FABIO BARBIERI

**LINEAR SYSTEMS WITH MARKOV JUMPS AND
MULTIPLICATIVE NOISES - THE CONSTRAINED TOTAL
VARIANCE PROBLEM**

Submitted in partial fulfillment for the degree
of Master in Science to the Escola Politéc-
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RESUMO

Neste trabalho, estudamos o problema do controle ótimo estocástico de sistemas lineares em tempo discreto sujeitos a saltos Markovianos e ruídos multiplicativos. Consideramos a otimização multiperíodo, com horizonte de tempo finito, de um funcional da média-variância sob um novo critério. Neste novo problema, maximizamos o valor esperado da saída do sistema ao mesmo tempo em que limitamos a sua variância total ponderada pelo seu parâmetro de risco. A lei de controle ótima é obtida através de um conjunto de equações de diferenças de Riccati interconectadas, estendendo resultados anteriores da literatura. São apresentadas simulações numéricas para uma carteira de investimentos com ações e um ativo de risco para exemplificarmos a aplicação de nossos resultados.

Palavras-chave: Controle estocástico. Sistemas lineares. Controle ótimo. Variância máxima. Otimização de carteiras de investimento.

ABSTRACT

In this work we study the stochastic optimal control problem of discrete-time linear systems subject to Markov jumps and multiplicative noises. We consider the multi-period and finite time horizon optimization of a mean-variance cost function under a new criterion. In this new problem, we apply a constraint on the total output variance weighted by its risk parameter while maximizing the expected output. The optimal control law is obtained from a set of interconnected Riccati difference equations, extending previous results in the literature. The application of our results is exemplified by numerical simulations of a portfolio of stocks and a risk-free asset.

Keywords: Stochastic control. Linear systems. Optimal control. Maximum variance. Portfolio optimization.

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1 INTRODUCTION

1.1 Objective

In this dissertation we consider a linear system with Markov jumps and multiplicative noises, and our goal is to develop an explicit optimal control policy of a mean-variance problem that maximizes the system's output when its total weighted variance is restricted by a fixed value.

In our formulation we use the concept of coupled difference Riccati equations and recursive equations in order to obtain sufficient conditions for the optimal control policy of a discrete time, multi-period, and finite time horizon mean-variance problem.

We apply our results to the management of a portfolio of financial assets and use our derived control strategy to find the optimal assets' allocation that maximizes the portfolio's value while keeping its total weighted variance limited by a value provided by the investor.

The control law we obtained complements the results developed in (COSTA; OLIVEIRA, 2012), where necessary and sufficient conditions for a solution were presented along with explicit control policies for the problems of optimizing a mean-variance cost function without restrictions, and then, with a restriction on the level of the output return. Our main results were presented and published in (BARBIERI; COSTA, 2016).

1.2 Motivation

Linear systems with Markov jumps and multiplicative noises have been the subject of many researches in the recent past. These models are better suited to describe systems that suffer abrupt changes in their dynamics. As examples of applications concerning Markov jump systems we can cite the autopilot of a ship, the pH control in a chemical reactor, the combustion control of a boiler, the fuel air control in a car

engine, the flight control system, the control of a solar-powered boiler, among others (COSTA; FRAGOSO; MARQUES, 2005).

In this study, we will apply this class of systems in the financial context, more specifically in the optimization of a portfolio of financial assets. A practical application of such models to the management of portfolios requires the possibility of imposing boundaries to the system such as limiting the output return, its variance, the amount of wealth allocated in each security, etc. These restrictions or policies arise from a natural attempt to control certain risks such as limiting the portfolio's leverage, its total loss or volatility, or just limiting the portfolio over exposure to assets with low liquidity or from a specific country or region.

Therefore, obtaining an explicit optimal asset allocation strategy that takes into consideration a new restriction on the total weighted variance of the portfolio's value shall improve the applicability of our system in real situations, specially the one faced by a hedge fund that must limit his/her portfolio's total volatility over time while maximizing the returns.

1.3 Structure

This dissertation is structured in seven Chapters. In Chapter 2 we present a brief literature review with a focus on the main theoretical achievements regarding our system's model. In Chapter 3 we show how to model a portfolio of risky assets and a reference security using our system's notation. The problem formulation and previous results are presented in Chapter 4 and the solution of our problem is described in Chapter 5. An example on how to estimate the model parameters and numerical simulations are described in Chapter 6 and, finally, Chapter 7 presents our final considerations.

2 LITERATURE REVIEW

One of the main goals in systems engineering is to model the dynamics of real systems through mathematical equations and to meet specific performance requirements through the control of such systems.

The following sections focus on the historical development of our system, highlighting the main issues and solutions that had to be pursued in order to achieve the current state of the art. Overall, the main milestones were: (i) the modeling of stochastic systems with Markov jumps; (ii) the identification of necessary and sufficient conditions for the solvability of indefinite stochastic linear systems; (iii) the introduction of an auxiliary cost function in order to find the analytical optimal solution; (iv) the explicit control policy for the many variations of our system (single or multi-period, finite or infinite time horizon, discrete or continuous, terminal or multi-period optimization, with or without noises, etc); and (v) the introduction of risk parameters and restrictions to our variables with the development of their respective optimal controls.

Throughout this chapter, the matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ will represent the dynamics of the corresponding system, continuous or discrete, and the matrices $Q(\cdot)$, $R(\cdot)$, and H will represent the input weights in the related cost function. The variables $x(\cdot)$ and $u(\cdot)$ are the usual state and control variables, respectively, and the meaning of other variables that appear along the text shall be described within their context.

Lastly, Linear Quadratic or "LQ", considers that the system can be described by linear equations and the control strategy is obtained through the minimization (or maximization) of a quadratic function.

2.1 Markov jumps and multiplicative noises

The performance of a control system is often challenged when the environment dynamics are subjected to abrupt and significant changes, forcing the model to operate under new conditions, or in other words, the system jumps to another operation mode.

Whenever those jumps can be modeled according to a Markov chain and the system's dynamics are described by linear equations, the system is called Markov Jumps Linear System or just "MJLS".

There are many examples of situations that would require such complex models, for instance: (i) aircraft control systems dealing with abrupt changes in pressure, altitude, and speed; (ii) economic models facing the burst of financial crisis or changes of governments and policies; (iii) population models with the advent of diseases, etc.

To illustrate how the MJLS works, consider a discrete-time system that is, in a certain moment, well described by

$$\begin{aligned} x(k+1) &= A_S(k)x(k) + B_S(k)u(k), \\ x(0) &= x_0, \quad k = 0, 1, \dots, T-1, \end{aligned}$$

where $x(k)$ denotes the system state at the instant k with initial condition x_0 , $u(k)$ represents the control or system input, and matrices A_S and B_S represent the environment dynamics under the operation mode S , which follows a Markov chain, with $S \in \{1, 2, \dots, N\}$.

