GUILHERME SCABIN VICINANSÁ

Algebraic Estimators with Applications

São Paulo
2018

GUILHERME SCABIN VICINANSÁ

Algebraic Estimators with Applications

Thesis submitted for the degree of Master in
Science to the Escola Politécnica of Universidade
de São Paulo.

São Paulo
2018

GUILHERME SCABIN VICINANSÁ

Algebraic Estimators with Applications

Thesis submitted for the degree of Master in
Science to the Escola Politécnica of Universidade
de São Paulo.

Concentration field:
Systems Engineering

Advisor:
Prof. Dr. Paulo Sérgio Pereira da Silva
Co-Advisor:
Prof. Dr. Claudio Garcia

São Paulo
2018

Este exemplar foi revisado e corrigido em relação à versão original, sob responsabilidade única do autor e com a anuência de seu orientador.

São Paulo, _____ de _____ de _____

Assinatura do autor: _____

Assinatura do orientador: _____

Catálogo-na-publicação

Vicinansa, Guilherme Scabin

Algebraic Estimators with Applications / G. S. Vicinansa -- versão corr. --
São Paulo, 2018.
58 p.

Dissertação (Mestrado) - Escola Politécnica da Universidade de São Paulo. Departamento de Engenharia de Telecomunicações e Controle.

1. Estimção paramétrica 2. Controle não linear 3. Estimadores algébricos
4. Controle baseado em lógica I. Universidade de São Paulo. Escola Politécnica.
Departamento de Engenharia de Telecomunicações e Controle II. t.

To my parents

ACKNOWLEDGEMENT

Formost I would like to express my sincere gratitude to my parents Fernando and Laura for their support, education, values and love. Without them I possibly wouldn't have had the opportunities I had to study and to do my research. For that and many other things I'm trully grateful.

I am thankful to the Professors from LAC for their friendship and encouragement that they continuously give me since my undergraduate studies. Especially I would like to thank my advisor Prof. Dr. Paulo Sérgio Pereira da Silva for all the discussions we had about mathematics and control, and all the insights he provided me through the development of this dissertation. I am also grateful to my co-advisor Prof. Dr. Claudio Garcia, who helped me with a better understanding of how process control works in practice. Last, but not least, I want to express my gratitude to Prof. Dr. Felipe Pait, for all the help provided during my graduate and undergraduate studies.

During the last years I made many friends who I would like to thank for the help, the nice conversations and the time we spent together. Especially the people from the LCA Arthur, Fabio, Gabriel, Matheus, Samuel and Bruno who always welcomed me, most of the times with a freshly brewed cup of coffee, and for their companionship in these past years. Also, many other friends have played an important role during the confection of this dissertation such as Daniel Noriaki, Bruno Giordano and Gustavo Leite.

I wish to thank my girlfriend Juliana for her support, comprehension and love. Without whom I wouldn't have had some many happy moments in these past few years.

I am grateful to CNPq, for supporting this work financially under grant 132806/2017-7 and CAPES for supporting my participation at COBEM 2017.

RESUMO

Nessa pesquisa, estudamos o problema de compensação de atrito em válvulas pneumáticas. É proposta uma lei de controle não linear que tem estimadores algébricos em sua estrutura, para adaptar o controlador ao envelhecimento da válvula. Para isso, estimam-se os valores de parâmetros relacionados ao modelo de Karnopp da válvula, necessários à compensação do atrito, de maneira *online*. Os estimadores e o controlador são validados através de simulações.

Palavras-chave: Estimadores Algébricos. Controle Não Linear. Controle Adaptativo. Controle Baseado em Lógica.

ABSTRACT

In this work we address the problem of friction compensation in a pneumatic control valve. It is proposed a nonlinear control law that uses algebraic estimators in its structure, in order to adapt the controller to the aging of the valve. For that purpose we estimate parameters related to the valve's Karnopp model, necessary to friction compensation, online. The estimators and the controller are validated through simulations

Keywords: Algebraic Estimators, Nonlinear Control, Adaptive Control, Logic-Based Control

LIST OF FIGURES

1	Results of the identification procedure	7
2	Results of the identification procedure	8
3	Results of the estimation procedure	10
4	Valve schematics	11
5	Block Diagram for the External Control System	17
6	Block Diagram of the Pneumatic Valve's Control System	18
7	State Machine for the Control Law	20
8	Comparison between a_e and a	23
9	Comparison between b_e and b	24
10	Comparison between c_e and c	24
11	Comparison between the estimated velocity and the true one	25
12	Comparison between the estimated acceleration and the true one	25
13	Comparison between k_e and k	26
14	Output Regulation Performance	27
15	Comparison between a_e and a	28
16	Comparison between b_e and b	28
17	Comparison between c_e and c	29
18	Comparison between the estimated velocity and the true one	30
19	Comparison between the estimated acceleration and the true one	30
20	Comparison between k_e and k	31
21	Output Regulation Performance	31

CONTENTS

1	Introduction	1
1.1	Objectives	1
1.2	Motivation and Justification	1
1.3	Introduction to Algebraic Estimators	2
1.3.1	Examples	2
2	Main results	11
2.1	System Model	11
2.2	Online Algebraic Identification and Estimation	12
2.2.1	Identification of the Model Parameters	12
2.2.2	Identifiability	14
2.2.3	Friction Estimation	15
2.3	Control Law	17
2.3.1	Control Logic	18
2.3.2	Stability and Performance	21
3	Numerical Results	22
3.1	Computational Procedure	22
3.1.1	Sinusoidal Noise	22
3.1.1.1	Identifier	22
3.1.1.2	Friction Estimation	23
3.1.1.3	System Performance	26
3.1.2	High Frequency Noise	27

3.1.2.1	Identifier	27
3.1.2.2	Friction Estimation	28
3.1.2.3	System Performance	29
4	Conclusion	32
	References	33
	Appendix A - Algebraic Properties of the Estimators	35
A.1	Algebraic Definitions	35
A.2	Relation with Laplace Transform	38
	Appendix B - Dynamic Properties	40
B.1	Validity of the Taylor Expansion	40
B.2	Stability and Performance	41
B.2.1	Input to State Stability (ISS)	41
B.2.2	Minimal Gap Between Desired Outputs	46

1 INTRODUCTION

1.1 Objectives

The objective of this dissertation is the study of algebraic estimators and identifiers, and their applications to pneumatic valve's friction compensation. This study involves the design of the estimators and identifiers, as well as the design of a control law that is suitable to the control of this system under conditions typically found in practice.

1.2 Motivation and Justification

The control of industrial processes usually is a complex task, due to the number of control loops in the plant, nonlinearities, pure delays, backlash among others (DESBOROUGH; MILLER, 2001; JELALI; HUANG, 2012). One of these problems is the friction in pneumatic control valves, that may cause oscillations and increase process' variability. It has been assessed that 20% to 30% of the oscillations in industrial processes are due to friction in pneumatic valves (BIALKOWSKI, 1992). Moreover, pneumatic valves are ubiquitous, being fundamental blocks in the process industry (SEBORG; MELLICHAMP, 2006). Hence, designing controllers that can maintain their correct operation is a relevant task from both academic and practical perspectives.

Many control techniques, which provide good performance, rely on the fact that a model for the process is known. Therefore, the knowledge of the valve physical model parameters is desirable. However, although there are identification methods for estimating these parameters, they are often complicated and require long periods of time with persistent signals as the process input (GARCIA, 2008; ROMANO; GARCIA, 2011).

Algebraic estimators are recent (FLIESS et al., 2002), nonetheless they have shown good results in simulations and practical applications (DIAO et al., 2013; MORAES; SILVA, 2015). These estimators can handle linear and nonlinear phenomena, in a very straightforward manner (FLIESS; JOIN; RAMIREZ, 2008). Unfortunately, algebraic estimators have a very abstract theory that forms a bottleneck for those who are not familiar with differential algebra or operational calculus.

Even though algebraic estimators present those disadvantages, it is important to mention their advantages that allow them to solve some problems at which the conventional techniques fail. Between those advantages we mention the ones that made us choose them over conventional techniques, which were the fact that they are not asymptotic, converging in finite time, that often is small, and the fact that they are capable of identifying model parameters without the usual persistent excitation hypothesis (FLIESS; JOIN; RAMIREZ, 2008).

In this thesis we address the problem of controlling the position of the valve stem under the effects of friction and stiction. In order to solve this problem, we present a controller that uses algebraic estimators in its structure, as well as a switched control logic to better handle the fact that the dynamic model of the stem position is non-smooth, due to friction. We validate our methodology through simulations. Some of these results were presented at the 24th ABCM International Congress of Mechanical Engineering in the paper (VICINANSÁ; GARCIA; SILVA, 2017) wrote by the author.

1.3 Introduction to Algebraic Estimators

The operational calculus bears a remarkable resemblance to the Laplace transform. Moreover, this resemblance can be precisely stated using results from algebra. We present in the Appendix A the necessary relations for the clear understanding of the mathematical concepts behind the working principles of these estimators. Nonetheless, the practitioner can understand easily how to design them without a heavy mathematical machinery.

Here we will present a few low-order examples of the use of algebraic estimators and will discuss its practical issues, such as robustness to measurement noise, and numerical issues, without relying on mathematical proofs. Also, we hope that with these examples we may help the reader to create intuition about the functioning of these estimators.

1.3.1 Examples

Example 1 (Robustness): This example is based on physics and is the simplest one that the author could think of. Consider a weighing balance with a weight of mass m over it. Assume that we perform the measurements continuously, such that the result of the measurement is a continuous function of time $\hat{m}(\cdot)$. Also, assume that those measurements are corrupted by noise

$n(\cdot)$, such that:

$$\hat{m}(t) = m + n(t) \quad (1.1)$$

To build an algebraic estimator for m , we must ignore the noise, and rewrite the equation using operational calculus notation. One can multiply both sides of the equation by the operator $\frac{1}{s}$, and we arrive at

$$\frac{\hat{M}}{s} = \frac{M}{s}, \quad (1.2)$$

which does not differ by any means from Equation (1.1) if the noise is set to zero. Finally, assuming that $\hat{m}$ is constant, which is true if the noise is zero, we write

$$\hat{m}(t) = \frac{\int_0^t m d\tau}{t}, \quad (1.3)$$

We must consider the effects from noise now, and using the same estimator as before, but with the term inside the integral being $m + n(t)$

$$\hat{m}(t) = \frac{\int_0^t m d\tau}{t} + \frac{\int_0^t n(\tau) d\tau}{t}. \quad (1.4)$$

The second term in the right hand side is the time average of the noise, which will converge to the noise's mean if the noise is mean stationary. This term will go to zero, in the stationary case, if and only if the balance tare is set to zero.

