omega} da^r + \int_{\Omega^r} (\bar{\mathbf{t}}_- \cdot \mathbf{e}_i \phi_{i-}) \cdot \dot{\mathbf{p}}_i da^r,\end{aligned}\quad (4.65)$$

where the upper surface has an equivalent result. The integral of the lateral surface is

$$\begin{aligned}
\int_{\Gamma_t^r} (\bar{\mathbf{t}}_\Gamma^r \cdot \dot{\mathbf{x}}_\Gamma) da^r &= \int_{\Gamma_t^r} (\bar{\mathbf{t}}_\Gamma^r \cdot (\dot{\mathbf{u}}_\Gamma + \boldsymbol{\omega} \times \mathbf{a}_\Gamma + (\boldsymbol{\phi}_{i\Gamma} \cdot \dot{\mathbf{p}}_i) \mathbf{e}_i)) da^r \\
&= \int_{\Gamma_t^r} \left(\int_{H^r} \bar{\mathbf{t}}_\Gamma^r dx^r \right) \cdot \dot{\mathbf{u}}_\Gamma dx^r + \int_{\Gamma_t^r} \left(\int_{H^r} \mathbf{a}_\Gamma \times \bar{\mathbf{t}}_\Gamma^r dx^r \right) \cdot \boldsymbol{\omega} dx^r \\
&\quad + \int_{\Gamma_t^r} \left(\int_{H^r} (\bar{\mathbf{t}}_\Gamma^r \cdot \mathbf{e}_i) \boldsymbol{\phi}_{i\Gamma} dx^r \right) \cdot \dot{\mathbf{p}}_i dx^r.
\end{aligned} \tag{4.66}$$

So, we can gather all the external work in generalized stress efforts as

$$P_{ext} = \int_{\Omega^r} (\bar{\mathbf{n}} \cdot \dot{\mathbf{u}} + \bar{\mathbf{m}} \cdot \dot{\boldsymbol{\omega}} + \bar{\boldsymbol{\beta}}_i \cdot \dot{\mathbf{p}}_i) da^r + \int_{\Gamma^r} (\bar{\mathbf{n}}^\Gamma \cdot \dot{\mathbf{u}} + \bar{\mathbf{m}}^\Gamma \cdot \dot{\boldsymbol{\omega}} + \bar{\boldsymbol{\beta}}_i^\Gamma \cdot \dot{\mathbf{p}}_i) dx^r, \tag{4.67}$$

where

$$\bar{\mathbf{n}} = \int_{H^r} \mathbf{b}^r dx^r + \bar{\mathbf{t}}_-^r + \bar{\mathbf{t}}_+^r, \tag{4.68}$$

$$\bar{\mathbf{m}} = \int_{H^r} \mathbf{a} \times \mathbf{b}^r dx^r + \mathbf{a}_- \times \bar{\mathbf{t}}_-^r + \mathbf{a}_+ \times \bar{\mathbf{t}}_+^r, \tag{4.69}$$

$$\bar{\boldsymbol{\beta}}_i = \int_{H^r} \mathbf{b}^r \cdot \mathbf{e}_i \boldsymbol{\phi}_i dx^r + (\bar{\mathbf{t}}_-^r \cdot \mathbf{e}_i) \boldsymbol{\phi}_{i-} + (\bar{\mathbf{t}}_+^r \cdot \mathbf{e}_i) \boldsymbol{\phi}_{i+}, \tag{4.70}$$

$$\bar{\mathbf{n}}^\Gamma = \left(\int_{H^r} \bar{\mathbf{t}}_\Gamma^r dx^r \right), \tag{4.71}$$

$$\bar{\mathbf{m}}^\Gamma = \left(\int_{H^r} \mathbf{a}_\Gamma \times \bar{\mathbf{t}}_\Gamma^r dx^r \right) \text{ and} \tag{4.72}$$

$$\bar{\boldsymbol{\beta}}_i^\Gamma = \left(\int_{H^r} (\bar{\mathbf{t}}_\Gamma^r \cdot \mathbf{e}_i) \boldsymbol{\phi}_{i\Gamma} dx^r \right). \tag{4.73}$$

In the above equations, $\bar{\mathbf{n}}$ is the applied external force, $\bar{\mathbf{m}}$ the applied external moment, all per unit area of the middle surface in the reference configuration. $\bar{\mathbf{n}}^\Gamma$ are the applied external force and $\bar{\mathbf{m}}^\Gamma$ external moment on the lateral surface. As in (Campello, 2003), $\bar{\boldsymbol{\mu}}$ is the applied external pseudo-moment that is truly power-conjugated to $\boldsymbol{\theta}$.

So, we can gather all the external work in generalized stress efforts as

$$P_{ext} = \int_{\Omega^r} \bar{\mathbf{q}} \cdot \dot{\mathbf{d}} da^r + \int_{\Gamma^r} \bar{\mathbf{q}}^\Gamma \cdot \dot{\mathbf{d}} dx^r \mu, \tag{4.74}$$

where

$$\bar{\mathbf{q}} = \begin{bmatrix} \bar{\mathbf{n}} & \bar{\boldsymbol{\mu}} & \bar{\boldsymbol{\beta}} \end{bmatrix}^T. \tag{4.75}$$