The system could start in any operation mode of the Markov chain and, in our example, we will suppose that this system starts in the environment condition S_4 and after passing through significant changes, it could start to operate under a different condition S_5 with the transition probability p_{45} . More generally, we could imagine this system over time jumping to a series of possible modes, for instance, $\{S_1, S_2, \dots, S_N\}$ with transition probabilities from mode i to j given by p_{ij} . In this way, we would have

that

$$x(1) = A_{S_4}(0)x_0 + B_{S_4}(0)u(0)$$

$$\text{jump} : S_4 \rightarrow S_5$$

$$x(2) = A_{S_5}(1)x(1) + B_{S_5}(1)u(1)$$

$$\text{jump} : S_5 \rightarrow S_1$$

$$x(3) = A_{S_1}(2)x(2) + B_{S_1}(2)u(2)$$

...

Note that we know the current operation mode and the transition probabilities from and to any mode, but we do not know a priori the exactly sequence of jumps.

Another special type of MJLS considers multiplicative white noises in the state and control variables. This new system takes the following general form:

$$x(k+1) = [A_S(k)w^x(k)] x(k) + [B_S(k)w^u(k)] u(k),$$

where $\{w^x(k)\}$ and $\{w^u(k)\}$ are both zero-mean random variables with unitary variance that operate directly on the dynamics and on the input of the system.

2.2 Solvability and well-posedness of indefinite stochastic systems

We start this section informally presenting a couple of necessary definitions to show the relevance of the following results. We say that a stochastic LQ control problem is "indefinite" whenever the cost weighting matrices for the state or the control are allowed to be indefinite. Given a cost function, $J(u(\cdot))$, the minimization/maximization of it is called a "problem" which yields the optimal control. The value of the cost function for this optimal control results in the value function, $V(J(\cdot))$. Therefore, the "solvability" of a stochastic LQ control problem regards the existence of the optimal control policy that solves it, and the problem is said to be "well-posed" whenever $V(J(\cdot)) > -\infty$.

Thereby, the introduction of noises and jumps in the state and control variables led to the possibility of indefinite stochastic LQ problems, which naturally raised concerns

about the existence of necessary and sufficient conditions for the solvability and well-posedness of such control problems. In (CHEN; LI; ZHOU, 1998), it was found for the first time that a stochastic LQ problem with indefinite weighting matrix in the control term may still be well-posed. There, the authors analyzed the continuous-time system described by

$$dx(t) = [A(t)x(t) + B(t)u(t)]dt + [C(t)x(t) + D(t)u(t)]dW(t),$$

where $W(t)$ is the standard Brownian motion in $[0, T]$.

They also considered the following stochastic LQ problem:

$$\text{Minimize } J(u(\cdot)) := E \left\{ \int_{t_0}^T (x(t)' Q(t)x(t) + u(t)' R(t)u(t))dt \right\} + E \{ x(T)' H x(T) \},$$

whose optimal control strategy is given by

$$u(t) = -[R(t) + D(t)' P(t) D(t)]^{-1} [B(t)' P(t) + D(t)' P(t) C(t)] x(t),$$

where $P(t)$ is the differentiable symmetric matrix solution of a Riccati equation that arises by assuming that the value function of $J(u(\cdot))$ leads to a quadratic equation with the term $x(t)' P(t)x(t)$. Therefore, finding $P(t)$ leads to solving the problem of minimizing $J(u(\cdot))$.

Thereby, the authors proved that if the following type of Riccati equation has a solution, then the indefinite stochastic LQ problem cited above is well-posed and an optimal feedback control can be constructed via $P(\cdot)$ (t is suppressed to ease the notation).

$$\begin{cases} \dot{P} = -PA - A'P - C'PC + (PB + C'PD)(R + D'PD)^{-1}(B'P + D'PC) - Q, \\ P(T) = H, \\ R + D'PD > 0, \quad t \in [0, T]. \end{cases}$$

The solvability problem continues if the above equation does not have a solution and, in (RAMI et al., 2001; RAMI; MOORE; ZHOU, 2001), it was shown that by relaxing the condition $R + D'PD > 0$, one would obtain the following generalized Riccati equation ("GRE") whose solvability are both necessary and sufficient for the solvability of the

underlying LQ problem in more general terms.

$$\left\{ \begin{array}{l} \dot{P} = -PA - A'P - C'PC + (PB + C'PD)(R + D'PD)^{-1}(B'P + D'PC) - Q, \\ P(T) = H, \\ ((R + D'PD)(R + D'PD)^{\dagger} - I)(B'P + D'PC) = 0, \\ R + D'PD \geq 0, \quad t \in [0, T], \end{array} \right.$$

where † is the Moore-Penrose pseudo inverse.

In (RAMI; CHEN; ZHOU, 2002) the authors introduced a new generalized difference Riccati equation ("GDRE") with no positiveness constraint, and a linear matrix inequality condition ("LMI") for the indefinite stochastic LQ problem. They considered the following discrete-time stochastic system where they also introduced the idea of multiplicative noises in comparison to the previous problem:

$$x(k+1) = [A(k) + w^x(k)C(k)]x(k) + [B(k) + w^u(k)D(k)]u(k) + w(k),$$

where $w(k)$, $w^x(k)$ and $w^u(k)$ are white noises and $k = 0, 1, \dots, T-1$.

The LQ problem of interest is given by

$$\text{minimize } J(u(\cdot)) := E \left\{ \sum_{k=0}^{T-1} [x(k)'Q(k)x(k) + u(k)'R(k)u(k)] + x(T)'Q(T)x(T) \right\},$$

whose optimal solution is

$$u(k) = -G(k)^{\dagger}M(k)x(k),$$

where $G(k)$ and $M(k)$ are gains defined in the GDRE below, and $G(k)^{\dagger}$ is the Moore-Penrose pseudo-inverse of $G(k)$.

Then, they obtained the following generalized difference Riccati equation with no

positiveness constraint (k is suppressed whenever possible to ease the notation).

$$GDRE : \begin{cases} P(T) = Q(T), \\ G^\dagger GM - M = 0, \\ M = B'P(k+1)A + \rho_i^{xu} D'P(k+1)C, \\ G = R + B'P(k+1)B + D'P(k+1)D, \end{cases}$$

where $\rho(k)^{xu} = E[w(k)^x w(k)^u]$.