Example 2: In this example we show a first-order system with an unknown constant perturbation. We also assume that our measurements are corrupted by additive white gaussian noise (AWGN). This system is described precisely in Equations (1.5). Our goal is to estimate the parameters a and b of the model.

$$\dot{x}(t) = ax(t) + bu(t) + k \quad (1.5a)$$

$$y(t) = x(t) + n(t) \quad (1.5b)$$

In order to design the estimators we ignore the noise and rewrite the first part of Equation (1.5) using operational calculus notation. One can think of it as rewriting the equation in the frequency domain (see Appendix A).

$$sX - x(0) = aX + bU + \frac{k}{s} \quad (1.6)$$

Our goal is to annihilate the initial condition and the constant perturbation. Therefore, we multiply both sides by the operator s ,

$$s^2X - sx(0) = asX + bsU + k, \quad (1.7)$$

and apply the formal differential d , which, for any practical purposes, can be thought as the derivative $\frac{d}{ds}$, and, for this reason, will be used interchangeably in this work.

$$d(s^2X) - x(0) = ad(sX) + bd(sU) \quad (1.8)$$

Hence, it is necessary to differentiate again to make the initial condition disappear.

$$d^2(s^2X) = ad^2(sX) + bd^2(sU) \quad (1.9)$$

Applying the Leibniz rule, we expand the multiple derivatives and arrive at the following:

$$2X + 4sdX + s^2d^2X = a(2dX + sd^2X) + b(2dU + sd^2U) \quad (1.10)$$

The formal derivative, as well as the derivative $\frac{d}{ds}$, corresponds to multiplying the time function by $-t$. We observe that there are terms multiplied by the operator s , therefore, we need to multiply both sides by $\frac{1}{s^2}$ to avoid any time differentiation.

$$2\frac{X}{s^2} + 4\frac{dX}{s} + d^2X = a\left(2\frac{dX}{s^2} + \frac{d^2X}{s}\right) + b\left(2\frac{dU}{s^2} + \frac{d^2U}{s}\right) \quad (1.11)$$

Renaming the term that multiplies the parameter a by $p_{11} = 2\frac{dX}{s^2} + \frac{d^2X}{s}$, the term that multiplies the parameter b by $p_{12} = 2\frac{dU}{s^2} + \frac{d^2U}{s}$, and the term in the left hand side by q_1 , we arrive

at:

$$q_1(t) = ap_{11}(t) + bp_{12}(t) \quad (1.12)$$

Now, to achieve a system with two unknowns and two equations we must integrate this equation, so that

$$\begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} \quad (1.13)$$

where, $\dot{p}_{2j} = p_{1j}$, for $j = 1, 2$, and $\dot{q}_2 = q_1$.

One should notice that q_1 , p_{11} , and p_{12} can be calculated as the output of linear time-varying filters described in the next equations.

$$\dot{\pi}_1^1(t) = -2tx \quad (1.14a)$$

$$\dot{\pi}_2^1(t) = t^2x + \pi_1^1 \quad (1.14b)$$

$$p_{11}(t) = \pi_2^1(t) \quad (1.14c)$$

$$\dot{p}_{21}(t) = p_{11}(t) \quad (1.14d)$$

$$\dot{\pi}_1^2(t) = -2tu \quad (1.15a)$$

$$\dot{\pi}_2^2(t) = t^2u + \pi_1^2 \quad (1.15b)$$

$$p_{12}(t) = \pi_2^2(t) \quad (1.15c)$$

$$\dot{p}_{22}(t) = p_{12}(t) \quad (1.15d)$$

$$\dot{\eta}_1(t) = 2x \quad (1.16a)$$

$$\dot{\eta}_2(t) = -2tx + \eta_1 \quad (1.16b)$$

$$\dot{\eta}_3(t) = t^2x + \eta_2 \quad (1.16c)$$

$$q_1(t) = \eta_3(t) \quad (1.16d)$$

$$\dot{q}_2(t) = q_1(t) \quad (1.16e)$$

With these equations it is simple to write a code that would calculate the parameters. One should bear in mind that the matrix in the left hand side of Equation (1.13) might not be invertible. Consequently, we say that we loose identifiability. More than that, it is easy to check that if the input is a step, there is loss of identifiability for every t , simply looking at Equations (1.15).

Fortunately, one can choose easily an input u , such that the identifiability is ensured for almost all $t > 0$ ((FLIESS; SIRA-RAMÍREZ, 2007)). However, in practice there can be numerical issues, for instance the matrix might be ill-conditioned. Therefore, any practical implementation might take these issues into account. For instance, for any system, the algebraic estimator is ill-conditioned for small t , since the matrix is ill-conditioned. Hence, we always plot the identified value only after a short interval, chosen by the user. Also, one can keep track of the moments when the matrix is ill-conditioned, through the condition number, and discard values for which the matrix can be considered ill-conditioned.

We implemented a numerical example for this system where the parameters are $a = -2$, $b = 3$, $k = 10$. The AWGN has variance $\sigma^2 = 10^{-8}$. The applied input is $u = \sin(t) + \frac{2}{s+1}pwm(t)$, where $pwm(t) = 1$ for $t \in [k, 0.7 + k]$, $k \in \mathbb{N}$, and $pwm(t) = 0$ otherwise. Therefore, a 70% of the PWM period as 1. The results are shown in Figure 1, where the dashed lines correspond to the parameters true values, while the solid lines correspond to estimated parameters. Also, we use the subscript e to indicate that those are estimated values.

In Figure 1 we notice a spike in the time interval $[0.7, 0.8]$, that corresponds to the loss of identifiability phenomenon aforementioned. One should notice that it occurred in an isolated point. Also, one should note that at the begining, the estimated values are far from the true ones, that is due to the ill-conditioning of the matrix and the effect of noise. In general, the noise might cause spikes, however it is hard to predict. We did not plot the estimated values for

Figure 1 - Results of the identification procedure

Source: Author

$t < 0.2$, since the graph explodes and its results are meaningless.

Example 3: This example is a second-order linear system without perturbations. The purpose of this example is to illustrate the normal functioning of an algebraic estimator for an unperturbed system. Being precise, we consider the system of Equation (1.17), where $a = -5$, $b = -7$ and $c = 3$. The identification input was a step $u = 1$, which shows that the identifiability conditions are not the same for algebraic estimators, with relation to previous estimators which depended on persistent excitation hypothesis (ÅSTRÖM; WITTENMARK, 2013). We can see in Figure 2 that there exists a negligible mismatch between the true values and the estimated ones, which shouldn't exist in the ideal case, however numerical approximations cause those discrepancies (FLIESS; JOIN; RAMIREZ, 2008).

$$\ddot{x} = a\dot{x} + bx + cu \quad (1.17)$$

Example 4 (Derivative Estimation): In this example we show how to estimate the derivative of a function through algebraic estimators. The goal is to approximate the measured function by its Taylor polynomial of a given order and estimate its parameters. The better the approximation, the better the result. Here, we consider a system that is described exactly by a second order polynomial, to illustrate the functioning of these estimators.

Consider a body undergoing a vertical free fall from an initial position $y(0)$, and an initial

Figure 2 - Results of the identification procedure

Source: Author

velocity $\dot{y}(0)$. Our goal is to estimate $\dot{y}(t)$. To do that, first we describe the equation that governs this motion

$$y(t) = y(0) + \dot{y}(0)t + \frac{g}{2}t^2, \quad (1.18)$$

then we rewrite the equation using operational calculus,

$$Y = \frac{y(0)}{s} + \frac{\dot{y}(0)}{s^2} + \frac{g}{2s^3}. \quad (1.19)$$

In the sequel we annihilate the term where the acceleration appears, by multiplying both sides by s^3 , and then applying the formal derivative

$$d(s^3 Y) = y(0)d(s^2) + \dot{y}(0)d(s) = 2y(0)s + \dot{y}(0). \quad (1.20)$$

Finally, we divide by s and apply the formal derivative to isolate the term $\dot{y}(0)$, arriving at

$$d\left(s^{-1}d\left(s^3Y\right)\right) = -\frac{\dot{y}(0)}{s^2}. \quad (1.21)$$

Algebraically, multiplying both sides by s^{-3} , changes nothing. Nonetheless, this multiplication affects the estimation when noise is present. Since s^{-3} corresponds to the triple integration operator it can be interpreted as a low-pass filter, which will attenuate high frequency measurement noise. One should notice that a DC noise corresponds to an ill-tuned sensor, which has a bias.

$$s^{-3}d\left(s^{-1}d\left(s^3Y\right)\right) = -\frac{\dot{y}(0)}{s^5}. \quad (1.22)$$

Since $\dot{y}(0)$ is a constant, $\frac{1}{s^5}$ is the operational representative of the time function $\frac{t^4}{4!}$, or, for those who prefer, the Laplace transform of $\frac{t^4}{4!}$. Therefore, Equation (1.22) can be written explicitly as the following formula

$$\dot{y}(0) = -\frac{12}{t^4} \int_0^t (3t^2 - 16t\tau + 15\tau^2)y(\tau)d\tau \quad (1.23)$$

One should notice that the initial moment is arbitrary. Therefore, one could estimate $\dot{y}(1)$ at instant $t = 1$, applying the same formula for $\dot{y}(1)$, only by considering $t = 1$, as the new instant $t = 0$. Hence, if one resets the integrator in the right hand sided from time to time, then we would have a good estimate for the velocity during the entire movement.

To emulate the resetting effect, we can implement the integration with a sliding window. The larger the integrating window, the better the filtering effect. However, large windows tend to enlarge the gap between the estimated value and the true one. Therefore, we need to adjust the window's size to every specific application.

To implement this estimator numerically it is necessary to approximate the integral in some sense. In this thesis we have chosen to use a trapezoidal numerical integration (ZEHETNER; REGER; HORN, 2007).

In this simulation we considered the initial parameters $y(0) = 200$, $\dot{y}(0) = 5$ and $g = 10$. The measurements are corrupted by an AWGN with variance $\sigma^2 = 10^{-4}$. It is important to mention that at the beginning, the estimated values are of little to no importance, since the term $\frac{1}{t^4}$, makes

our calculations ill-conditioned, and this is the reason why there is a spike at the beginning of the estimation in Figure 3.

Figure 3 - Results of the estimation procedure

Source: Author

2 MAIN RESULTS

2.1 System Model

The valve structure is presented in Figure 4, where one can see the presence of a spring, with elastic coefficient k_s , that moves the stem upwards, the diaphragm where the pressure u is applied, the gasket that avoids leakings, and where the friction occurs.