The stress matrix $\bar{\mathbf{q}}$ can be obtained by

$$\begin{aligned}
\bar{\mathbf{q}} &= \int_{H^r} \hat{\boldsymbol{\Lambda}}^T \boldsymbol{\Sigma}^{rT} \mathbf{Q}^T \mathbf{b} dx^r + \hat{\boldsymbol{\Lambda}}^T (\boldsymbol{\Sigma}^r|_-)^T \mathbf{Q}^T \mathbf{t}_- \\
&\quad + \hat{\boldsymbol{\Lambda}}^T (\boldsymbol{\Sigma}^r|_+)^T \mathbf{Q}^T \mathbf{t}_+
\end{aligned} \tag{4.76}$$

and

$$\bar{\mathbf{q}}^\Gamma = \int_{H^r} \hat{\mathbf{\Lambda}}^T \mathbf{\Sigma}^{rT} \mathbf{Q}^T(\mathbf{t}|_\Gamma) d\mathbf{x}^r \quad (4.77)$$

where

$$\mathbf{\Sigma}^r = \begin{bmatrix} \mathbf{I} & -\mathbf{A}^r & \mathbf{\Psi}^r \end{bmatrix} \quad (4.78)$$

and

$$\hat{\mathbf{\Lambda}} = \begin{bmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O}_{3 \times np} \\ \mathbf{O} & \mathbf{\Gamma}^T & \mathbf{O}_{3 \times np} \\ \mathbf{O}_{np \times 3} & \mathbf{O}_{np \times 3} & \mathbf{I}_{np \times np} \end{bmatrix}. \quad (4.79)$$

4.8 Balance and weak form

The Weak form is given by the sum of the internal and external powers that yields

$$\delta W = \int_{\Omega^r} (\boldsymbol{\sigma}^r \cdot \delta \boldsymbol{\epsilon}^r) d\mathbf{a}^r - \int_{\Omega^r} (\mathbf{q} \cdot \delta \mathbf{d}) d\mathbf{a}^r - \int_{\partial\Omega^r} (\mathbf{q}^\Gamma \cdot \delta \mathbf{d}) d\mathbf{x}^r. \quad (4.80)$$

Introducing the eq. (4.56) and performing partial integration on the terms with $\dot{\mathbf{u}}_{,\alpha}^r$, $\dot{\boldsymbol{\omega}}_{,\alpha}^r$ and $\dot{\mathbf{p}}_{i,\alpha}$ then

$$\begin{aligned} P_{int} &= \int_{\Omega^r} (\mathbf{n}_\alpha^r \cdot \delta \boldsymbol{\eta}_\alpha^r + \mathbf{m}_\alpha^r \cdot \delta \boldsymbol{\kappa}_\alpha^r + \boldsymbol{\beta}_i \cdot \delta \mathbf{p}_i + \boldsymbol{\rho}_{i\alpha} \cdot \delta \mathbf{p}_{i,\alpha}) d\mathbf{a}^r \\ &= \int_{\Omega^r} (\mathbf{n}_\alpha \cdot (\delta \mathbf{u}_{,\alpha} + \mathbf{Z}_{,\alpha} \delta \boldsymbol{\omega}) + \mathbf{m}_\alpha \cdot \delta \boldsymbol{\omega}_{,\alpha} + \boldsymbol{\beta}_i \cdot \dot{\mathbf{p}}_i + \boldsymbol{\rho}_{i\alpha} \cdot \dot{\mathbf{p}}_{i,\alpha}) d\mathbf{a}^r \\ &= \int_{\Omega^r} (-\mathbf{n}_{\alpha,\alpha} \cdot \delta \mathbf{u} - ((\mathbf{z}_{,\alpha} \times \mathbf{n}_\alpha) + \mathbf{m}_{\alpha,\alpha}) \cdot \delta \boldsymbol{\omega} - (\boldsymbol{\rho}_{i\alpha,\alpha} - \boldsymbol{\beta}_i) \cdot \dot{\mathbf{p}}_i) d\mathbf{a}^r \\ &\quad + \int_{\Gamma^r} (\mathbf{n}_\alpha \nu_\alpha) d\mathbf{x}^r + \int_{\Gamma^r} (\mathbf{m}_\alpha \nu_\alpha) d\mathbf{x}^r + \int_{\Gamma^r} (\boldsymbol{\rho}_{i\alpha} \nu_\alpha) d\mathbf{x}^r. \end{aligned} \quad (4.81)$$

If we do the same with eq. (4.74) and apply the fundamental lemma of variational calculus, the following local equilibrium equations are obtained

$$\mathbf{n}_{\alpha,\alpha} + \bar{\mathbf{n}} = \mathbf{0}, \quad (4.82)$$

$$\mathbf{m}_{\alpha,\alpha} + (\mathbf{z}_{,\alpha} \times \mathbf{n}_\alpha) + \bar{\mathbf{m}} = \mathbf{0} \text{ and} \quad (4.83)$$

$$\boldsymbol{\rho}_{i\alpha,\alpha} - \boldsymbol{\beta}_i + \bar{\boldsymbol{\beta}}_i = \mathbf{0}. \quad (4.84)$$

The Neumann boundary conditions for this set of differential equations are given by

$$\mathbf{n}_\alpha \nu_\alpha = \bar{\mathbf{n}}^\Gamma \quad (4.85)$$

$$\mathbf{m}_\alpha \nu_\alpha = \bar{\mathbf{m}}^\Gamma \quad (4.86)$$

$$\boldsymbol{\rho}_{i\alpha} \nu_\alpha = \bar{\boldsymbol{\beta}}_i^\Gamma \quad (4.87)$$

where ν_α are the components of the vector $\boldsymbol{\nu}$ normal to the boundary Γ .