Finally, the linear matrix inequality condition is given by

LMI :

$$\begin{bmatrix} A'P(k+1)A - P(k) + C'P(k+1)C + Q & A'P(k+1)B + \rho^{xu} C'P(k+1)D \\ B'P(k+1)A + \rho^{xu} D'P(k+1)C & R + B'P(k+1)B + D'P(k+1)D \end{bmatrix} \geq 0,$$

for $k = 0, \dots, T-1$, and $P(T) \leq Q(T)$.

In this case, it was proven that the solvability of the GDRE, the LMI condition, and the feasibility and well-posedness of the LQ problem are all equivalent.

Follow-up researches on indefinite stochastic LQ control have been carried out extensively. For instance, we have studies that considered such systems with integral quadratic constraints (LIM; ZHOU, 1999), or cross terms in the functional cost (LUO; FENG, 2004), or even studies that took different paths to solve the Riccati equation such as algorithms, near optimal controls, or approximations to the Riccati equation (LIU; YIN; ZHOU, 2005; LI; ZHOU; RAMI, 2003; ZHU, 2005) .

2.3 The auxiliary problem

In this section, we will describe an auxiliary problem that is very relevant to solve our mean-variance optimization problem. Consider the following discrete-time linear

system with Markov jumps

$$\begin{aligned}x(k+1) &= A_S(k)x(k) + B_S(k)u(k), \\x(0) &= x_0, \quad k = 0, 1, \dots, T-1, \\y^u(t) &= L_s(t)x(t), \quad t = 1, \dots, T,\end{aligned}$$

where $y^u(t)$ is the system's output subjected to the control policy u . Let's also consider the functional cost, $\mathcal{J}(u)$, given by

$$\mathcal{J}(u) := \sum_t \left[\text{Var}[y^u(t)] - E[y^u(t)] \right] = \sum_t \left[E[y^u(t)^2] - E[y^u(t)]^2 - E[y^u(t)] \right].$$

In the portfolio optimization problem, $y(t)$ represents the value of the portfolio and u the investment strategy, see Chapter 3. Thus the first term, $\text{Var}[y^u(t)]$, represents the risk of the investment (we desire to minimize), while the second term, $E[y^u(t)]$, represents the expected value of the wealth (we desire to maximize).

Note that the mean-variance optimization is not readily solvable due to the quadratic term that arises from the variance in $\mathcal{J}(u)$. This issue led to the introduction of an auxiliary problem, $AP(\cdot)$ defined below, framed without this quadratic term and whose solution is also a solution of $\min_u \mathcal{J}(u)$. See (LI; NG, 2000) for further details and proofs.

$$AP(u) := \min_u \sum_t \left[E[y^u(t)^2] - E[y^u(t)] \right].$$

Later, it was proved that the auxiliary problem can also be applied when the state weighting matrix depends on a stochastic market depicted by a Markov chain (CAK-MAK; OZECKICI, 2006) and, in the same paper, the authors also provided an explicit control strategy for this case.

Once more, extensive researches have been carried out on the application of the above technique and the reader may find more detailed information about the theory and examples of its application in (YIN; ZHOU, 2004; ZHOU; YIN, 2003; CELIKYURT; OZECKICI, 2007; CANAKOGLU; OZECKICI, 2010; COSTA; PAULO, 2007; COSTA; PAULO, 2008).

2.4 The introduction of risk parameters

The next step regards the introduction of different risk preferences, which are vital to a proper application of our model. We can find in the literature a broad variety of approaches on how to incorporate risk preferences in a control theory perspective. In particular, regarding the application on portfolio selection, the reader is referred to (MARKOWITZ, 1952; MARKOWITZ, 1959; ELTON; GRUBER, 1995; SATCHEL; SCOWCROFT, 2003; CAKMAK; OZECKICI, 2006) for more detailed information.

Notice that the cost function defined above, $\mathcal{J}(u)$, is not flexible enough to take into account different risk aversion characteristics, such as preferences about expected outputs and variances.

Therefore, to tackle this problem, a more flexible cost function was introduced and the control problem solved for the terminal optimization with single and multi-period models (COSTA; NABHOLZ, 2007; COSTA; ARAUJO, 2008; COSTA; OKIMURA, 2007; COSTA; OKIMURA, 2009).

In this new cost function, illustrated below for the most general case, we define the risk aversion parameters, set by $\nu(t)$ and $\xi(t)$, as inter-temporal weights of their respective variables associated with the expected value of the output and its variance. In this way, we have that

$$\mathcal{J}_2(u) := \sum_t \left[\nu(t) \text{Var}[y^u(t)] - \xi(t) E[y^u(t)] \right].$$

Those inter-temporal parameters are chosen according to the relative relevance that we attribute to each associated variable. For instance, in order to obtain an optimal solution that penalizes the control inputs with higher variance in the first three periods, we should consider a higher weight for $\nu(1)$, $\nu(2)$ and $\nu(3)$ in comparison with the weights we had considered on the other periods. Thus, in a portfolio management perspective, it would lead to a solution that allocate less wealth in assets with higher expected volatility in the first three periods.

In a similar reasoning, in order to obtain an optimal solution that benefits the con-

trol inputs that lead to a higher expected output in the first three periods, we should consider a higher weight for $\xi(1)$, $\xi(2)$ and $\xi(3)$ in comparison with the weights we had considered on the other periods. Once more, in a portfolio management perspective, it would mean a solution that allocates more wealth in assets with higher expected output in the first three periods.

As we will see in this dissertation, the introduction of such parameters plays a central role in obtaining the optimal strategy of either the unconstrained and constrained problems.

2.5 The state of the art and problem overview

Finally, a generalized multi-period optimal control policy was developed in (COSTA; OLIVEIRA, 2012) for two performance criteria. The first one, denoted by $PU(\nu, \xi)$ and formally defined in Section 4.4, is composed by an unconstrained linear combination of the expected output and its variance as stated below.

$$PU(\nu, \xi) : \min_u \sum_{t=1}^T \left[\nu(t) \text{Var}[y^u(t)] - \xi(t) E[y^u(t)] \right].$$

In this problem, the optimal solution, u , depends on the input parameters, ν and ξ , and it provides a policy with no boundaries regarding neither the output nor its variance.

On the other hand, the second performance criterion, denoted by $PC(\nu, \epsilon)$, considers the minimization of the variance while keeping the expected output of the system higher than some specified value.

$$PC(\nu, \epsilon) := \min_u \sum_t \nu(t) \text{Var}[y^u(t)],$$

$$\text{subject to: } E[y^u(t)] \geq \epsilon(t).$$

In this problem, they establish the risk aversion regarding the output's variance through ν and the minimum required expected output return, ϵ . It has been shown that the optimal solution of $PC(\nu, \epsilon)$ can be obtained using the same formulation of the

optimal solution of $PU(\nu, \xi)$ by simply finding an appropriate ξ that complies with the restriction imposed.