Figure 4 - Valve schematics

Source: (ROMANO, 2010)

We will base our calculations on the Classical physical model for stiction (KARNOPP, 1985) with the Stribeck correction term. This system modelling is deduced from force-balance equations based on the Newton's second law. In addition, we also discarded negligible terms (JELALI; HUANG, 2012), therefore arriving at Equation (2.1) when the stem is moving.

$$m\ddot{x}(t) = -F_v\dot{x}(t) - k_s x(t) - \left(F_c - (F_s - F_c) e^{-(\dot{x}(t)/v_s)^2} \right) \text{sign}(\dot{x}(t)) + Au(t), \quad (2.1)$$

where m is the mass of the moving parts (stem and plug), x is the stem position, F_v is the viscous friction coefficient, F_s is the static friction coefficient, F_c is the Coulomb friction coefficient, v_s is the Stribeck velocity, k_s is the Hooke's law constant of the spring, A is the diaphragm area and u is the air pressure applied. When the stem is stopped ($\dot{x}=0$), using the

notation $F_e = Au - k_s x$ we have two possible situations:

$$m\ddot{x}(t) = \begin{cases} 0 & \text{if } |F_e(t)| \leq F_s, \\ -k_s x(t) + Au(t) + F_s \text{sign}(F_e(t)) & \text{otherwise.} \end{cases} \quad (2.2)$$

In the simulations in Section 3.1 we consider the usual Karnopp model, with dead zone for the velocity as described in (GARCIA, 2008). The second condition refers to the valve in the eminence of movement. It should be emphasized that this part of the model does not interfere in the dynamics when the valve is in movement ($\dot{x} \neq 0$).

Aiming to simplify notation and for identifiability purposes, that will be explained at Subsection 2.2.2, we will define $a = -\frac{F_v}{m}$, $b = -\frac{k_s}{m}$, $c = \frac{A}{m}$, $k = -\frac{F_c}{m}$, and $\gamma = \frac{F_s - F_c}{m}$.

$$\ddot{x}(t) = a\dot{x}(t) + bx(t) + \left(k + \gamma e^{-(\dot{x}(t)/v_s)^2}\right) \text{sign}(\dot{x}(t)) + cu(t) \quad (2.3)$$

It should be remarked that, at high velocities, $|\dot{x}| \gg v_s$, the Stribeck term vanishes for all practical purposes. Consequently the system model can be well approximated by Equation (2.4). More than that, this approximation is widely used in the literature ((GARCIA, 2008; ROMANO; GARCIA, 2008)), since the Stribeck effect is often negligible.

$$\ddot{x}(t) = a\dot{x}(t) + bx(t) + k \text{sign}(\dot{x}(t)) + cu(t) \quad (2.4)$$

2.2 Online Algebraic Identification and Estimation

2.2.1 Identification of the Model Parameters

In this section we will assume that the high velocity condition is satisfied, also that $\dot{x}$ is positive, without loss of generality, in order to make the *sign* function disappear from the left hand side in Equation (2.4). One can then rewrite the approximate model using Yosida's operational calculus ((YOSIDA, 2012)) notation, hence obtaining Equation (2.5).

$$s^2 X - sx(0) - \dot{x}(0) = asX - ax(0) + bX + k\frac{1}{s} + cU \quad (2.5)$$

Multiplying both sides by s and differentiating with respect to s three times in order to

cancel the effects of the constant disturbance and the effects of the initial conditions, we arrive at Equation (2.6):

$$\frac{d^3(s^3X)}{ds^3} = a\frac{d^3(s^2X)}{ds^3} + b\frac{d^3(sX)}{ds^3} + c\frac{d^3(sU)}{ds^3} \quad (2.6)$$

At last, we multiply both sides by s^{-4} in order to avoid any time derivatives and filter the signals involved.

$$\begin{aligned} \frac{1}{s} \frac{d^3X}{ds^3} + \frac{9}{s^2} \frac{d^2X}{ds^2} + \frac{18}{s^3} \frac{dX}{ds} + \frac{6}{s^4} X &= a \left(\frac{1}{s^2} \frac{d^3X}{ds^3} + \frac{6}{s^3} \frac{d^2X}{ds^2} + \frac{6}{s^4} \frac{dX}{ds} \right) \\ &+ b \left(\frac{1}{s^3} \frac{d^3X}{ds^3} + \frac{1}{s^4} \frac{d^2X}{ds^2} \right) + c \left(\frac{1}{s^3} \frac{d^3U}{ds^3} + \frac{1}{s^4} \frac{d^2U}{ds^2} \right) \end{aligned} \quad (2.7)$$

We now can rewrite Equation (2.7) in time domain as:

$$q_1(t) = ap_{11}(t) + bp_{12}(t) + cp_{13}(t) \quad (2.8)$$

where:

$$\begin{aligned} q_1(t) &= \int_0^t (-\sigma_1^3 x(\sigma_1)) d\sigma_1 + 9 \int_0^t \int_0^{\sigma_1} (\sigma_2^2 x(\sigma_2)) d\sigma_2 d\sigma_1 + 18 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} (-\sigma_3 x(\sigma_3)) d\sigma_3 d\sigma_2 d\sigma_1 \\ &+ 6 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} \int_0^{\sigma_3} (x(\sigma_4)) d\sigma_4 d\sigma_3 d\sigma_2 d\sigma_1 \end{aligned} \quad (2.9)$$

$$\begin{aligned} p_{11}(t) &= \int_0^t \int_0^{\sigma_1} (-\sigma_2^3 x(\sigma_2)) d\sigma_2 d\sigma_1 + 6 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} (-\sigma_3^2 x(\sigma_3)) d\sigma_3 d\sigma_2 d\sigma_1 \\ &+ 6 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} \int_0^{\sigma_3} (-\sigma_4 x(\sigma_4)) d\sigma_4 d\sigma_3 d\sigma_2 d\sigma_1 \end{aligned} \quad (2.10)$$

$$\begin{aligned} p_{12}(t) &= \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} (-\sigma_3^3 x(\sigma_3)) d\sigma_3 d\sigma_2 d\sigma_1 + 3 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} \int_0^{\sigma_3} (\sigma_4^2 x(\sigma_4)) d\sigma_4 d\sigma_3 d\sigma_2 d\sigma_1 \end{aligned} \quad (2.11)$$

$$p_{13}(t) = \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} (-\sigma_3^3 u(\sigma_3)) d\sigma_3 d\sigma_2 d\sigma_1 + 3 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} \int_0^{\sigma_3} (\sigma_4^2 u(\sigma_4)) d\sigma_4 d\sigma_3 d\sigma_2 d\sigma_1 \quad (2.12)$$

Finally, integrating Equation (2.8) once and twice, we reach the following system of equations:

$$\begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} \quad (2.13)$$

where:

$$p_{21}(t) = \int_0^t p_{11}(\sigma_1) d\sigma_1; \quad p_{31}(t) = \int_0^t p_{21}(\sigma_1) d\sigma_1 \quad (2.14)$$

$$p_{22}(t) = \int_0^t p_{12}(\sigma_1) d\sigma_1; \quad p_{32}(t) = \int_0^t p_{22}(\sigma_1) d\sigma_1 \quad (2.15)$$

$$p_{23}(t) = \int_0^t p_{13}(\sigma_1) d\sigma_1; \quad p_{33}(t) = \int_0^t p_{23}(\sigma_1) d\sigma_1 \quad (2.16)$$

2.2.2 Identifiability

In this Subsection we address the identifiability problem for the algebraic identification procedure presented in the last subsection. To ensure that our model is identifiable, we need to impose conditions in the input signal, such that the determinant of the matrix on the left hand side in Equation (2.13) is different from zero. Fortunately, as proved in (FLIESS; SIRA-RAMÍREZ, 2003), almost every input signal will work. However, a few remarks must be made, because a constant input, which would correspond to setpoint regulation, does not provide the necessary conditions.

To see that a constant input in fact does not verify those conditions one just have to look at Equation (2.6) and notice that if $u(t)$ was constant, then the term that multiplies the parameter c

would be equal to zero. Therefore the term $p_{13}(t)$ would also be zero, which implies our claim. An input that circumvents this difficulty is proposed in Section 2.3.

Furthermore, we defined the constants a, b, c, k and γ , in Section 2.1, claiming that it would be important for the identifiability. This is the case, because if we had not done so, the system of Equation (2.13) would be overparametrized (FLIESS; SIRA-RAMÍREZ, 2007).

2.2.3 Friction Estimation

Friction estimation is achieved by using algebraic derivative estimators described in (ZEHETNER; REGER; HORN, 2007; FLIESS; JOIN; RAMIREZ, 2008). Their main advantages are that they are non-assympotic and that they do not require the knowledge of the model. Hence, they are independent from the algebraic identifier outputs, and can be implemented separately.

One should notice that a function that is n times differentiable has a Taylor polynomial expansion up to order n , and can be approximated by Equation (2.17) (HIRSCH, 1997).

$$x(t) \approx \sum_{i=0}^n x^{(i)}(0) \frac{t^i}{i!} \quad (2.17)$$

Therefore, using an apposite procedure to that of Subsection 2.2.1, one can construct an algebraic estimator for the term that multiplies the i -th power in Equation (2.17). Building this estimator, by its turn, is equivalent to constructing an estimator for $x^{(i)}(0)$. Hence, using this estimate only for a short period of time should bare consistent results. Nonetheless, since these estimators have no dynamics, the instant $t = 0$ can be chosen arbitrarily. Thus these estimators can be reinitialized as often as possible, achieving very good estimates. Also, to shorten the notation we will write $a_i = x^{(i)}(0)$.

We must remark here that these coefficients a_i , have no direct relation to the model's parameters a, b , and c previously defined. This fact gives idependence between the identification proceedure and the friction estimation, allowing us to perform them separately.