The Dirichlet boundary conditions for this set of differential equations are given by

$$\mathbf{u} = \bar{\mathbf{u}} \quad (4.88)$$

$$\boldsymbol{\theta} = \bar{\boldsymbol{\theta}} \quad (4.89)$$

$$\mathbf{p} = \bar{\mathbf{p}}. \quad (4.90)$$

The equilibrium equations and its boundary conditions can be simplified by the relation

$$\mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q} = \mathbf{0} \text{ in } \Omega, \quad (4.91)$$

$$\mathbf{n}\mathbf{H}\boldsymbol{\sigma}^r = \mathbf{q}^\Gamma \text{ on } \partial\Omega, \quad (4.92)$$

$$\mathbf{d} = \bar{\mathbf{d}} \text{ on } \partial\Omega, \quad (4.93)$$

where $\mathcal{T}_e^r(\bullet)$ is the adjoint of $\mathcal{T}_c^r(\bullet)$ and $\mathbf{H}$ is the matrix that gather the the boundary information.

4.9 Linearization of the Weak form

For the solution of the weak form of the problem, stated by eq. 4.80 , within a Newton's type of iterative process, it is crucial to explicitly know the exact tangent operator. We utilize the operator Δ to represent the linearization needed in the Newton method.

The consistent linearization of the weak form is

$$\Delta\delta W = \int_{\Omega^r} (\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\epsilon}^r + \boldsymbol{\sigma}^r \cdot \Delta\delta\boldsymbol{\epsilon}^r) d\mathbf{a}^r - \int_{\Omega^r} (\Delta\mathbf{q} \cdot \delta\mathbf{d}) d\mathbf{a}^r - \int_{\partial\Omega^r} (\Delta\mathbf{q}^\Gamma \cdot \delta\mathbf{d}) d\mathbf{x}^r. \quad (4.94)$$

4.9.1 Linearization of the material part of the virtual internal work

The material tangent matrix is the part related to the linearization of the stress vector $\boldsymbol{\sigma}^r$. Its linearization results in the matrix $\mathbf{D}$ where

$$\delta\boldsymbol{\epsilon}^r \cdot \Delta\boldsymbol{\sigma}^r = \Delta\mathbf{n}_\alpha^r \cdot \delta\boldsymbol{\eta}_\alpha^r + \Delta\mathbf{m}_\alpha^r \cdot \delta\boldsymbol{\kappa}_\alpha^r + \Delta\boldsymbol{\beta} \cdot \delta\mathbf{p} + \Delta\boldsymbol{\rho}_\alpha \cdot \delta\mathbf{p}_{,\alpha} \quad (4.95)$$

$$= \delta\boldsymbol{\epsilon}^r \cdot \frac{\partial\boldsymbol{\sigma}^r}{\partial\boldsymbol{\epsilon}^r} \cdot \Delta\boldsymbol{\epsilon}^r = \delta\boldsymbol{\epsilon}^r \cdot \mathbf{D} \cdot \Delta\boldsymbol{\epsilon}^r. \quad (4.96)$$

If we substitute eq. 4.59 and eq. 4.32 we arrive at

$$\mathbf{D} = \begin{bmatrix} \frac{\partial\mathbf{n}_1^r}{\partial\boldsymbol{\eta}_1^r} & \frac{\partial\mathbf{n}_1^r}{\partial\boldsymbol{\eta}_2^r} & \frac{\partial\mathbf{n}_1^r}{\partial\boldsymbol{\kappa}_1^r} & \frac{\partial\mathbf{n}_1^r}{\partial\boldsymbol{\kappa}_2^r} & \frac{\partial\mathbf{n}_1^r}{\partial\mathbf{p}} & \frac{\partial\mathbf{n}_1^r}{\partial\mathbf{p}_{,1}} & \frac{\partial\mathbf{n}_1^r}{\partial\mathbf{p}_{,2}} \\ \frac{\partial\mathbf{n}_2^r}{\partial\boldsymbol{\eta}_1^r} & \frac{\partial\mathbf{n}_2^r}{\partial\boldsymbol{\eta}_2^r} & \frac{\partial\mathbf{n}_2^r}{\partial\boldsymbol{\kappa}_1^r} & \frac{\partial\mathbf{n}_2^r}{\partial\boldsymbol{\kappa}_2^r} & \frac{\partial\mathbf{n}_2^r}{\partial\mathbf{p}} & \frac{\partial\mathbf{n}_2^r}{\partial\mathbf{p}_{,1}} & \frac{\partial\mathbf{n}_2^r}{\partial\mathbf{p}_{,2}} \\ \frac{\partial\mathbf{m}_1^r}{\partial\boldsymbol{\eta}_1^r} & \frac{\partial\mathbf{m}_1^r}{\partial\boldsymbol{\eta}_2^r} & \frac{\partial\mathbf{m}_1^r}{\partial\boldsymbol{\kappa}_1^r} & \frac{\partial\mathbf{m}_1^r}{\partial\boldsymbol{\kappa}_2^r} & \frac{\partial\mathbf{m}_1^r}{\partial\mathbf{p}} & \frac{\partial\mathbf{m}_1^r}{\partial\mathbf{p}_{,1}} & \frac{\partial\mathbf{m}_1^r}{\partial\mathbf{p}_{,2}} \\ \frac{\partial\mathbf{m}_2^r}{\partial\boldsymbol{\eta}_1^r} & \frac{\partial\mathbf{m}_2^r}{\partial\boldsymbol{\eta}_2^r} & \frac{\partial\mathbf{m}_2^r}{\partial\boldsymbol{\kappa}_1^r} & \frac{\partial\mathbf{m}_2^r}{\partial\boldsymbol{\kappa}_2^r} & \frac{\partial\mathbf{m}_2^r}{\partial\mathbf{p}} & \frac{\partial\mathbf{m}_2^r}{\partial\mathbf{p}_{,1}} & \frac{\partial\mathbf{m}_2^r}{\partial\mathbf{p}_{,2}} \\ \frac{\partial\boldsymbol{\beta}}{\partial\boldsymbol{\eta}_1^r} & \frac{\partial\boldsymbol{\beta}}{\partial\boldsymbol{\eta}_2^r} & \frac{\partial\boldsymbol{\beta}}{\partial\boldsymbol{\kappa}_1^r} & \frac{\partial\boldsymbol{\beta}}{\partial\boldsymbol{\kappa}_2^r} & \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{p}} & \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{p}_{,1}} & \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{p}_{,2}} \\ \frac{\partial\boldsymbol{\rho}_1}{\partial\boldsymbol{\eta}_1^r} & \frac{\partial\boldsymbol{\rho}_1}{\partial\boldsymbol{\eta}_2^r} & \frac{\partial\boldsymbol{\rho}_1}{\partial\boldsymbol{\kappa}_1^r} & \frac{\partial\boldsymbol{\rho}_1}{\partial\boldsymbol{\kappa}_2^r} & \frac{\partial\boldsymbol{\rho}_1}{\partial\mathbf{p}} & \frac{\partial\boldsymbol{\rho}_1}{\partial\mathbf{p}_{,1}} & \frac{\partial\boldsymbol{\rho}_1}{\partial\mathbf{p}_{,2}} \\ \frac{\partial\boldsymbol{\rho}_2}{\partial\boldsymbol{\eta}_1^r} & \frac{\partial\boldsymbol{\rho}_2}{\partial\boldsymbol{\eta}_2^r} & \frac{\partial\boldsymbol{\rho}_2}{\partial\boldsymbol{\kappa}_1^r} & \frac{\partial\boldsymbol{\rho}_2}{\partial\boldsymbol{\kappa}_2^r} & \frac{\partial\boldsymbol{\rho}_2}{\partial\mathbf{p}} & \frac{\partial\boldsymbol{\rho}_2}{\partial\mathbf{p}_{,1}} & \frac{\partial\boldsymbol{\rho}_2}{\partial\mathbf{p}_{,2}} \end{bmatrix}. \quad (4.97)$$