In this dissertation, our goal is to extend these results just described by finding an optimal control policy to a new constrained problem denoted by $PC(\nu, \beta, \alpha)$ which informally reads

$$PC(\nu, \beta, \alpha) : \max_u \sum_{t=1}^T [\beta(t)E[y^u(t)]] ,$$

$$\text{subject to: } \sum_{t=1}^T \nu(t)Var[y^u(t)] \leq \alpha .$$

In $PC(\nu, \beta, \alpha)$, we wish the optimal control strategy, u , that maximizes the system's expected output while its total weighted variance is kept lower than a maximum value, α . Here, the coefficients β and ν are input parameters associated with the risk aversion towards the system's output and its variance, respectively.

In this new constrained problem we will follow a similar approach to the one adopted in (COSTA; OLIVEIRA, 2012) and obtain its optimal control law through the solution of $PU(\nu, \xi)$ by finding an appropriate ξ that complies with both the new restriction α and the new input parameter β .

3 PORTFOLIO MANAGEMENT MODEL

A specific problem of great interest regards the management of a portfolio of assets. This challenge is probably as old as economy itself, but only with Markowitz it was framed in proper scientific terms and improved in many ways since then.

In this chapter we show how to model the dynamics of a portfolio of assets using a specific notation that will allow us to use the state-of-the-art results in control theory in order to find the optimal allocation of its assets. Therefore, this chapter can be considered a motivation to the more general problem we will describe in detail in Chapter 4.

This chapter is laid out as follows: In Section 3.1 we cite some examples on how the complexity of portfolio management models evolved over time to meet specific needs such as the consideration of changes in expectation, use of benchmarks, computation of cash flows, etc. Then, in Section 3.2, we develop a portfolio selection formulation that matches the notation of our system.

3.1 Brief historical overview

The seminal works of Markowitz (MARKOWITZ, 1952; MARKOWITZ, 1959) verified the benefits of diversification and framed the asset allocation in a way to maximize the expected portfolio's return while minimizing its variance. In his model all expectations about the future had to be incorporated in a single period and the liabilities and leverages were not considered.

Naturally, subsequent studies took into consideration more characteristics such as leverage (TOBIN, 1958), liabilities (SHARPE; TINT, 1990), and a multi-period investment horizon (MOSSIN, 1968; SAMUELSON, 1969; HAKANSSON, 1970).

A variety of portfolio planning models have been proposed and investigated besides the mean-variance model of Markowitz. They include the mean absolute variance, the weighted goal programming, the minimax model which use alternative metrics for risk,

the use of genetic algorithms for efficiently selecting a subset of stocks to trade, etc. The reader is referred to (SATCHEL; SCOWCROFT, 2003) for detailed information on the subject.

Other relevant characteristics of portfolio management models include the possibility of considering a benchmark, cash flows within the investment period, and a risk-free security. The relevance of these characteristics becomes evident due to their practical applications, exemplified below.

Exchange traded funds or pension funds with a mandate to track the return of an index is a classical example of a practical problem that led to the portfolio management's formulation with a benchmark. In this model, the optimization takes into consideration the maximization of the excess return over the benchmark and the minimization of its variance.

Another example of model regards the Asset Liability Management (ALM) theory in which we must consider cash inflows and outflows besides the benchmark and risky assets. This type of model would be of great value for pension funds that must provide returns higher than inflation in a long time horizon while creating wealth to honor the actuarial liabilities.

A very relevant model, and that will be described in detail in the next section, regards a portfolio with risky assets and a risk-free security. This type of model is a classical application of portfolio management and it was chosen to exemplify our results due to the high volatility easily achieved in their applications.

3.2 Model formulation

In this section we will consider a portfolio with $m + 1$ securities following random prices represented by the vector $\bar{S}(t) \in \mathbb{R}^{m+1}$ and with relative returns represented by the vector $\bar{R}(t) \in \mathbb{R}^{m+1}$. Thus, we have that

$$\bar{S}(t) = [S_1(t) \cdots S_{m+1}(t)]', \quad (3.1)$$

$$\bar{R}(t) = [R_1(t) \cdots R_{m+1}(t)]', \quad (3.2)$$

where

$$R_i(t) = \frac{S_i(t+1)}{S_i(t)}, \quad i = 1, \dots, m+1. \quad (3.3)$$

We will also assume that, for each market operation mode, the assets' returns are described by Equation (3.4). Notice that this is very relevant because it defines the dynamics of each security and, therefore, the dynamics of the portfolio.

Hence, we set the assets' returns between the steps t and $t+1$ as

$$\bar{R}_{\theta(t)}(t) = (\bar{e} + \bar{\mu}_{\theta(t)}(t)) + \bar{\sigma}_{\theta(t)}(t)w(t), \quad (3.4)$$

where, $\{\theta(t); t = 0, \dots, T-1\}$ is a Markov chain with a finite number of discrete operation modes over time and taking values in $\{1, \dots, N\}$. Thus, the parameter $\theta(t)$ represents the Markov jumps that the assets' prices can take and, therefore, it will greatly influence how the portfolio's value evolve until the investment time horizon, T , is reached.

The vectors $\{w(t)' = [w_1(t) \dots w_{m+1}(t)]; t = 0, \dots, T-1\}$ constitute a sequence of random and independent vectors of $m+1$ dimension with zero mean and covariance equal to the identity matrix. They are also independent of the market operation modes.

We also define the vector $\bar{e} = [1 \ e]'$, where $e \in \mathbb{R}^m$ is a vector with unitary elements. The vector $\bar{\mu}_{\theta(t)}(t) \in \mathbb{R}^{m+1}$ is formed by the expected returns of each security and $\bar{\sigma}_{\theta(t)}(t) \in \mathbb{R}^{m+1, m+1}$ represents the standard deviation matrix of the assets' returns at the time t .