Since we want to approximate the first derivative of the stem position, we have to consider that the function x is at least three times differentiable, a fact that is true for every open neighborhood far from the line $\dot{x} = 0$ (see Appendix B). Furthermore, consider the second order Taylor

polynomial of x around zero, which exists by the previous assumption on the differentiability.

$$x(t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} \quad (2.18)$$

Our derivative estimate is the term a_1 . In order to obtain it, we perform a similar procedure as in Subsection 2.2.1. Hence we apply the operator $s^{-3} \frac{d}{ds} s^{-1} \frac{d}{ds} s^3$ to obtain the following derivative estimator:

$$a_1 = -\frac{12}{t^4} \int_0^t (3t^2 - 16t\tau + 15\tau^2)x(\tau)d\tau \quad (2.19)$$

For the second order derivative estimator we will consider the third order Taylor polynomial expansion around zero.

$$x(t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} \quad (2.20)$$

This time we will apply the operator $s^{-4} \frac{d^2}{ds^2} s^{-1} \frac{d}{ds} s^4$ in order to estimate a_2 . After these calculations we end up with the following estimator.

$$a_2 = \frac{120}{t^6} \int_0^t (4t^3 - 45t^2\tau + 108t\tau^2 - 70\tau^3)x(\tau)d\tau \quad (2.21)$$

It is then straightforward to derive a friction estimator from Eq. (2.4). For the validity region of the approximation, one can isolate the friction term and use both the parameters identified and the first and second order derivative estimates in order to estimate the friction. Considering that $\dot{x}(t) > 0$, then we arrive at the following estimator:

$$k_e = \ddot{x}_e(t) + a_e \dot{x}_e(t) + b_e x_e(t) - c_e u(t) \quad (2.22)$$

From this point on, all the parameters with subscript e , such as a_e , b_e , c_e , and k_e , denote parameter estimates. For example a_e represents the estimate for the parameter a . Also, for the real system one has to consider the measurement noise. Therefore, the subscripts are necessary.

It is important to mention that the second order derivative estimate is less robust to measurement noise (MORAES; SILVA, 2015), for this reason its results might be unreliable. However,

for the control law presented in the Section (2.3) we can make the approximation that $\ddot{x}_e = 0$ if needed, due to the fact that the valve dynamics presents high accelerations in a very small window of time, only at the change of reference points, therefore this approximation is not unrealistic.

2.3 Control Law

The control law proposed here does not intend to solve the problem of controlling the flow of fluids going through the valve, but just the problem of controlling the stem position. Therefore, we consider that our system is part of a larger control system, which involve an external controller for the flow, as depicted in Figure 5.

Figure 5 - Block Diagram for the External Control System

Source: Author

We can interpret the pneumatic valve block in Figure 5 as an actuator for that block diagram. In Figure 6 we depict the valve control system block in detail.

It can be seen in Figure 6 several signals, where θ_e represents the estimated parameters (a_e , b_e , c_e , and k_e), r is the reference signal that we wish to track (the desired stem position), u is the control input (the applied pressure), y is the system output (stem position) and the η_i , $i = 1, 2, 3$ variables are external binary signals (flags) sent by the supervisory system in order to switch the identifier on and off.

The supervisory system structure is flexible. It can be simply a clock that switches the system from a normal controlled functioning, to an identification procedure from time to time, or something more elaborated as a logic-based switching that depends on the variability of the flow that passes through the valve. Hence, the parameters η are basically binary inputs that make

Figure 6 - Block Diagram of the Pneumatic Valve's Control System

Source: Author

the system switch from an identification state, to a controlled state. Moreover, it is important to mention that all η_i are synchronized, in the sense that they always switch from the controlled state to the identification state whenever one of them switches, and vice versa.

The proposed control law makes the following reasonable assumptions: (i) the reference signal r sent by the input generator is kept constant during the system's transient regime (piecewise constant input), (ii) the identification procedure is hold on for a time window large enough to ensure the convergence of the identifiers (FLIESS; SIRA-RAMÍREZ, 2007), (iii) the Karnopp's model parameters are such that the linear part is overdamped. (iv) The gap between two consecutive reference values is such that $|u| > \frac{F_s + k_s x_1(t_0)}{A}$.

The first assumption is ensured by the fact that the valve's dynamics is considerably faster than the flows, while the second is ensured by the design of the supervisory system and can be easily fulfilled. The third claim is based on data available in the usual values for those parameters. The final assumption, $|u| > \frac{F_s + k_s x_1(t_0)}{A}$, is only a requirement to ensure that our system will start moving, in most of the real cases it will be met, however this assumption cannot be dropped. We present an alternative to circumvent the necessity of assumption(iv) in the Appendix B.

2.3.1 Control Logic

The control logic is divided into 4 different states. It involves both an identification phase and controlled phases. Denote by Δr the difference between the last value of r and the current value, keeping this difference until the next change of reference happens. The binary variable

η represents the external signals η_i , sent by the supervisory system in order to start or stop the identification.

The reason to define Δr is that if its value is positive (resp. negative), then the velocity of the stem will be positive (resp. negative) if we depart from the transient regime (at which the stem will be practically at rest). This will occur due to our hypothesis that the system is overdamped. In this situation, the stem will monotonically approach the desired value r either monotonically increasing, or decreasing, depending exclusively on the previous value of r , which justifies the Δr definition. It is important to mention that if the system is not overdamped one can find a state feedback, which depends only on the measured output and the estimated velocity, that ensures the hypothesis (iii).

Another remark on the definition of Δr , is that one could change the definition of Δr to be the difference of the last value of y and the current value of r , with no change in the functioning of the system in the ideal case. Nonetheless, if there is noise the last definition is less robust than the former. On the other hand, as a consequence of Theorem 1, the first definition is not suitable, even in the ideal case, if $\Delta r < \gamma$, but since γ is very small it will not obey assumption (iv), therefore we will use Δr as defined in the previous paragraph.

The monotonic convergence of the stem position means that the velocity does not change signs, then one can ignore the term $\text{sign}(\dot{x})$, in Equation (2.3) and work with a constant term. This fact helps to cancel the friction, once that its sign is known.

In the sequel we present the description of the state machine presented in figure (7).

- In state S_0 , the system is started, the purposes of this state are to allow the practitioner to start the process with any desired input and to avoid any indeterminations in the value of Δr .
- In state S_1 , which is triggered by $\Delta r > 0$ and $\eta \neq 0$, we apply the control law given by:

$$u(t) = \frac{-r(t) - b_e k_e}{c_e} \quad (2.23)$$

- In state S_2 , which is triggered if $\Delta r < 0$ and $\eta \neq 0$, we apply the control law given by:

$$u(t) = \frac{-r(t) + b_e k_e}{c_e} \quad (2.24)$$

- In state S_3 , which is triggered if $\eta = 0$, is the state where the identification happens. Firstly, we must bring the valve to the lowest position, and then we apply an input that ensures identifiability, in our case we have chosen

$$r(t) = \alpha 1(t) + \beta \sin(\omega t), \quad (2.25)$$

where α is a constant greater than F_s/A , $1(t)$ is the step function, $\beta > 0$ is a small parameter chosen by the practitioner and ω is a small positive number to avoid high frequency oscillations. The reason for the sinusoidal term is only to circumvent the loss of identifiability pointed out in Subsection 2.2.2, which would occur if we only applied the step function and the motive for $\alpha > F_s/A$ is that it ensures that the system will start moving.

We bring the valve stem to the lowest position just to ensure that we will not start the identification phase close to the highest position. Moreover, we chose the lowest position to avoid arbitrariness on the choice of the initial position in state S_3 and another position might be used if desirable for practical or security purposes.

One should notice that the parameter α from the identification phase can be chosen, in some cases, as an educated guess to approximate the control u from the controlled phase, reducing the loss in performance during the identification phase. All these rules can be synthesized in Figure 7. Also, one should remark that the typical poles for the uncontrolled system are usually fast and overdamped, hence there is no need for their replacement.

Figure 7 - State Machine for the Control Law

Source: Author

2.3.2 Stability and Performance

In this section we will only consider the stability and performance of the system while in the states S_1 and S_2 , since they are the states at which the controller is actually trying to solve the output regulation problem. Also in the remaining of this subsection we will consider that we are at S_1 , since for S_2 is analogous. Therefore, we will make the following change of coordinates in Eq. (2.1), $y = x - r$, and assume that r is a piecewise constant function, which is what one should expect in practice. Then $\dot{y} = \dot{x}$.

The rational of this choice comes from the following development. By the certainty equivalence principle, analyzing the equilibrium point of Eq. (2.1) with the control law described, we get

$$\ddot{x}_{eq} = a\dot{x}_{eq} + bx_{eq} + c(-rb_e - \text{sign}(\Delta r)k_e)/c_e = 0, \quad (2.26)$$

$$\dot{x}_{eq} = 0_+, \quad (2.27)$$

where, the $\dot{x}_{eq} = 0_+$ is interpreted as $\lim_{x \rightarrow 0_+}$. It was done only to avoid the discontinuity that our model possesses at the velocity zero. We finally arrive at

$$x_{eq} = r \quad (2.28)$$

In Appendix B we show in Lemma 3 that the position of the valve is a monotonic function if we start the movement from rest, that means that either the valve position increases or decreases monotonically, hence there are no oscillations. Furthermore, we show that the signal of the velocity remains the same, showing that this assumption holds.

Under the hypothesis that we have correctly identified the parameters, we showed in Theorem 1 the error achieved in steady state by our control system is smaller than or equal to the parameter γ in Eq. (2.3), in the Stribeck correction term. If the parameters are not correct, then one can still give an upper bound for the steady state error through the inequality given in Theorem 1. Finally we make a few remarks on the problem when assumption (iv) is dropped in Appendix B.

3 NUMERICAL RESULTS

3.1 Computational Procedure

In this chapter we considered the parameters values as being $m = 1.3608\text{kg}$, $A = 0.0645\text{ m}^2$, $F_c = 1.4234 \cdot 10^3\text{N}$, $F_v = 612.944\text{N}$, $F_s = 284.6862\text{N}$, $k_s = 5.2538 \cdot 10^4\text{N}$, $v_s = 2.54 \cdot 10^{-4}\text{m/s}$. Also, the output value was converted from meters to p.u.. Furthermore, the dead zone for $|\dot{x}|$, from the Karnopp model, was chosen to be $0.6v_s$, as suggested by ((GARCIA, 2008)). These parameter values were obtained previously through experiments with the pneumatic control valve at the Laboratory of Automation and Control from the Escola Politécnica of the University of São Paulo in the dissertation ((CARVAJAL, 2015)).

3.1.1 Sinusoidal Noise

3.1.1.1 Identifier

In order to explain how the identifier calculations were performed in the simulations, we will exemplify how to implement them through the calculation of the term $p_{12}(t)$ in Eq.(2.13). Because the other terms are computed in a similar fashion, we omit their expressions in this text and refer to the article ((FLIESS; SIRA-RAMÍREZ, 2007)) and to the Chapter 1 for further clarifications.

$$\dot{\pi}_1(t) = 3t^2x \quad (3.1)$$

$$\dot{\pi}_2(t) = -t^3x + \pi_1 \quad (3.2)$$

$$\dot{\pi}_3(t) = \pi_2(t) \quad (3.3)$$

$$\dot{\pi}_4(t) = \pi_3(t) \quad (3.4)$$

$$p_{12}(t) = \pi_4(t) \quad (3.5)$$

Basically, we rewrote the integrals as a time-varying linear system realization, where the input is x , the states are π_i , with $i = 1, \dots, 4$, and p_{12} is the output.