The linearization of the stress vector done in components. We start with the linearization of the

normal stress $\mathbf{n}_\alpha^r$. Its linearization is given by

$$\delta \boldsymbol{\eta}_\alpha^r \cdot \Delta \mathbf{n}_\alpha^r = \delta \boldsymbol{\eta}_\alpha^r \cdot \frac{\partial \mathbf{n}_\alpha^r}{\partial \boldsymbol{\eta}_\beta^r} \cdot \Delta \boldsymbol{\eta}_\beta^r + \delta \boldsymbol{\eta}_\alpha^r \cdot \frac{\partial \mathbf{n}_\alpha^r}{\partial \boldsymbol{\kappa}_\beta^r} \cdot \Delta \boldsymbol{\kappa}_\beta^r + \delta \boldsymbol{\eta}_\alpha^r \cdot \frac{\partial \mathbf{n}_\alpha^r}{\partial \mathbf{p}} \cdot \Delta \mathbf{p} + \delta \boldsymbol{\eta}_\alpha^r \cdot \frac{\partial \mathbf{n}_\alpha^r}{\partial \mathbf{p}_{,\beta}} \cdot \Delta \mathbf{p}_{,\beta} \quad (4.98)$$

where each component is given by

$$\frac{\partial \mathbf{n}_\alpha^r}{\partial \boldsymbol{\eta}_\beta^r} = \int_{H^r} \mathbf{C}_{\alpha\beta}^r \, dx^r, \quad (4.99)$$

$$\frac{\partial \mathbf{n}_\alpha^r}{\partial \boldsymbol{\kappa}_\beta^r} = - \int_{H^r} \mathbf{C}_{\alpha\beta}^r \mathbf{A}^r \, dx^r, \quad (4.100)$$

$$\frac{\partial \mathbf{n}_\alpha^r}{\partial \mathbf{p}} = \int_{H^r} \mathbf{C}_{\alpha\beta}^r \mathbf{K}_\beta^r \boldsymbol{\Psi} + \mathbf{C}_{\alpha 3}^r \boldsymbol{\Psi}' \, dx^r \text{ and} \quad (4.101)$$

$$\frac{\partial \mathbf{n}_\alpha^r}{\partial \mathbf{p}_{,\beta}} = \int_{H^r} \mathbf{C}_{\alpha\beta}^r \boldsymbol{\Psi} \, dx^r. \quad (4.102)$$

The same is done for the moment stress $\mathbf{m}_\alpha^r$ what results

$$\delta \boldsymbol{\kappa}_\alpha^r \cdot \Delta \mathbf{m}_\alpha^r = \delta \boldsymbol{\kappa}_\alpha^r \cdot \frac{\partial \mathbf{m}_\alpha^r}{\partial \boldsymbol{\eta}_\beta^r} \cdot \Delta \boldsymbol{\eta}_\beta^r + \delta \boldsymbol{\kappa}_\alpha^r \cdot \frac{\partial \mathbf{m}_\alpha^r}{\partial \boldsymbol{\kappa}_\beta^r} \cdot \Delta \boldsymbol{\kappa}_\beta^r + \quad (4.103)$$

$$\delta \boldsymbol{\kappa}_\alpha^r \cdot \frac{\partial \mathbf{m}_\alpha^r}{\partial \mathbf{p}} \cdot \Delta \mathbf{p} + \delta \boldsymbol{\kappa}_\alpha^r \cdot \frac{\partial \mathbf{m}_\alpha^r}{\partial \mathbf{p}_{,\beta}} \cdot \Delta \mathbf{p}_{,\beta} \quad (4.104)$$