For a convenience that will be apparent later, we consider the first security as the reference asset and decompose $\bar{\mu}_{\theta(t)}(t)$ in the following way:

$$\bar{\mu}_{\theta(t)}(t) = \begin{bmatrix} \mu_{\theta(t),1}(t) \\ \hat{\mu}_{\theta(t)}(t) \end{bmatrix}, \quad (3.5)$$

where,

$$\hat{\mu}_{\theta(t)}(t) = \begin{bmatrix} \mu_{\theta(t),2}(t) \\ \vdots \\ \mu_{\theta(t),m+1}(t) \end{bmatrix}. \quad (3.6)$$

Repeating the decomposition above to $\bar{\sigma}_{\theta(t),1}(t)$, we have that

$$\bar{\sigma}_{\theta(t)}(t) = \begin{bmatrix} \sigma_{\theta(t),1}(t) \\ \hat{\sigma}_{\theta(t)}(t) \end{bmatrix}, \quad (3.7)$$

where,

$$\hat{\sigma}_{\theta(t)}(t) = [\sigma_{\theta(t),1,1}(t) \cdots \sigma_{\theta(t),1,m+1}(t)], \quad (3.8)$$

and

$$\hat{\sigma}_{\theta(t)}(t) = \begin{bmatrix} \sigma_{\theta(t),2,1}(t) & \cdots & \sigma_{\theta(t),2,m+1}(t) \\ \vdots & \ddots & \vdots \\ \sigma_{\theta(t),m+1,1}(t) & \cdots & \sigma_{\theta(t),m+1,m+1}(t) \end{bmatrix}. \quad (3.9)$$

Notice that the assets' returns vector, $\bar{R}_{\theta(t)}(t)$, can also be rewritten as

$$\bar{R}_{\theta(t)}(t) = \begin{bmatrix} R_{\theta(t),1}(t) \\ \hat{R}_{\theta(t)}(t) \end{bmatrix}, \quad (3.10)$$

where,

$$\hat{R}_{\theta(t)}(t) = \begin{bmatrix} R_{\theta(t),2}(t) \\ \vdots \\ R_{\theta(t),m+1}(t) \end{bmatrix}. \quad (3.11)$$

Using the decomposition above, the assets' returns over time will be described in a way that leads to the formulation of our system as defined in (4.1).

In order to determine how the dynamics of the portfolio's value evolve, we will define the wealth allocated to each security over time given the market operation mode in every instant.

Let $U_i(t)$ be the wealth allocated to the i^{th} security, $i = 1, \dots, m+1$. We define the vector $U(t)$ as

$$U(t) = [U_1(t) \cdots U_{m+1}(t)]'. \quad (3.12)$$

Applying the same decomposition as above, we have that

$$U(t) = \begin{bmatrix} U_1(t) \\ \hat{U}(t) \end{bmatrix}, \quad (3.13)$$

where,

$$\hat{U}(t) = \begin{bmatrix} U_2(t) \\ \vdots \\ U_{m+1}(t) \end{bmatrix}. \quad (3.14)$$

We then represent the portfolio's value associate with the strategy U as $X^U(t)$ and to simplify our notation we will omit the index U every time it does not raise any ambiguity or misunderstanding. Hence, the portfolio's value at time t can be described as

$$X(t) = U_1(t) + \hat{U}(t)' e', \quad (3.15)$$

and the wealth allocated in the reference asset will be given by

$$U_1(t) = X(t) - \hat{U}(t)' e'. \quad (3.16)$$

Considering there are neither cash inflows nor cash outflows, the portfolio is self-financed and the wealth process is given by

$$X(t+1) = R_{\theta(t),1}(t)U_1(t) + \hat{R}_{\theta(t)}(t)' \hat{U}(t). \quad (3.17)$$

Substituting (3.16) into (3.17), we obtain that

$$X(t+1) = R_{\theta(t),1}(t)X(t) + \hat{P}_{\theta(t)}(t)' \hat{U}(t), \quad (3.18)$$

where,

$$\hat{P}_{\theta(t)}(t) = \hat{R}_{\theta(t)}(t)' - R_{\theta(t),1}(t)' e'. \quad (3.19)$$

Finally, in order to represent the model using the same notation developed in Chapter 4, consider that $x(t) = X(t)$, $u(t) = \hat{U}(t)$, and rewriting Equation (3.18) we have that

$$x(t+1) = R_{\theta(t),1}(t)x(t) + \hat{P}_{\theta(t)}(t)' u(t). \quad (3.20)$$

Applying the relation (3.4) into (3.20) and (3.19), we obtain respectively that

$$x(t+1) = [(1 + \bar{\mu}_{\theta(t),1}(t)) + \sigma_{\theta(t),1}(t)w^x(t)] x(t) + \hat{P}_{\theta(t)}(t)' u(t), \quad (3.21)$$

and

$$\begin{aligned}\hat{P}_{\theta(t)}(t) &= \hat{R}_{\theta(t)}(t)' - R_{\theta(t),1}(t)' e' \\ &= (\hat{\mu}_{\theta(t)}(t) - \mu_{\theta(t),1}(t) e') + (\hat{\sigma}_{\theta(t)}(t) - \sigma_{\theta(t),1}(t) w^u(t) e').\end{aligned}\quad (3.22)$$

Defining D^s as the s^{th} column vector of matrix D , we obtain that

$$\hat{P}_{\theta(t)}(t) = (\hat{\mu}_{\theta(t)}(t) - \mu_{\theta(t),1}(t) e') + \sum_{s=1}^{m+1} (\hat{\sigma}_{\theta(t)}^s(t) - \sigma_{\theta(t),1}^s(t) e') w_s^u(t), \quad (3.23)$$

and

$$\begin{aligned}x(t+1) &= \left[(1 + \bar{\mu}_{\theta(t),1}(t)) + \sum_{s=1}^{m+1} \sigma_{\theta(t),1}^s(t) w_s^x(t) \right] x(t) \\ &\quad + \left[(\hat{\mu}_{\theta(t)}(t) - \mu_{\theta(t),1}(t) e')' + \sum_{s=1}^{m+1} (\hat{\sigma}_{\theta(t)}^s(t) - \sigma_{\theta(t),1}^s(t) e')' w_s^u(t) \right] u(t).\end{aligned}\quad (3.24)$$

Lastly, we make the following definitions for $s = 1, \dots, m+1$,

$$\begin{aligned}\bar{A}_{\theta(t)}(t) &= 1 + \mu_{\theta(t),1}(t), \\ \tilde{A}_{\theta(t),s}(t) &= \sigma_{\theta(t),1}^s(t), \\ \bar{B}_{\theta(t)}(t) &= (\hat{\mu}_{\theta(t)}(t) - \mu_{\theta(t),1}(t) e')', \\ \tilde{B}_{\theta(t),s}(t) &= (\hat{\sigma}_{\theta(t)}^s(t) - \sigma_{\theta(t),1}^s(t) e')',\end{aligned}\quad (3.25)$$

which turn our model (3.24) into the Equations (4.1) and (4.2) as described below:

$$\begin{aligned}x(k+1) &= \left[\bar{A}_{\theta(k)}(k) + \sum_{s=1}^{\varepsilon^x} \tilde{A}_{\theta(k),s}(k) w_s^x(k) \right] x(k) + \left[\bar{B}_{\theta(k)}(k) + \sum_{s=1}^{\varepsilon^u} \tilde{B}_{\theta(k),s}(k) w_s^u(k) \right] u(k), \\ x(0) &= x_0, \quad \theta(0) = \theta_0, \\ y(t) &= L_{\theta(t)}(t) x(t), \\ L_{\theta(t)}(t) &= 1.\end{aligned}\quad (3.26)$$

Note that $\bar{A}_{\theta(t)}(t)$ and $\tilde{A}_{\theta(t),s}(t)$ are related to the wealth process creation of the reference asset, and $\bar{B}_{\theta(t)}(t)$ and $\tilde{B}_{\theta(t),s}(t)$ are related to the wealth process creation of the remaining securities.