The simulations were performed using the model from Eq. (2.1), to account for the Stribeck

effect. Also, the output of the process was affected by an additive high frequency noise modeled by a sinusoidal function with frequency 10^4 Hz, and amplitude 0.01 p.u.. All the simulations used fixed steps of size $T_s = 10^{-5}$ with the solver running the Dorman-Prince method (edo8) in Matlab. The system remains in the identification state S_3 for the first 0.8s and afterwards, it moves to state S_1 . The signal applied during the identification state was chosen to be $u = 1.92\sin(t) + 6.4$. One should also notice that the graphs only begin after a brief time, in our case 0.4 s, due to numerical issues ((FLIESS; JOIN; RAMIREZ, 2008)).

Figure 8 - Comparison between a_e and a

Source: Author

As can be seen from Figures 8, 9, and 10, the identified parameters are good estimates of the true ones. Any mismatch between the real values and the estimated ones cannot be noticed in the pictures.

3.1.1.2 Friction Estimation

To implement the derivative estimators we proceed in a similar fashion as in the paper ((ZEHETNER; REGER; HORN, 2007)). Hence, we approximate the integral by a trapezoidal rule and generate a time-varying FIR filter of length N , with input x . In order to implement this filter, we must keep the last N samples of the stem position x stored.

The length of the filters used in the simulations was 500 each and the simulations were performed with fixed step size of $T_s = 10^{-5}$, and using the Dorman-Prince method (edo8).

Figure 9 - Comparison between b_e and b

Source: Author

Figure 10 - Comparison between c_e and c

Source: Author

Furthermore, the output of the process was affected by an additive high frequency noise modeled by a sinusoidal function with frequency 10^5 Hz, and amplitude 0.01 p.u.. The system performs the friction estimation while in state S_3 , which occurs at the first 0.8s. One should notice that we keep the final value of k_e as our estimate, but some other option could have been made. Also, experiments show that choosing any other value inside the blue region from 0.4s to 0.8s would give similar results.

From Figure 12 and Figure 11 we observe that the first derivative estimator performed well, however the second order derivative performance, although satisfactory for controlling the system, is very noisy. We presented only the results for the first 0.1s, since the remainder of the period of 0.8s presents a similar behavior.

Figure 11 - Comparison between the estimated velocity and the true one

Source: Author

Figure 12 - Comparison between the estimated acceleration and the true one

Source: Author

In addition, the estimate for the friction term k_e , was estimated using Eq. (2.22) and it is presented in Figure 13. For this reason, the noise that affects both the velocity estimator and the acceleration diminishes the quality of our estimate for k_e . However, it does not compromise sig-

nificantly the controller performance, as shown in subsection 3.1.1.3. It is important to mention that due to the dynamic of the system we can ignore the second derivative and assume $\ddot{x} = 0$, which is a good choice due to the bad performance of the acceleration estimate. In the following Subsection this approximation is not necessary, it only occurs due to the spectral composition of the sinusoidal signal.

Figure 13 - Comparison between k_e and k

Source: Author

3.1.1.3 System Performance

In this subsection the simulations were performed with fixed step size of $T_s = 10^{-5}$ and using the Dorman-Prince (edo8). We modeled the additive measurement high frequency noise at the output of the process as a sinusoidal function with frequency 10^4 Hz and amplitude 0.01 p.u.. These simulations refer to the states S_1 , which starts at 0.8s and S_2 , which starts at 1.8s. Our goal was to show the controller going through all states in the simulations.

We applied a reference signal r , that we want the output to follow, that was equal to 1 from time 0.8s to 1.8s, and afterwards we changed the reference to 0.5 at time 1.8s onwards. In the described conditions, we achieved a steady state error smaller than 0.01 p.u. between the system's output and the reference we wanted to track. One should notice that our tracking error is smaller than the amplitude of the measurement noise. These results are shown in Figure 14.

Figure 14 - Output Regulation Performance

Source: Author

3.1.2 High Frequency Noise

In this Subsection we present the results for the same simulations as before, but with the high frequency noise modeled by a filtered AWGN instead of a sinusoidal function. Explicitly the white Gaussian noise, with variance $\sigma^2 = 10^{-6}$, is filtered by a first order transfer function given by $\frac{s}{s+1}$ and is introduced additively in the measured output. The choice of this filter was done based on the typical passband of the dynamics of a pneumatic valve, without the knowledge of the actual valve model's parameters.

3.1.2.1 Identifier

Once again, the simulations were performed using the model from Eq. (2.1). In these simulations the output of the process was affected by an additive high frequency noise modeled by a filtered AWGN noise. All the simulations used fixed steps of size $T_s = 10^{-5}$ with the solver running the Dorman-Prince method (edo8) in Matlab. The system remains in the identification state S_3 for the first 0.8s and afterwards, it moves to state S_1 . The signal applied during the identification state was chosen to be $u = 1.92\sin(t) + 6.4$. The same remarks regarding the beginning of the simulation made in the Subsection 3.1.1 hold for the figures in this subsection.

As can be seen from Figures 15, 16, and 17, now there is an evident mismatch between the estimated values and the true one, even though they are small.

Figure 15 - Comparison between a_e and a

Source: Author

Figure 16 - Comparison between b_e and b

Source: Author

3.1.2.2 Friction Estimation

Here we implement the derivative estimators in the same way as in the Subsection 3.1.1

The length of the filters used in the simulations was 500 each and the simulations were performed with fixed step size of $T_s = 10^{-5}$, and using the Dorman-Prince method (edo8). Furthermore, the output of the process was affected by an additive high frequency noise modeled

Figure 17 - Comparison between c_e and c

Source: Author

by a filtered AWGN process with variance $\sigma^2 = 10^{-6}$, with filter represented by the transfer function $\frac{s}{s+1}$. The system performs the friction estimation while in state S_3 , which occurs at the first 0.8s. Once more, we keep the final value of k_e as our estimate.

From Figure 19 and Figure 18, we observe that the first derivative estimator is noisier while the second order derivative estimate presents a better performance than in Subsection 3.1.1. Also, there exists a sharp increase in the acceleration near $t = 1.6$, that happens due to the dead zone in Karnopp model. That spike does not reflect the acceleration that a real valve would suffer in this conditions and its only caused by our model choice. Nevertheless, this fact does not affect our simulations since it occurs at an isolated point. Furthermore, it is important to bare in mind that the estimates follow the shape of the true velocity and acceleration for most of the time respectively.

Since the estimate for the friction term k_e , was estimated using Eq. (2.22) it also will be noisier than before. Also, due to the mismatch in the indentified values the friction estimate will stay farther than in the sinusoidal case. These phenomena can be seen in Figure 20.

3.1.2.3 System Performance

In this subsection the simulations were performed with fixed step size of $T_s = 10^{-5}$ and using the Dorman-Prince (edo8). We modeled the additive measurement high frequency noise

Figure 18 - Comparison between the estimated velocity and the true one

Source: Author

Figure 19 - Comparison between the estimated acceleration and the true one

Source: Author

at the output of the process as a filtered AWGN noise with variance $\sigma^2 = 10^{-6}$ and it is filtered by a filter which transfer function is $\frac{s}{s+1}$. These simulations refer to the states S_1 , which starts at 0.8s and S_2 , which starts at 1.8s. Our goal was to show the controller going through all states in the simulations. Just as before, we applied a reference signal r , that we want the output to follow, that was equal to 1 from time 1.6s to 1.8s, and afterwards we changed the reference to 0.5 at time 1.8s onwards. Surprisingly, even in these conditions, we also achieved a steady state

Figure 20 - Comparison between k_e and k

Source: Author

error smaller than 0.01 p.u. between the system's output and the reference we wanted to track, as can be seen in Figure 21. It is important to notice that even when there is a mismatch between the identified values and the true ones, the controller can show a very satisfactory performance. This is the case as predicted in Theorem 1 in the Appendix B.

Figure 21 - Output Regulation Performance

Source: Author

4 CONCLUSION

In this work we addressed the problem of controlling a pneumatic valve with friction using algebraic identifiers and estimators for the system's parameters and a term that is related to the Coulomb friction. We obtained the equations for the identifiers, as well as explained how to implement them. Also, we described the procedure to obtain the derivative estimators and presented proofs for their proper working.

Moreover, we designed a control logic that ensures monotonic convergence of the valve's stem position, avoiding oscillations that may accelerate the valve's aging. More than that, we provided a bound for the error that we might have between the output and the reference even if possible mistakes occurred during the identification.

Furthermore, we simulated the pneumatic valve under the influence of the controller and the parameter estimates. Both the identifiers and the friction estimator performed well, even in the presence of high frequency noise, although the former had a much better performance in comparison to the later. Moreover, the controlled system's performance obtained was considerably more than satisfactory for the noise conditions imposed.

As future research directions, investigations will be held with the goal of finding manners to better understand the robustness of our estimators and to find ways to reduce the sampling frequency necessary for those estimators to work properly. Also, we wish to find manners to reduce the noise in the estimate k_e , aiming to improve our result.