and its components are

$$\frac{\partial \mathbf{m}_\alpha^r}{\partial \boldsymbol{\eta}_\beta^r} = \frac{\partial \mathbf{n}_\beta^{rT}}{\partial \boldsymbol{\kappa}_\alpha^r} = \int_{H^r} \mathbf{A}^r \mathbf{C}_{\alpha\beta}^r \, dx^r, \quad (4.105)$$

$$\frac{\partial \mathbf{m}_\alpha^r}{\partial \boldsymbol{\kappa}_\beta^r} = - \int_{H^r} \mathbf{A}^r \mathbf{C}_{\alpha\beta}^r \mathbf{A}^r \, dx^r, \quad (4.106)$$

$$\frac{\partial \mathbf{m}_\alpha^r}{\partial \mathbf{p}} = \int_{H^r} \mathbf{A}^r \mathbf{C}_{\alpha\beta}^r \mathbf{K}_\beta^r \boldsymbol{\Psi} + \mathbf{A}^r \mathbf{C}_{\alpha 3}^r \boldsymbol{\Psi}' \, dx^r - \text{skew}(\boldsymbol{\rho}_\alpha) \text{ and} \quad (4.107)$$

$$\frac{\partial \mathbf{m}_\alpha^r}{\partial \mathbf{p}_{,\beta}} = \int_{H^r} \mathbf{A}^r \mathbf{C}_{\alpha\beta}^r \boldsymbol{\Psi}' \, dx^r. \quad (4.108)$$

The next component of the stress vector, see eq. 4.59, is the bi-shear stress $\boldsymbol{\beta}$. Its linearization is directly given by

$$\delta \mathbf{p} \cdot \Delta \boldsymbol{\beta} = \delta \mathbf{p} \cdot \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\eta}_\beta^r} \cdot \Delta \boldsymbol{\eta}_\beta^r + \delta \mathbf{p} \cdot \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\kappa}_\beta^r} \cdot \Delta \boldsymbol{\kappa}_\beta^r + \delta \mathbf{p} \cdot \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{p}} \cdot \Delta \mathbf{p} + \delta \mathbf{p} \cdot \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{p}_{,\beta}} \cdot \Delta \mathbf{p}_{,\beta} \quad (4.109)$$

and its components follows as

$$\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\eta}_\beta^r} = \frac{\partial \mathbf{n}_\beta^{rT}}{\partial \mathbf{p}} = \int_{H^r} -\boldsymbol{\Psi}^T \mathbf{K}_\alpha^r \mathbf{C}_{\alpha\beta}^r + \boldsymbol{\Psi}'^T \mathbf{C}_{3\beta}^r \, dx^r, \quad (4.110)$$

$$\frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\kappa}_\beta^r} = \frac{\partial \mathbf{m}_\beta^{rT}}{\partial \mathbf{p}} = \int_{H^r} \boldsymbol{\Psi}^T \mathbf{K}_\alpha^r \mathbf{C}_{\alpha\beta}^r \mathbf{A}^r - \boldsymbol{\Psi}'^T \mathbf{C}_{3\beta}^r \mathbf{A}^r \, dx^r + \text{skew}(\boldsymbol{\rho}_\alpha), \quad (4.111)$$

$$\frac{\partial \beta}{\partial \mathbf{p}} = \int_{H^r} -\Psi^T \mathbf{K}_\alpha^r C_{\alpha\beta}^r \mathbf{K}_\beta^r \Psi + \Psi^T C_{33}^r \Psi' dx^r \text{ and} \quad (4.112)$$

$$\frac{\partial \beta}{\partial \mathbf{p}_{,\beta}} = \int_{H^r} -\Psi^T \mathbf{K}_\alpha^r C_{\alpha\beta}^r \Psi + \Psi'^T C_{3\beta}^r \Psi dx^r. \quad (4.113)$$

Finally, the linearization of ρ_α is given by

$$\delta \mathbf{p}_{,\alpha} \cdot \Delta \rho_\alpha = \delta \mathbf{p}_{,\alpha} \cdot \frac{\partial \rho_\alpha}{\partial \eta_\beta^r} \cdot \Delta \eta_\beta^r + \delta \mathbf{p}_{,\alpha} \cdot \frac{\partial \rho_\alpha}{\partial \kappa_\beta^r} \cdot \Delta \kappa_\beta^r \quad (4.114)$$

$$+ \delta \mathbf{p}_{,\alpha} \cdot \frac{\partial \rho_\alpha}{\partial \mathbf{p}} \cdot \Delta \mathbf{p} + \delta \mathbf{p}_{,\alpha} \cdot \frac{\partial \rho_\alpha}{\partial \mathbf{p}_{,\beta}} \cdot \Delta \mathbf{p}_{,\beta} \quad (4.115)$$

where its components are

$$\frac{\partial \rho_\alpha}{\partial \eta_\beta^r} = \frac{\partial \mathbf{n}_\beta^r}{\partial \mathbf{p}_{,\alpha}}^T = \int_{H^r} \Psi^T C_{\alpha\beta}^r dx^r, \quad (4.116)$$

$$\frac{\partial \rho_\alpha}{\partial \kappa_\beta^r} = \frac{\partial \mathbf{m}_\beta^r}{\partial \mathbf{p}_{,\alpha}}^T = - \int_{H^r} \Psi^T C_{\alpha\beta}^r \mathbf{A}^r dx^r, \quad (4.117)$$

$$\frac{\partial \rho_\alpha}{\partial \mathbf{p}} = \frac{\partial \beta}{\partial \mathbf{p}_{,\alpha}}^T = \int_{H^r} \Psi^T C_{\alpha\beta}^r \mathbf{K}_\beta^r \Psi + \Psi^T C_{\alpha 3}^r \Psi' dx^r \text{ and} \quad (4.118)$$

$$\frac{\partial \rho_\alpha}{\partial \mathbf{p}_{,\beta}} = \int_{H^r} \Psi^T C_{\alpha\beta}^r \Psi dx^r. \quad (4.119)$$