In the case of a portfolio with a reference security and risk assets, $L_{\theta(t)}(t)$ is a scalar and, therefore, the output we optimize considers only the portfolio's value regarding the strategy chosen. When considering benchmarks or cash flows, the state variable, $x(k)$, becomes either a vector or a matrix, respectively, and $L_{\theta(t)}(t)$ becomes a vector.

Even though we did not include other models to exemplify our results, they are valid to any model whose dynamics can be described by the system we used to develop our control strategy.

3.3 $PC(\nu, \beta, \alpha)$ problem overview under the portfolio management perspective

In this section we will provide an interpretation of our goal under the portfolio management perspective. The more general problem will be described in detail in Chapter 4.

Thus, regarding our objective in this dissertation, solving the $PC(\nu, \beta, \alpha)$ problem means finding the optimal assets' allocation policy, $u(\cdot)$, that maximizes the portfolio's value, $y(\cdot)$, while restricting its total weighted variance to an absolute maximum quantity provided by the investor and denoted by α .

This problem, which is formally defined in Section 4.3, can be enunciated as follows.

$$\begin{aligned} PC(\nu, \beta, \alpha) : \max_u & \sum_{t=1}^T \left[\beta(t) E[y^u(t)] \right], \\ \text{subject to: } & \sum_{t=1}^T \nu(t) \text{Var}[y^u(t)] \leq \alpha. \end{aligned}$$

Here, the weights $\beta(t)$ and $\nu(t)$ are parameters defined by the investor representing his/her risk aversion over time regarding the portfolio's value and its variance, respectively.

4 THE OPTIMAL CONTROL OF LINEAR SYSTEM WITH MARKOV JUMPS AND MULTIPLICATIVE NOISES

This chapter presents the notations and definitions necessary for the development of our optimal control policy.

We start by formally specifying our system (Sections 4.1 and 4.2) and the constrained problem we will solve in this thesis (Section 4.3). Then we describe the unconstrained problem (Section 4.4) providing that its control law will be useful in obtaining the solution of our $PC(\nu, \beta, \alpha)$ problem.

We also establish a set of operators and definitions (Section 4.5) in order to simplify the notation of our system's control policy. In Section 4.6, we present all relevant previous results we need to develop a solution to our problem. Finally, in Section 4.7, we provide an algorithm to find the optimal control law of the unconstrained problem.

4.1 Notation and definitions

Throughout this work the n -dimensional real Euclidean space will be denoted by $\mathbb{R}^n$ and the linear space of all $m \times n$ real matrices by $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, with $\mathbb{B}(\mathbb{R}^n) := \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$.

We denote by $\mathbb{H}^{n,m}$ the linear space made up of all N -sequences of real matrices $V = (V_1, \dots, V_N)$ with $V_i \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, for $i = 1, \dots, N$ and, for simplicity, set $\mathbb{H}^n := \mathbb{H}^{n,n}$.

We say that $V = (V_1, \dots, V_N) \in \mathbb{H}^{n,+}$ if $V \in \mathbb{H}^n$ and $V_i \geq 0$, for each $i = 1, \dots, N$. The space of all bounded linear operators from $\mathbb{H}^n$ to $\mathbb{H}^m$ will be represented by $\mathbb{B}(\mathbb{H}^n, \mathbb{H}^m)$ and, in particular, $\mathbb{B}(\mathbb{H}^n) := \mathbb{B}(\mathbb{H}^n, \mathbb{H}^n)$.

We use the standard notation for $\text{tr}(A)$, A' , and $A^\dagger$ to represent the trace, transpose, and Moore-Penrose inverse of A respectively.

The Kronecker product between two matrices A and B will be denoted by $A \otimes B$, and the identity matrix (of appropriate dimension from the context) will be represented by I .

For a sequence of n -dimensional square matrices $A(0), \dots, A(t)$, we use the nota-

tion:

$$\prod_{l=s}^t A(l) = \begin{cases} A(t) \cdots A(s) & \text{for } t \geq s, \\ I & \text{for } t < s. \end{cases}$$

For a set S we define 1_s as the usual indicator function, that is,

$$1_s(\omega) = \begin{cases} 1 & \text{if } \omega \in S, \\ 0 & \text{otherwise.} \end{cases}$$

4.2 The linear system with Markov jumps and multiplicative noises

We consider the following linear system with multiplicative noises and Markov jumps on a probabilistic space $(\Omega, \mathbf{P}, \mathcal{F})$ for $k = 0, \dots, T-1$ and $t = 1, \dots, T$:

$$\begin{aligned} x(k+1) &= \left[\bar{A}_{\theta(k)}(k) + \sum_{s=1}^{\varepsilon^x} \tilde{A}_{\theta(k),s}(k) w_s^x(k) \right] x(k) + \left[\bar{B}_{\theta(k)}(k) + \sum_{s=1}^{\varepsilon^u} \tilde{B}_{\theta(k),s}(k) w_s^u(k) \right] u(k), \\ x(0) &= x_0, \quad \theta(0) = \theta_0, \end{aligned} \tag{4.1}$$

$$y(t) = L_{\theta(t)}(t)x(t), \tag{4.2}$$

where $L_{\theta(t)}(t) \in \mathbb{H}^{1,n}$.

Here, s refers to the s^{th} column vector and $\theta(k)$ denotes the operation modes of a time-varying Markov chain taking values in $\{1, \dots, N\}$ with transition probability matrix $\mathbb{P}(k) = [p_{ij}(k)]$.