REFERENCES

- ARTIN, M. **Algebra**. [S.l.]: Pearson Prentice Hall, 2011. ISBN 9780132413770.
- ÅSTRÖM, K.; WITTENMARK, B. **Adaptive Control**. 2. ed. [S.l.]: Dover Publications, 2013. (Dover Books on Electrical Engineering). ISBN 9780486319148.
- BARREIRA, L.; VALLS, C. **Ordinary Differential Equations: qualitative theory**. [S.l.]: American Mathematical Society, 2012. (Graduate studies in mathematics). ISBN 9780821887493.
- BIALKOWSKI, W. L. Dreams vs. reality: a view from both sides of the gap. In: **Control Systems**. Whistler, BC, Canada: [s.n.], 1992. p. 283–294.
- CARVAJAL, R. D. A. **Avaliação da eficiência de compensadores de atrito implementados em um sistema digital de controle distribuído**. 2015. 128p.
- DESBOROUGH, P. N. L.; MILLER, R. Process out of control. **InTech-International Journal for Measurement Control**, v. 48, n. 8, p. 52–56, 2001.
- DIAO, S. et al. A differential algebraic approach for position/speed estimation in PMSM. In: **INTERNATIONAL ELECTRIC MACHINES & DRIVES CONFERENCE**, 2013. **Proceedings...** [S.l.]: IEEE, 2013.
- EVANS, L.; SOCIETY, A. M. **Partial Differential Equations**. [S.l.]: American Mathematical Society, 1998. (Graduate studies in mathematics). ISBN 9780821807729.
- FLIESS, M.; JOIN, C.; RAMIREZ, H. S. Non-linear estimation is easy. **International Journal of Modelling, Identification and Control**, Inderscience Publishers, v. 4, n. 1, p. 12, 2008.
- FLIESS, M. et al. Correcteurs proportionnels-intégraux généralisés. **ESAIM: Control, Optimisation and Calculus of Variations**, EDP Sciences, v. 7, p. 23–41, 2002.
- FLIESS, M.; SIRA-RAMÍREZ, H. Closed-loop parametric identification for continuous-time linear systems via new algebraic techniques. In: **Identification of Continuous-time Models from Sampled Data**. [S.l.]: Springer London, 2007. p. 363–391.
- FLIESS, M.; SIRA-RAMÍREZ, H. An algebraic framework for linear identification. **ESAIM: Control, Optimisation and Calculus of Variations**, EDP Sciences, v. 9, p. 151–168, jan 2003.
- GARCIA, C. Comparison of friction models applied to a control valve. **Control Engineering Practice**, Elsevier BV, v. 16, n. 10, p. 1231–1243, oct 2008.
- HAHN, W.; BAARTZ, A. **Stability of motion**. [S.l.]: Springer, 1967. (Grundlehren der mathematischen Wissenschaften).
- HIRSCH, M. **Differential Topology**. [S.l.]: Springer New York, 1997. (Graduate Texts in Mathematics).

- HIRSCH, M.; DEVANEY, R.; SMALE, S. **Differential Equations, Dynamical Systems, and Linear Algebra**. [S.l.]: Elsevier Science, 1974. (Pure and Applied Mathematics). ISBN 9780080873763.
- JELALI, M.; HUANG, B. **Detection and Diagnosis of Stiction in Control Loops: State of the Art and Advanced Methods**. [S.l.]: Springer London, 2012. (Advances in Industrial Control).
- KARNOPP, D. Computer simulation of stick-slip friction in mechanical dynamic systems. **Journal of Dynamic Systems, Measurement, and Control**, ASME International, v. 107, n. 1, p. 100, 1985.
- MORAES, M. S.; SILVA, P. S. P. da. Model-free control of magnetic levitation systems through algebraic derivative estimation. In: ABCM INTERNATIONAL CONGRESS OF MECHANICAL ENGINEERING, 23. **Proceedings...** [S.l.]: ABCM Brazilian Society of Mechanical Sciences and Engineering, 2015.
- OPPENHEIM, A.; WILLSKY, A.; NAWAB, S. **Sinais e sistemas**. [S.l.]: Prentice-Hall, 2010. ISBN 9788576055044.
- ROMANO, R. A. **Identificação de processos não-lineares e quantificação de atrito em válvulas de controle**. Tese (Doutorado), Escola Politécnica da Universidade de São Paulo, São Paulo, 2010. 160p.
- ROMANO, R. A.; GARCIA, C. Karnopp friction model identification for a real control valve. **IFAC Proceedings Volumes**, Elsevier BV, v. 41, n. 2, p. 14906–14911, 2008.
- _____. Valve friction and nonlinear process model closed-loop identification. **Journal of Process Control**, Elsevier BV, v. 21, n. 4, p. 667–677, apr 2011.
- SEBORG, D.; MELLICHAMP, T. **Process Dynamics & Control**. 2. ed. [S.l.]: Wiley India Pvt, 2006.
- SONTAG, E. **Mathematical Control Theory: Deterministic Finite Dimensional Systems**. [S.l.]: Springer New York, 2013. (Texts in Applied Mathematics).
- SONTAG, E. D. Smooth stabilization implies coprime factorization. **IEEE Transactions on Automatic Control**, v. 34, p. 435–443, 1989.
- STRUBLE, R. A. Functions meromorphic in right half-planes are laplace transforms. **Journal of Mathematical Analysis and Applications**, Elsevier BV, v. 70, n. 1, p. 304–307, jul 1979.
- VICINANSA, G. S.; GARCIA, C.; SILVA, P. S. P. da. Control of pneumatic valves with friction using algebraic estimators. In: ABCM INTERNATIONAL CONGRESS OF MECHANICAL ENGINEERING, 24. **Proceedings...** [S.l.]: ABCM Brazilian Society of Mechanical Sciences and Engineering, 2017.
- YOSIDA, K. **Operational Calculus: a theory of hyperfunctions**. [S.l.]: Springer New York, 2012. (Applied Mathematical Sciences).
- ZEHETNER, J.; REGER, J.; HORN, M. A derivative estimation toolbox based on algebraic methods - theory and practice. In: **2007 IEEE INTERNATIONAL CONFERENCE ON CONTROL APPLICATIONS**, 2007. **Proceedings...** [S.l.]: IEEE, 2007.

APPENDIX A - ALGEBRAIC PROPERTIES OF THE ESTIMATORS

A.1 Algebraic Definitions

Definition 1. A ring is a triple $(R, +, *)$, where R is a set, $+$ is a binary operation that is commutative, and $*$ is a binary operation on R . Also, those operations must satisfy the following conditions:

- $(a + b) + c = a + (b + c), \forall a, b, c \in R,$
- $\exists 0 \in R, \text{ such that } a + 0 = a, \forall a \in R,$
- $\forall a \in R, \exists -a \in R, \text{ such that } a + (-a) = 0,$
- $(a * b) * c = a * (b * c), \forall a, b, c \in R,$
- $\exists 1 \in R, \text{ such that } a * 1 = a, \forall a \in R,$
- $a * (b + c) = (a * b) + (a * c), \forall a, b, c \in R,$
- $(b + c) * a = (b * a) + (c * a) \forall a, b, c \in R,$

if it is also true that $a * b = b * a, \forall a, b \in R$, we call the ring R commutative or abelian.

Definition 2. A field is a triple $(F, +, *)$, such that $(F, +, *)$ is an abelian ring and $\forall a \in F \setminus \{0\}, \exists a^{-1} \in F$, such that $a * a^{-1} = 1$.

Examples of rings are:

- The integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, the reals $\mathbb{R}$, complex numbers $\mathbb{C}$ with the usual addition and multiplication.
- Polynomials with coefficients on a ring $R[x]$

- Square matrices with entries in an arbitrary ring $M_n(R)$.
- Real or complex functions with the usual multiplication and addition of functions.

Examples of fields are:

- The rationals $\mathbb{Q}$, the reals $\mathbb{R}$, complex numbers $\mathbb{C}$ with the usual addition and multiplication.
- Rational functions with coefficients on a field R , $R[x]$
- Square invertible matrices with entries in an arbitrary field F , $M_n(F)$.
- Strictly positive real functions with the usual multiplication and addition.

Here we will consider the ring of continuous functions with the usual addition and the convolution product. This is in fact a ring as shown in (YOSIDA, 2012). We also need the definition of a integral domain and the definition of a differential in a ring.

Definition 3. *Let $(R, +, *)$ be a ring. If $a * b = 0$ for some $a, b \in R$, implies $a = 0$ or $b = 0$, then we call this ring an integral domain.*

The name integral domain comes from the integers that form a ring with this property. In fact this property is very useful since it makes it possible to define abstract fractions, i.e. inverses, for the elements of the ring in a simple way. Furthermore, we can construct the smallest field that contains that ring from those fractions and equivalence relations, in a precise sense, and work in that field. We refer to (ARTIN, 2011) for the construction of this field, called fraction field.

One should be familiar with this construction in the case where the ring is $\mathbb{Z}$, and we construct $\mathbb{Q}$ the rationals as the smallest field that contains all of the integers. By a theorem in (YOSIDA, 2012), we know that the ring of continuous functions with the usual addition and the convolution product is an integral domain, therefore it has a field of fractions.

Definition 4. *Let $(R, +, *)$ an integral domain. We call the map $d : R \rightarrow R$ a differential in the ring, if $\forall a, b \in R$*

- $d(a+b)=d(a)+d(b)$,

- $d(a*b)=a*d(b)+d(a)*b,$

where the second relation is often called Liebniz rule.

The differential in the ring we are interested in is the usual mutiplication by $-t$, which resembles the differential of the Laplace transform regarding the frequency variable s . Another definition necessary are that of ring homomorphisms and isomorphisms.

Definition 5. Let $(R_1, +_1, *_1)$ and $(R_2, +_2, *_2)$ be two rings. Denote by 1_{R_1} the identity in the ring R_1 and by 1_{R_2} the identity in the ring R_2 . Then, we define a homomorphism as a map $\phi : (R_1, +_1, *_1) \rightarrow (R_2, +_2, *_2)$, such that, $\forall a, b \in R_1$

-

$$\phi(a +_1 b) = \phi(a) +_2 \phi(b) \quad (\text{A.1})$$

-

$$\phi(a *_1 b) = \phi(a) *_2 \phi(b) \quad (\text{A.2})$$

-

$$\phi(1_{R_1}) = 1_{R_2} \quad (\text{A.3})$$

If, ϕ is surjective and $\phi(a) = 0_2$ if and only if $a = 0_1$, then we say that ϕ is a ring isomorphism.

The role that ring homomorphisms play in algebra resembles the role that linear transformations play in linear algebra. Two rings that are isomorphic are basically the same from the algebraic point of view, hence one can treat two such objects as the same object.

Examples of ring homomorphisms are:

- The inclusion map from $\mathbb{Q}$ to $\mathbb{R}$.
- The evaluation homomorphism from the ring of polynomials with complex coefficients, to $\mathbb{C}$. It takes the abstract polynomial $p \in \mathbb{C}[x]$ and maps it to its value at a , i.e. $\phi_a(p) = p(a)$.
- The map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$, that maps an integer m to the remainder after division by n , i.e., $\phi(m) = m \mod n$.

Finally we define holomorphic and meromorphic functions.

Definition 6. *A holomorphic function is a complex analytic function, i.e. it can be written as a power series. A meromorphic function is a holomorphic function on a domain $\Omega = \mathbb{C} \setminus S$, where S is a discrete set of points.*

Meromorphic functions are important in our study since the Laplace transforms found in practice are meromorphic, for instance $\frac{1}{s+a}$, $\frac{e^{-s\tau}}{s+b}$, etc. It can be shown that every meromorphic function on the half right plane corresponds to a Laplace transform of some function (STRUBLE, 1979).

Our goal is to show that there exists an isomorphism between a subring of continuous functions with usual addition and convolution product, and the image of the Laplace transform of continuous functions, which is a ring.