4.9.2 Linearization of the geometrical part of the virtual internal work

The geometrical part of the linearization of the weak form is the part related to the the strain vector ϵ^r in .It linearization is

$$\sigma^r \cdot \Delta \delta \epsilon^r = \mathbf{n}_\alpha^r \cdot \Delta \delta \eta_\alpha^r + \mathbf{m}_\alpha^r \cdot \Delta \delta \kappa_\alpha^r + \beta \cdot \Delta \delta \mathbf{p} + \rho_\alpha \cdot \Delta \delta \mathbf{p}_{,\alpha} \quad (4.120)$$

$$= \mathbf{n}_\alpha^r \cdot \Delta \delta \eta_\alpha^r + \mathbf{m}_\alpha^r \cdot \Delta \delta \kappa_\alpha^r = \mathbf{\Delta}(\delta \mathbf{d}) \cdot \mathbf{G} \cdot \mathbf{\Delta}(\Delta \mathbf{d}), \quad (4.121)$$

where $\mathbf{G}$ is geometrical tangent matrix and $\mathbf{\Delta}$ is the operator presented in eq. 4.44.

The linearization of $\Delta \delta \eta_\alpha^r$ is given by

$$\mathbf{n}_\alpha^r \cdot \Delta \delta \eta_\alpha^r = \delta \mathbf{u}_{,\alpha} \cdot \mathbf{G}_\alpha^{u'\theta} \cdot \Delta \boldsymbol{\theta} + \delta \boldsymbol{\theta} \cdot \mathbf{G}_\alpha^{\theta u'} \cdot \Delta \mathbf{u}_{,\alpha} \quad (4.122)$$

$$+ \delta \boldsymbol{\theta} \cdot (\mathbf{\Gamma}^T \text{skew}(\mathbf{z}_{,\alpha}) \text{skew}(\mathbf{n}_\alpha) \mathbf{\Gamma} - \mathbf{V}(\boldsymbol{\theta}, \mathbf{z}_{,\alpha} \times \mathbf{n}_\alpha)) \cdot \Delta \boldsymbol{\theta}, \quad (4.123)$$

where its components are

$$\mathbf{G}_\alpha^{u'\theta} = (\mathbf{G}_\alpha^{\theta u'})^T = -\text{skew}(\mathbf{n}_\alpha) \mathbf{\Gamma}. \quad (4.124)$$

And the linearization of $\Delta \delta \kappa_\alpha^r$ is given by

$$\mathbf{m}_\alpha^r \cdot \Delta \delta \kappa_\alpha^r = \delta \boldsymbol{\theta} \cdot \mathbf{G}_\alpha^{\theta\theta'} \cdot \Delta \boldsymbol{\theta}_{,\alpha} + \delta \boldsymbol{\theta}_{,\alpha} \cdot \mathbf{G}_\alpha^{\theta'\theta} \cdot \Delta \boldsymbol{\theta} + \delta \boldsymbol{\theta} \cdot (\mathbf{V}_{,\alpha}(\boldsymbol{\theta}, \boldsymbol{\theta}_{,\alpha}, \mathbf{m}_\alpha) - \mathbf{\Gamma}_{,\alpha}^T \mathbf{M}_\alpha \mathbf{\Gamma}) \cdot \Delta \boldsymbol{\theta}, \quad (4.125)$$

where its components are

$$\mathbf{G}_\alpha^{\theta'\theta} = (\mathbf{G}_\alpha^{\theta\theta'})^T = \mathbf{V}(\boldsymbol{\theta}, \mathbf{m}_\alpha) - \boldsymbol{\Gamma}^T \text{skew}(\mathbf{m}_\alpha) \boldsymbol{\Gamma} = \mathbf{V}^T(\boldsymbol{\theta}, \mathbf{m}_\alpha). \quad (4.126)$$

We can define another tensor

$$\mathbf{G}^{\theta\theta} = \mathbf{V}_{,\alpha}(\boldsymbol{\theta}, \boldsymbol{\theta}_{,\alpha}, \mathbf{m}_\alpha) - \boldsymbol{\Gamma}_{,\alpha}^T \mathbf{M}_\alpha \boldsymbol{\Gamma} + \boldsymbol{\Gamma}^T \text{skew}(\mathbf{z}_{,\alpha}) \text{skew}(\mathbf{n}_\alpha) \boldsymbol{\Gamma} - \mathbf{V}(\boldsymbol{\theta}, \mathbf{z}_{,\alpha} \times \mathbf{n}_\alpha), \quad (4.127)$$

based on terms that multiply $\Delta\boldsymbol{\theta}$ and $\delta\boldsymbol{\theta}$ in eq. 4.126 and eq. 4.122.