We define $\mathcal{F}_\tau$ as the σ -field generated by $\{(\theta(s), x(s)); s = 0, \dots, \tau\}$, $\mathcal{F}_k$ - measurable for each $k = \tau, \dots, T-1$ and write $\mathbb{U}(\tau) = \{u_\tau = (u(\tau), \dots, u(T-1))\}$, where $u(k)$ is an m -dimensional random vector with finite second moments.

Without loss of generality, we assume that $\varepsilon = \varepsilon^x = \varepsilon^u$, and the superscript u will indicate that the control law u is being applied to (4.1) and (4.2).

We have that, for each $s = 1, \dots, \varepsilon$ and $k = 0, 1, \dots, T$,

$$\begin{aligned}\bar{A}(k) &= (\bar{A}_1(k), \dots, \bar{A}_N(k)) \in \mathbb{H}^n, \\ \tilde{A}_s(k) &= (\tilde{A}_{s,1}(k), \dots, \tilde{A}_{s,N}(k)) \in \mathbb{H}^n, \\ \bar{B}(k) &= (\bar{B}_1(k), \dots, \bar{B}_N(k)) \in \mathbb{H}^{m,n}, \\ \tilde{B}_s(k) &= (\tilde{B}_{s,1}(k), \dots, \tilde{B}_{s,N}(k)) \in \mathbb{H}^{m,n}.\end{aligned}$$

The multiplicative noises $\{w_s^x(k); s = 1, \dots, \varepsilon^x, k = 0, 1, \dots, T - 1\}$ and $\{w_s^u(k); s = 1, \dots, \varepsilon^u, k = 0, 1, \dots, T - 1\}$ are both zero-mean random variables with variance equal to 1 and also independent of the Markov chain $\{\theta(k)\}$. The independence among their elements are set as $E[w_i^x(k)w_j^x(l)] = 0$ and $E[w_i^u(k)w_j^u(l)] = 0$, $\forall k = l$ and $i \neq j$, or $\forall k \neq l$ and $\forall i, j$.

The mutual correlation between $w_{s1}^x(k)$ and $w_{s2}^u(k)$ is denoted by $E[w_{s1}^x(k)w_{s2}^u(k)] = \rho_{s1,s2}(k)$.

The initial conditions θ_0 and x_0 are assumed to be independent of $\{w_s^x(k)\}$ and $\{w_s^u(k)\}$, with x_0 an n -dimensional random vector with finite second moments.

We also set the following expected values regarding the state variable.

$$\begin{aligned}\mu_i(0) &= E(x_0 1_{\{\theta_0=i\}}), \\ \mu(0) &= [\mu_1(0)' \dots \mu_N(0)'] \in \mathbb{H}^{n,1}, \\ Q_i(0) &= E(x_0 x_0' 1_{\{\theta_0=i\}}), \text{ and} \\ Q(0) &= [Q_1(0) \dots Q_N(0)] \in \mathbb{H}^{n+}.\end{aligned}$$

4.3 The constrained problem formulation, $PC(\nu, \beta, \alpha)$

Our goal in this work is to find the optimal control policy, u , to the constrained problem denoted by $PC(\nu, \beta, \alpha)$ and defined as:

$$\begin{aligned} PC(\nu, \beta, \alpha) : \max_{u \in \mathbb{U}} & \sum_{t=1}^T [\beta(t) E[y^u(t)]], \\ \text{subject to: } & \sum_{t=1}^T \nu(t) \text{Var}[y^u(t)] \leq \alpha, \end{aligned} \quad (4.3)$$

where $\beta = [\beta(1) \cdots \beta(T)]'$, $\beta(t) \geq 0$, is the input parameter associated with system's output, $\nu = [\nu(1) \cdots \nu(T)]'$, $\nu(t) \geq 0$, is the input parameter associated with the variance of the system's output, and $\alpha \geq 0$ is the maximum total weighted variance of the system's output.

In this problem, the parameters β , ν , and α are provided by the user and can be seen as risk aversion coefficients reflecting a trade-off preference between the expected output and the associated risk (variance) level.

It is worth to mention that there is no formula to compute these parameters and they depend on the user's sensibility and capacity to "translate" risk aversion preferences between the expected output and the associated variance level into coefficients we can use in our model.

To assist someone on the task of computing these coefficients, we provide in Section 6.2 and Appendix C a detailed sensitivity analysis regarding ν and β for a certain level of α . However, just to illustrate the effects of these coefficients, we will also provide the following rather simple example.

Let's assume that $y^u(t)$ represents the absolute value, in monetary terms, of a portfolio of risk investment securities, where u corresponds to the optimal allocation of the assets over the period of time T . For simplicity, we will only consider a three-week period, $T = 3$, an initial portfolio's value of \$1,000, and that the investor wishes to limit the total variance of the portfolio's value to \$30 over the three-week period.

Let's also consider, in our example, that the investor is less averse to fluctuations

in the expected value of the portfolio in the first week than in the the last two weeks by a multiple of 1.4. In the same way, consider that he/she is more averse to fluctuations in the variance in the second week than in the first or third week by a multiple of 1.7.

In this way, the investor could define $\alpha = 30$, $\beta = [1.4 \ 1 \ 1]$, and $\nu = [1 \ 1.7 \ 1]$ in order to take into consideration the aversion towards risks as stated in the specific example above and to find an optimal allocation using $PC(\nu, \beta, \alpha)$.

Note that the relative weights between the elements of ν and β are also relevant. However, as we will see in Chapter 5, when we impose a total weighted variance α , there must be an adjustment between these relative weights in order to accommodate the new restriction on the variance. Intuitively, if we want to fix a lower total variance it could only be achieved if we accept lower returns and conversely, if we want to fix a higher total variance it could only be achieved if we "accept" higher returns.

Notwithstanding, lower or higher returns can be set by adjusting the coefficients of β as we will see when we present the solution of $PC(\nu, \beta, \alpha)$ in Chapter 5. Thereby, our objective can be rephrased to finding the exactly expected level of the output that leads to the desired total weighted variance restriction, α , which in turn can be achieved by properly adjusting β .

4.4 The unconstrained problem formulation, $PU(\nu, \xi)$

The mean-variance cost function is defined as, for all $u \in \mathbb{U}$,

$$C(u) := \sum_{t=1}^T \left[\nu(t) \text{Var}[y^u(t)] - \xi(t) E[y^u(t)] \right], \quad (4.4)$$

where $\xi = [\xi(1) \cdots \xi(T)]'$, $\xi(t) \geq 0$, is the input parameter associated with the system's output and $\nu(t) \geq 0$ is the input parameter as defined in problem (4.3). In the same way as in problem (4.3), the input parameters ξ and ν can be seen as risk aversion coefficients reflecting a trade-off preference between the expected output and the associated risk (variance) level, respectively.