A.2 Relation with Laplace Transform

The resemblance between operational calculus and the Laplace transform is blatant. In this section we will show that this fact is not a coincidence, in fact it is necessarily true due to their algebraic properties.

Consider the set of all Laplace transformable functions which are continuous with domain in the nonnegative reals, $C_L(\mathbb{R}_+, \mathbb{R})$. Notice that from this subset of $C(\mathbb{R}_+, \mathbb{R})$, one can define the ring $R_L = (C_L(\mathbb{R}_+, \mathbb{R}), *, +)$, where $*$ denotes the convolution product (ring multiplication), and $+$ denotes the usual function sum (ring sum). It is a simple exercise to show that the convolution and the sum of two continuous Laplace transformable functions is again Laplace transformable (OPPENHEIM; WILLISKY; NAWAB, 2010). Therefore, one can define the operational calculus in this ring.

Now it is straight forward to see that the Laplace transform is a ring homomorphism from R_L to a subring of the complex functions ring, $\mathcal{F} = (\mathcal{F}(\mathbb{C}, \mathbb{C}), \cdot, +)$, with $\mathcal{F}(\mathbb{C}, \mathbb{C})$ being the set of all complex functions, $(\cdot)$ the usual pointwise function multiplication, and the sum $+$ as the usual function sum. We denote $\mathcal{F}_L = (F_L(\mathbb{C}, \mathbb{C}), \cdot, +)$, the subring of complex functions with inverse Laplace transform that are continuous. We also denote the subring of meromorphic functions that admit inverse to the Laplace transform by $\mathcal{M}_L = (M_L(\mathbb{C}, \mathbb{C}), \cdot, +)$. Note that $\mathcal{M}_L \subseteq \mathcal{F}_L$. The fact that it is a ring is evident.

We now show that $\mathcal{L}$ is a ring isomorphism. Since the Laplace transform is linear and the convolution's Laplace transform equals the product of the Laplace transforms, we arrive at:

$$\mathcal{L}((f * g)(t)) = \mathcal{F}(s)\mathcal{G}(s), \quad \forall f, g \in R_L$$

$$\mathcal{L}((f + g)(t)) = \mathcal{F}(s) + \mathcal{G}(s), \quad \forall f, g \in R_L$$

Also, the unilateral Laplace transform is injective, because it is invertible. Therefore the Laplace transform is an isomorphism. Hence $\mathcal{F}_L \cong R_L$. In particular, this is true that we can, up to an homomorphism, say that $\mathcal{M}_L \subseteq R_L$.

From those facts, and the fact that most of the signals found in practice are indeed in $C_L(\mathbb{R}_+, \mathbb{R})$, since most of them give rise to meromorphic Laplace transforms, it is correct to use the Laplace transform intuition, because those two rings R_L and $\mathcal{F}_L$ are the same mathematical object from the algebraic point of view.

With this we aim to give a formal justification for the use of the Laplace transform in the algebraic estimators design, making it easier for the practitioner to learn and understand this tool.

APPENDIX B - DYNAMIC PROPERTIES

In this Chapter, we will consider Equations (2.1) and (2.2), after a convenient change of notation, putting $x_1 = x$ and $x_2 = \dot{x}$. In this manner Equation (2.1) become:

$$\dot{x}_1(t) = x_2(t), \quad (\text{B.1a})$$

$$\dot{x}_2(t) = ax_2(t) + bx_1(t) + (k + \gamma e^{-(x_2(t))^2/\nu_s}) + cu(t), \quad (\text{B.1b})$$

and Equation (2.2) is rewritten as:

$$m\dot{x}_2(t) = \begin{cases} 0 & \text{if } |F_e(t)| \leq F_s, \\ -k_s x_1(t) + Au(t) + F_s \text{sign}(F_e(t)) & \text{otherwise.} \end{cases} \quad (\text{B.2})$$

B.1 Validity of the Taylor Expansion

Lemma 1. *Consider the dynamic system determined by Equations (B.1) and (B.2). Also consider $t_0 \in \mathbb{R}$, the initial time, and any initial condition $(x_1(t_0), x_2(t_0)) \in S$, where $S = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$. Furthermore, consider that the function $u : \mathbb{R} \rightarrow \mathbb{R}$ is analytic. Then, the solution of the initial value problem $x(t)$, is an analytic function in some neighborhood of $S \times \{t_0\}$.*

Proof. For $(x_1(t_0), x_2(t_0)) \in S$, the dynamics of the system is given by Equation (B.1). Also $\text{sign}(x_2)$ is constant in a neighborhood of $(x_1(t_0), x_2(t_0))$, since x is continuous. Denoting the

right hand side of Equation (B.1) as

$$F_1(x_1, x_2, t) = x_2(t), \quad (\text{B.3a})$$

$$F_2(x_1, x_2, t) = ax_2(t) + bx_1(t) + (k + \gamma e^{-(x_2(t))^2/v_s}) + cu(t), \quad (\text{B.3b})$$

one can notice that $F(x_1, x_2, t) = (F_1(x_1, x_2, t), F_2(x_1, x_2, t))$ is analytic, therefore by the Cauchy-Kowalevski theorem (EVANS; SOCIETY, 1998), the solution is analytic for some neighborhood of $(x_1(0), x_2(0)) \times 0 \in S \times \{t_0\}$ $\square$

One should notice that this is true for any initial condition in S . However, if one define $t'_0 = t' \in \mathbb{R}$, such that $x(t') \in S$, then it is true that the solution of the initial value problem with initial condition $x(t'_0) = x(t') \in S$ is analytic for some neighborhood of $x(t') \times t'$.

Another usefull result that follows the same lines as the above is Lemma 2, which shows regularity for C^k inputs, if one does not have an analytic input. It follows from a similar result for equations with C^k right hand side (HIRSCH; DEVANEY; SMALE, 1974).

Lemma 2. *Consider the dynamic system determined by Equations (B.1) and (B.2). Also consider $t_0 \in \mathbb{R}$, the initial time, and any initial condition $(x_1(t_0), x_2(t_0)) \in S$, where $S = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$. Furthermore, consider that the function $u : \mathbb{R} \rightarrow \mathbb{R}$, $u \in C^k(\mathbb{R}, \mathbb{R})$. Then, the solution of the initial value problem $x(t)$, is a $C^{k+1}(\mathbb{R}, \mathbb{R}^2)$ function in some neighborhood of $S \times \{t_0\}$*

B.2 Stability and Performance

B.2.1 Input to State Stability (ISS)

Before proving that the control law makes the system stable even under mismatch, we will proof Lemma 3, which implies the desired monotonicity of the stem position, i.e. lack of oscillations.

Lemma 3. *Consider the dynamic system determined by Equations (B.1) and (B.2). Also consider $t_0 \in \mathbb{R}$, the initial time, and any initial condition $(x_1(t_0), x_2(t_0)) \in M$, where $M = \{(x, y) \in \mathbb{R}^2 : y \neq 0, x = 0\}$. Furthermore, consider that the function $u : \mathbb{R} \rightarrow \mathbb{R}$ is constant. Then, $\text{sign}(x_2)$ is positively invariant for the flow and x_1 is an strictly monotonic function.*

Proof. Consider that $\text{sign}(x_2) = 1$, for the other case is analogous. Also consider the change of coordinates in $(z_1, z_2) = (x_1 - \frac{(k-\gamma)+cu}{b}, x_2)$. Hence we arrive at

$$\dot{z}_1(t) = z_2(t), \quad (\text{B.4a})$$

$$\dot{z}_2(t) = az_2(t) + bz_1(t) + \gamma(1 - e^{-(z_2(t))^2/v_s}), \quad (\text{B.4b})$$

such that the origin is an equilibrium point.

Consider then the linear part of Equations (B.4), which can be written as

$$\dot{\bar{z}}_1(t) = \bar{z}_2(t), \quad (\text{B.5a})$$

$$\dot{\bar{z}}_2(t) = a\bar{z}_2(t) + b\bar{z}_1(t), \quad (\text{B.5b})$$

where one can denote the right hand side as $A\bar{z}$, where A is a 2×2 matrix. By hypothesis the eigenvalues of the matrix A are both negative and distinct (overdamped assumption) λ_1 and λ_2 , with corresponding eigenvectors $(1, \lambda_1)'$ and $(1, \lambda_2)'$ in the current coordinate system, respectively. It is known that e^{At} has the same eigenvectors, moreover, the subspaces $\text{span}\{(1, \lambda_i)'\}, i = 1, 2$ are invariant under the flow of (B.5). We need to prove that, even under the presence of the Stribeck term, z_2 won't change sign for any $t > 0$ if the system's initial condition is in M .

We divided the proof in items to make it more clear. The main idea is to show that there exists a solution $\psi(t)$, for initial value problem (B.4), that remains in the second (fourth) quadrant for every $t \in \mathbb{R}$. Therefore, this solution divides the second (fourth) quadrant in two connected components. Then, we show that every solution that has initial condition in M , enters the second (fourth) quadrant. Finally, since the solution $\psi(t)$ divides the second (fourth) quadrant the solution that started in one of the connected components must remain in that connected component for every $t \in \mathbb{R}$, due to the uniqueness theorem applied to autonomous systems. In what follows we will restrict our attention to the second quadrant, since the proof for the fourth quadrant is analogous.

- Step 1: Show that there exists a trajectory that remains in the second quadrant.

Consider the following, $\forall \epsilon > 0$, $z_0 = -\epsilon \{\alpha(1, \lambda_1)' + (1 - \alpha)(1, \lambda_2)'\}$, with $\alpha \in [0, 1]$. We show that the flow $\phi_t(z_0)$ remains in the second quadrant of the coordinate system. Use now the parameter variation formula (BARREIRA; VALLS, 2012).

$$z(t) = -\epsilon \left\{ \alpha e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + (1 - \alpha) e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \right\} + \int_0^t e^{A(t-s)} \begin{bmatrix} 0 \\ g(z(s)) \end{bmatrix} ds, \quad (\text{B.6})$$

where $g(z(s))$ is a function that is nonnegative and $\lim_{z \rightarrow 0} g(z(s))/\|z(s)\| = 0$. In our case $g(z(s)) = \gamma(1 - e^{-(z_2(t))^2/v_s})$, however more general perturbation terms could be used. The first term in Equation (B.6) is always in the second quadrant since the semi-infinite line segments that passes through 0 and have the directions $-(1, \lambda_i)'$, $i = 1, 2$ are in the second quadrant. It suffices now to show that the perturbation term remains in the second quadrant as well.