We can gather all terms in the Geometrical tangent matrix, given by

$$\mathbf{G} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{G}_1^{u'\theta} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{G}_2^{u'\theta} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{G}_1^{\theta\theta'} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{G}_2^{\theta\theta'} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{G}_1^{\theta u'} & \mathbf{G}_2^{\theta u'} & \mathbf{G}_1^{\theta'\theta} & \mathbf{G}_2^{\theta'\theta} & \mathbf{G}^{\theta\theta} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix}. \quad (4.128)$$

4.9.3 Linearization of the virtual external work

The Linearization of the virtual external work has two components, one of the terms integrated in the domain Ω^r and one of the terms integrated on its frontier $\partial\Omega^r$. This can be seen in the equation

$$\Delta\delta W_{\text{ext}} = \int_{\Omega^r} (\Delta\mathbf{q} \cdot \delta\mathbf{d}) \, da^r + \int_{\partial\Omega^r} (\Delta\mathbf{q}^\Gamma \cdot \delta\mathbf{d}) \, dx^r. \quad (4.129)$$

The linearization of the the first part is

$$\Delta\mathbf{q} \cdot \delta\mathbf{d} = \delta\mathbf{u} \cdot \Delta\bar{\mathbf{n}} + \delta\boldsymbol{\theta} \cdot \Delta\bar{\boldsymbol{\mu}} + \delta\mathbf{p} \cdot \Delta\bar{\boldsymbol{\beta}} = \delta\boldsymbol{\theta} \cdot \frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{\theta}} \cdot \Delta\boldsymbol{\theta} + \delta\boldsymbol{\theta} \cdot \frac{\partial\bar{\boldsymbol{\mu}}}{\partial\mathbf{p}} \cdot \Delta\mathbf{p} + \delta\mathbf{p} \cdot \frac{\partial\bar{\boldsymbol{\beta}}}{\partial\boldsymbol{\theta}} \cdot \Delta\boldsymbol{\theta}, \quad (4.130)$$

where its components are

$$\frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{\theta}} = \boldsymbol{\Gamma}^T (\bar{\mathbf{T}}^- \mathbf{A}_- + \bar{\mathbf{T}}^+ \mathbf{A}_+ + \int_{H^r} \mathbf{B} \mathbf{A} \, dx^r) \boldsymbol{\Gamma} + \mathbf{V}(\boldsymbol{\theta}, \mathbf{m}_\alpha) \text{ and} \quad (4.131)$$

$$\frac{\partial\bar{\boldsymbol{\mu}}}{\partial\mathbf{p}} = \frac{\partial\bar{\boldsymbol{\beta}}}{\partial\boldsymbol{\theta}}^T = \boldsymbol{\Gamma}^T (\bar{\mathbf{T}}^- \mathbf{Q} \boldsymbol{\Psi}_- + \bar{\mathbf{T}}^+ \mathbf{Q} \boldsymbol{\Psi}_+ + \int_{H^r} \mathbf{B} \mathbf{Q} \boldsymbol{\Psi} \, dx^r). \quad (4.132)$$

We can define an external tangent matrix of the stresses inside as

$$\mathbf{L} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \frac{\partial\bar{\boldsymbol{\mu}}}{\partial\boldsymbol{\theta}} & \frac{\partial\bar{\boldsymbol{\mu}}}{\partial\mathbf{p}} \\ \mathbf{O} & \frac{\partial\bar{\boldsymbol{\beta}}}{\partial\boldsymbol{\theta}} & \mathbf{O} \end{bmatrix}, \quad (4.133)$$

so that,

$$\Delta\mathbf{q} \cdot \delta\mathbf{d} = \delta\mathbf{d} \cdot \mathbf{L} \cdot \Delta\mathbf{d}. \quad (4.134)$$

The same thing can be done with the second term, so that, we can define an external tangent matrix of the stresses on the Neumann boundary as

$$\mathbf{L}^\Gamma = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \frac{\partial \bar{\boldsymbol{\mu}}^\Gamma}{\partial \boldsymbol{\theta}} & \frac{\partial \bar{\boldsymbol{\mu}}^\Gamma}{\partial \mathbf{p}} \\ \mathbf{O} & \frac{\partial \bar{\boldsymbol{\beta}}^\Gamma}{\partial \boldsymbol{\theta}} & \mathbf{O} \end{bmatrix}, \quad (4.135)$$

where its components are

$$\frac{\partial \bar{\boldsymbol{\mu}}^\Gamma}{\partial \boldsymbol{\theta}} = \boldsymbol{\Gamma}^T \left(\int_{H^r} \mathbf{T}^\Gamma \mathbf{A} \, dx^r \right) \boldsymbol{\Gamma} + \mathbf{V}(\boldsymbol{\theta}, \mathbf{m}_\alpha^\Gamma) \text{ and} \quad (4.136)$$

$$\frac{\partial \bar{\boldsymbol{\mu}}^\Gamma}{\partial \mathbf{p}} = \frac{\partial \bar{\boldsymbol{\beta}}^\Gamma}{\partial \boldsymbol{\theta}}{}^T = \boldsymbol{\Gamma}^T \left(\int_{H^r} \mathbf{T}^\Gamma \mathbf{Q} \boldsymbol{\Psi} \, dx^r \right). \quad (4.137)$$

4.9.4 Final linearization

The evaluation of each term of the eq. 4.94 results in

$$\begin{aligned} \Delta \delta W = & \int_{\Omega^r} (\delta \boldsymbol{\epsilon}^r \cdot \mathbf{D} \cdot \Delta \boldsymbol{\epsilon}^r + \boldsymbol{\Delta}(\delta \mathbf{d}) \cdot \mathbf{G} \cdot \boldsymbol{\Delta}(\Delta \mathbf{d}) \, da^r \\ & - \int_{\Omega^r} ((\delta \mathbf{d}) \cdot \mathbf{L} \cdot (\Delta \mathbf{d})) \, da^r - \int_{\partial \Omega^r} ((\delta \mathbf{d}) \cdot \mathbf{L}^\Gamma \cdot (\Delta \mathbf{d})) \, dx^r. \end{aligned} \quad (4.138)$$

4.10 Finite element discretization

The approximate solution of the problem defined in the eq. 4.80 is sought by the FEM discretization. The discretization used is the same presented in the linear case, see section 2.10.

The stiffness matrix is given by

$$\mathbf{K}^e = \int_{\Omega_e} (\boldsymbol{\Phi} \mathbf{Y} \mathbf{B})^T \mathbf{D} \boldsymbol{\Phi} \mathbf{Y} \mathbf{B} + \mathbf{B}^T \mathbf{G} \mathbf{B} \, da^r - \int_{\Omega_e \cap \Omega_n} \mathbf{N}^T \mathbf{L} \mathbf{N} \, da^r - \int_{\partial \Omega_e \cap \partial \Omega_n} \mathbf{N}^T \mathbf{L}^\Gamma \mathbf{N} \, dx^r, \quad (4.139)$$

where $\mathbf{B} = \boldsymbol{\Delta} \mathbf{N}$.

And the local residuum is

$$\mathbf{r}^e = \int_{\Omega_e} (\boldsymbol{\Phi} \mathbf{Y} \mathbf{B})^T \boldsymbol{\sigma}^r \, da^r - \int_{\Omega_e \cap \Omega_n} \mathbf{N}^T \mathbf{q} \, da^r - \int_{\partial \Omega_e \cap \partial \Omega_n} \mathbf{N}^T \mathbf{q}^\Gamma \, dx^r. \quad (4.140)$$

4.11 Results

It's difficult to obtain analytical solutions in nonlinear elasticity. The known solutions are restricted to simple problems. That's why we chose to compare with a commercial code. In this section we intend to compare the results with a commercial code, in our case we used the Abaqus [30].

The first benchmark is presented in 4.6. The data is presented in the table 4.2. This a simple example that exploit the classical kinematical hypothesis of a shell model but at the same time have some a localized solution, better captured by a three dimensional model.

The mesh used was regular with 84 x 30 x 6 elements.

Figure 4.6: *First benchmark*

Table 4.2: *Data of the first benchmark*

Data name	Value
(force) $q =$	1
(width) $w =$	10
(Length) $L =$	100
(height) $h =$	2
$E =$	20000
$\nu =$	0.25

The results were taken in a transverse cross section of the solid figure 4.6. The displacements are given in the figure 4.7. It can be seen that the displacement values don't shown any clear border effect, in this specific section.

However, the stress components S_{13} , S_{22} , S_{23} and S_{33} , see figure 4.8 and 4.9, shows a clear localized effect in the region where the displacement were prescribed. The enrichments in the thickness are presented in the fig. 4.10.

Figure 4.7: *Abaqus stress results of the first benchmark*

Figure 4.8: Abaqus stress results of the first benchmark

Figure 4.9: Abaqus stress results of the first benchmark

Figure 4.10: Enrichment in the thickness

Figure 4.11: *Deformed configuration*

Conclusion

The study was set out to explore the use of hierarchical models to represent three-dimensional solids in a computationally inexpensive way.

The first chapter presents the linear theory for hierarchical models. This chapter explores the choice of the finite element spaces and how it affects the convergence in relation to the thickness parameter. It was studied three different models, i.e. d^1 , d^2 and d^3 . In the section 2.11.1, it was evaluated five types of discretization. It was shown that the use of non-conform element as presented in [18] is restricted to a range of thickness since it loses convergence as thickness grows in the rotation field. It should be remarked that the element still converges in energy. The best lowest order suitable combination of spaces were the use of second order degrees polynomials to all fields. In the section 2.11.2, there is an increase in the order of lowest polynomial to three and in 2.11.3 to 4 – 5. This increase shows that the higher is the model; the bigger is the order of the lowest needed polynomial. For thin plate, high order models are only required close to the boundary. A good solution would be the use of hp-refinement strategy. This procedure allows the improvement of the element order just where it is necessary. Another important remark is that within we used just apriori kind of enrichment in the thickness. It is difficult to forecast locking for different kind of enrichments, what would be interesting if someone would try to use a posteriori enrichments techniques, i.e. proper generalized decomposition (PGD).

In the second chapter, it is presented a theory to evaluate the error in the discretization and the kinematical hypothesis. For calculate the error in the discretization, we present two techniques. First the ZZ super-patch recovery and after the explicit error in energy. These techniques together with mesh adaptativity are necessary to capture the boundary layer and avoid apriori mesh construction techniques, as cited in [53]. We showed the method is capable of achieving the boundary layer in automated way for any model d^n . After we presented a posteriori error technique for kinematical hypothesis based on the equilibrium error of higher order models. This method gave good results but showed a problem that the total energy of additional models is much lower than the total energy. This has been already presented in the first chapter. This imposes the necessity of an error in a quantity of interest, e.g. stress. This is important since the error in Reissner-Mindlin models can be of 30%, see [13], and this is even more remarkable in multi-layered plates. To implement the error in quantity of interest, an equilibrated flux technique must be used. At the end of the chapter, we present the theory to apply this kind of approach.

In the third chapter, it is presented a geometrical nonlinear hierarchical shell model and its discretization. It is shown that the model succeeds in representing the three-dimensional solution when compared with Abaqus using solid elements. The comparison was made with an d^2 model. It was clear in figure 4.10 that the enriched degrees of freedom have an extremely localized solution typical of a boundary layer. This enforces the need for a kind of hp-refinement to profit from the

hierarchical model approach.

The future directions of my work would be in the study of error in quantity of interest for stress in higher order models and the use of hp-refinement strategy.

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