Consider the relation:

$$\left(\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} - \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \right) \frac{g(x(s))}{\lambda_1 - \lambda_2}, \quad (\text{B.7})$$

also assume without loss of generality that $\lambda_2 < \lambda_1 < 0$. Now we arrive at

$$e^{A(t-s)} \left(\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} - \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \right) \frac{g(x(s))}{\lambda_1 - \lambda_2} = \left(e^{\lambda_1(t-s)} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} - e^{\lambda_2(t-s)} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \right) \frac{g(x(s))}{\lambda_1 - \lambda_2}, \quad (\text{B.8})$$

we now consider the sign of the equations as $t < 0$. From a few verifications using the monotonicity of the exponential we get that $e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)} \frac{g(x(s))}{\lambda_1 - \lambda_2} < 0$ and $\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)} \frac{g(x(s))}{\lambda_1 - \lambda_2} > 0$. Therefore, $\phi_t(z_0)$, the flow induced by Equation (B.4), for $t < 0$ remains in the second quadrant. Since $\epsilon > 0$ was arbitrary we can choose a value for ϵ such that $g(z(s)) < \delta \|z(s)\|$ for some $\delta > 0$ depending on ϵ for which the solution must remain in the second quadrant for $t > 0$ (BARREIRA; VALLS, 2012). Therefore, there exists a solution $\psi(t)$ to Equation (B.4) that is contained in the second quadrant. Note that, the solution is defined for $t < 0$, must go to infinity in a direction inside the second quadrant, hence the curve $\psi(t)$ divides the second quadrant in two components.

- Proof that the solutions that begin at M remain in the II or IV quadrant.

If $z_0 \in M$, and in the second quadrant, then it is easy to see that the vector field generated

by Equation (B.4) points inwards the second quadrant. Hence the solution must enter the second quadrant. However, there exists a nontrivial solution that is entirely in the second quadrant, which implies that in order to any solution that begins in M must intersect this solution $\psi(t)$. Nonetheless, this is impossible by Picard-Lindelöf's uniqueness theorem and the fact that Equation (B.4) is autonomous (BARREIRA; VALLS, 2012). The proof is analogous for the fourth quadrant.

Finally, going back to the original system. Since $x_1(t) = \int_0^t x_2(s)ds$, and $\text{sign}(x_2) = \text{sign}(z_2)$, implies that x_1 is strictly monotonic. $\square$

Now we prove that the error between the desired output and the real one is bounded for any trajectory that starts in M . Moreover, we give an asymptotic estimate for the error. Before doing so we need to define the classes $\mathcal{K}_\infty$ and $\mathcal{KL}$ (HAHN; BAARTZ, 1967).

Definition 7 (Class $\mathcal{K}_\infty$). *A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a class $\mathcal{K}_\infty$ function if $\alpha(0) = 0$, and α is strictly increasing, and α is unbounded.*

Definition 8 (Class $\mathcal{KL}$). *A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a class $\mathcal{KL}$ function if for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing, and $\lim_{s \rightarrow 0^+} \beta(s, t) = 0$, and for each $s \geq 0$ $\beta(s, \cdot)$ is nonincreasing and $\lim_{t \rightarrow \infty} \beta(s, t) = 0$.*

Theorem 1. *Consider the dynamic system determined by Equations (B.1) and (B.2). Also consider $t_0 \in \mathbb{R}$, the initial time, and initial condition $(x_1(t_0), x_2(t_0)) \in M$. Furthermore, consider that the function $u : \mathbb{R} \rightarrow \mathbb{R}$ is constant. Then, the origin is Input to State Stable (ISS). Moreover, the asymptotic error between the origin and the solution $x(t)$ is $\|cu + k + \gamma\|_\infty$*

In what follows we deal with the distance from the origin, but we can change the origin via a constant control input, therefore keeping the theorem valid up to a coordinate change.

Proof. We now that from the signal invariance if $(x_1(t_0), x_2(t_0)) \in \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$, then the system dynamics is given by

$$\dot{x}_1(t) = x_2(t), \tag{B.9a}$$

$$\dot{x}_2(t) = ax_2(t) + bx_1(t) + (k + \gamma e^{-(x_2(t))^2/v_s})\text{sign}(x_2(t_0)) + cu(t), \tag{B.9b}$$

here we assume that the origin will be our equilibrium point if $u = -(k + \gamma)/c$. Nonetheless, we do not lose generality, since we can change the equilibrium point to $(\bar{x}_1, \bar{x}_2)$ by changing the coordinates $(\bar{x}_1, \bar{x}_2) = (x_1 - \frac{(k-\gamma)+cu}{b}, x_2)$. Then one can argue that this change of coordinates can be compensated by changing the input to another constant.

We can analyse the system dynamics in each connected component $H_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, and $H_- = \{(x, y) \in \mathbb{R}^2 : y < 0\}$ separately. Consider then that $(x_1(t_0), x_2(t_0)) \in H_+ \cap M$. One should notice that our differential equation will only be valid in H_+ . However, we can define an extended system that obeys Equation (B.9) globally, which we will call global extension. If the global extension is ISS stable, then we prove that our original system is ISS on H_+ , and analogously on H_- . Hence consider the dynamics

$$\dot{x}_1(t) = x_2(t), \quad (\text{B.10a})$$

$$\dot{x}_2(t) = ax_2(t) + bx_1(t) + (k + \gamma e^{-(x_2(t))^2/v_s}) + cu(t), \quad (\text{B.10b})$$

and consider the candidate Lyapunov Function, with $\beta > 0$, a parameter such that $|\beta| < |a_1|$

$$V(x) = \frac{x_1^2}{2} + \beta \frac{x_2^2}{2}, \quad (\text{B.11})$$

and notice that its equation becomes

$$\dot{V}(x) = x_1x_2 + \beta x_2 \left(bx_1 + ax_2 \gamma \left(e^{-x_2^2/v_s} - 1 \right) \right) + x_2(cu + k + \gamma), \quad (\text{B.12})$$

here we omitted the free variable for clarity. After regrouping we arrive at

$$\dot{V}(x) = x_1x_2(\beta + b) + a\beta x_2^2 + \beta\gamma(e^{-x_2^2/v_s} - 1)x_2 + \beta x_2(cu + k + \gamma) \quad (\text{B.13})$$

Completing the squares we get

$$\dot{V}(x) = x_1x_2(\beta + b) + \beta b \left(x_2 + \frac{\gamma}{2a} \left(e^{-x_2^2/v_s} - 1 \right) \right)^2 - \beta b \left(\frac{\gamma}{2a} \left(e^{-x_2^2/v_s} - 1 \right) \right)^2 + \beta x_2(cu + k + \gamma). \quad (\text{B.14})$$

Now, notice that $\left(x_2 + \frac{\gamma}{2a} e^{-x_2^2/v_s} \right) > 0$, since $a < 0$, from the stability of the linearized system, also note that $e^{-x_2^2/v_s} - 1 < 0$. Furthermore, $\left(x_2 + \frac{\gamma}{2a} \left(e^{-x_2^2/v_s} - 1 \right) \right)^2 - \beta b \left(\frac{\gamma}{2a} \left(e^{-x_2^2/v_s} - 1 \right) \right)^2 > 0$.

We need Young's inequality in the sequel. This inequality states that $ab < \epsilon a^2 + (1/\epsilon)b^2$, for every $\epsilon > 0$. This is straightforward from expanding $(a\sqrt{\epsilon} + b/\sqrt{\epsilon})^2 > 0$. We now apply Young's inequality to the terms with the products x_1x_2 and $x_2(cu + k + \gamma)$, considering $\epsilon_1 > 0$ and $\epsilon_2 > 0$ respectively. Then

$$\dot{V}(x) \leq (\beta+b)(\epsilon_1 x_1^2 + \frac{1}{\epsilon_1} x_2^2) + \beta \left(b \left(x_2 - \frac{\gamma}{2a} (e^{-x_2^2/v_s} - 1) \right) - b \left(\frac{\gamma}{2a} (e^{-x_2^2/v_s} - 1) \right) \right)^2 + \epsilon_2 \beta x_2^2 + \frac{\beta}{\epsilon_2} (cu + k + \gamma)^2, \quad (\text{B.15})$$

therefore, there exists $\epsilon_2 > 0$ such that $\dot{V} \leq \alpha(\|x\|) + \nu(\|cu + k + \gamma\|)$, where $\alpha \in \mathcal{K}_\infty$, and $\nu \in \mathcal{K}_\infty$ (SONTAG, 1989). Hence, our system admits an ISS Lyapunov function, for initial conditions outside the line $x_2 = 0$. We can affirm that, $\|x\| \leq \max\{\|cu + k + \gamma\|_\infty, \delta(\|x(0)\|, t)\}$, where $\delta \in \mathcal{KL}$. (SONTAG, 2013).

For the sake of completeness. If the initial condition is on the line $x_2 = 0$, the system either remains on it or the system trajectory gets into one of the connected component and the later development holds.

□

Corollary 1. *The control law described in Chapter 2, ensure an asymptotic error smaller than or equal to γ .*

B.2.2 Minimal Gap Between Desired Outputs

Still under the hypothesis that the reference we want to follow is piecewise constant. If u in the control phase is such that $-F_s \leq Au - k_s x_1(t_0) \leq F_s$, then the system will not start to move. Hence we need that $|u| > \frac{F_s + k_s x_1(t_0)}{A}$ to start the movement. We provide two different solutions to this problem.

- Solution 1: The simplest solution to this problem consist of applying an input $\tilde{u}(t) = u(t) + p(t)$, where

$$p(t) = \begin{cases} 0 & \text{if } t \geq T_{step}, \\ \delta & \text{otherwise,} \end{cases} \quad (\text{B.16})$$

where δ is such that $|\delta + u(t)| > \frac{F_s + k_s x_1(t_0)}{A}$, for $t < Tstep$. In this way, if $Tstep$ is small enough the system will start moving and it will not change significantly the system's performance.

- Solution 2:

If however, it is unfeasible to apply the input $\tilde{u}$ described earlier, due to the impossibility of using a sufficiently small $Tstep$ by the hardware, we can adopt a different strategy that does not depend on the hardware clock.

The condition $-F_s \leq Au - k_s x_1(t_0) \leq F_s$ only happens if $|\Delta r|$ is too small, hence we must enlarge it making the system go in the opposite direction. In this manner it suffices to apply $\tilde{u} = -\text{sign}(\Delta r)(\frac{F_s + k_s x_1(t_0)}{A} + \epsilon)$, $\epsilon > 0$ arbitrary, and after the system settles again go back to our algorithm. One should notice that if such $\tilde{u}$ does not exists, then the valve would never start moving, hence it should be replaced.