The optimal control strategy of problem $PC(\nu, \beta, \alpha)$ will be obtained through the so-

lution of the mean-variance unconstrained problem denoted by $PU(\nu, \xi)$ and defined as:

$$PU(\nu, \xi) : \min_{u \in \mathcal{U}} C(u). \quad (4.5)$$

Since problem $PU(\nu, \xi)$ involves a nonlinear function of an expectation term in $\text{Var}[y^u(t)] = E[y^u(t)^2] - E[y^u(t)]^2$, it cannot be directly solved by dynamic programming and the following tractable auxiliary problem is solved instead.

$$A(\nu, \lambda) := \min_{u \in \mathcal{U}} E \left\{ \sum_{t=1}^T \left[\nu(t) y^u(t)^2 - \lambda(t) y^u(t) \right] \right\}, \quad (4.6)$$

where $\lambda = [\lambda(1) \cdots \lambda(T)]'$, $\lambda(t) \geq 0$.

Thus, the solution of problem (4.6) leads to the same solution of problem (4.5) and the reader is referred to (COSTA; OLIVEIRA, 2012) for more detailed information and proofs.

4.5 Mathematical operators

For $k = 0, \dots, T - 1$, $X \in \mathbb{H}^n$, and $i = 1, \dots, N$ the following operators will be useful in the sequel to obtain the optimal control strategy of Problem (4.5) through the solution of Problem (4.6).

- **Expected value operator, $\mathcal{E}(k, \cdot)$:**

$\mathcal{E}(k, \cdot) \in \mathbb{B}(\mathbb{H}^n)$:

$$\mathcal{E}_i(k, X) = \sum_{j=1}^N p_{ij}(k) X_j. \quad (4.7)$$

- **Auxiliary operators, $\mathcal{A}(k, \cdot)$, $\mathcal{G}(k, \cdot)$, and $\mathcal{R}(k, \cdot)$:**

$\mathcal{A}(k, \cdot) \in \mathbb{B}(\mathbb{H}^n)$:

$$\mathcal{A}_i(k, X) = \bar{A}_i(k)' \mathcal{E}_i(k, X) \bar{A}_i(k) + \sum_{s=1}^{\varepsilon} \tilde{A}_{i,s}(k)' \mathcal{E}_i(k, X) \tilde{A}_{i,s}(k). \quad (4.8)$$

$\mathcal{G}(k, \cdot) \in \mathbb{B}(\mathbb{H}^{n,1}, \mathbb{H}^{n,m}) :$

$$\mathcal{G}_i(k, X) = \left[\bar{A}_i(k)' \mathcal{E}_i(k, X) \bar{B}_i(k) + \sum_{s1=1}^{\varepsilon} \sum_{s2=1}^{\varepsilon} \rho_{s1,s2}(k) \tilde{A}_{i,s1}(k)' \mathcal{E}_i(k, X) \tilde{B}_{i,s2}(k) \right]'. \quad (4.9)$$

$\mathcal{R}(k, \cdot) \in \mathbb{B}(\mathbb{H}^n, \mathbb{H}^m) :$

$$\mathcal{R}_i(k, X) = \bar{B}_i(k)' \mathcal{E}_i(k, X) \bar{B}_i(k) + \sum_{s=1}^{\varepsilon} \tilde{B}_{i,s}(k)' \mathcal{E}_i(k, X) \tilde{B}_{i,s}(k). \quad (4.10)$$

- **Operator associated with the feedback gain of the optimal control law, $\mathcal{K}(k, \cdot)$:**

$$\begin{aligned} \mathcal{K}_i(k, X) &= \mathcal{R}_i(k, X)^{\dagger} \mathcal{G}_i(k, X), \text{ and} \\ K_i(k) &= \mathcal{K}_i(k, P(k+1)). \end{aligned} \quad (4.11)$$

- **Operator associated with the generalized coupled Riccati difference equation, $\mathcal{P}(k, \cdot)$:**

$$\begin{aligned} \mathcal{P}_i(k, X) &= \mathcal{A}_i(k, X) - \mathcal{G}_i(k, X)' \mathcal{R}_i(k, X)^{\dagger} \mathcal{G}_i(k, X) + v(k) L_i(k)' L_i(k), \text{ and} \\ P_i(k) &= \mathcal{P}_i(k, P(k+1)). \end{aligned} \quad (4.12)$$

- **Operators related to the presence of the linear term in problem (4.6), $\mathcal{V}(k, \cdot, \cdot)$ and $\mathcal{H}(k, \cdot)$:**

$\mathcal{V}(k, \cdot, \cdot) \in \mathbb{B}(\mathbb{H}^{n,1}, \mathbb{H}^{n,m}) :$

$$\begin{aligned} \mathcal{V}_i(k, X, V) &= \mathcal{E}_i(k, V) [\bar{A}_i(k) - \bar{B}_i(k) \mathcal{K}_i(k, X)] + \lambda(k) L_i(k), \text{ and} \\ V_i(k) &= \mathcal{V}_i(k, P(k+1), V(k+1)). \end{aligned} \quad (4.13)$$

$\mathcal{H}(k, \cdot) \in \mathbb{B}(\mathbb{H}^{n,1}, \mathbb{H}^{n,m}) :$

$$\begin{aligned} \mathcal{H}_i(k, V) &= \bar{B}_i(k)' \mathcal{E}_i(k, V)', \text{ and} \\ H_i(k) &= \mathcal{H}_i(k, V(k+1)). \end{aligned} \quad (4.14)$$

The following definitions are used to obtain a simplified formula for some parame-

ters that will be used in the sequel.

$$\Gamma = \begin{bmatrix} \nu(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu(T) \end{bmatrix}. \quad (4.15)$$

$$A_i^{cl}(k) = \bar{A}_i(k) - \bar{B}_i(k)K_i(k),$$

$$\mathbb{A}(k) = \begin{bmatrix} p_{11}(k)A_1^{cl}(k) & \cdots & p_{N1}(k)A_N^{cl}(k) \\ \vdots & \ddots & \vdots \\ p_{1N}(k)A_1^{cl}(k) & \cdots & p_{NN}(k)A_N^{cl}(k) \end{bmatrix}. \quad (4.16)$$

$$\mathbb{Q}_i(k) = \pi_i(k)\bar{B}_i(k)\mathcal{R}_i(k, P(k+1))^\dagger \bar{B}_i(k)' \geq 0,$$

$$\mathbb{D}(k) = \begin{bmatrix} \mathbb{Q}_1(k) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{Q}_N(k) \end{bmatrix} \geq 0. \quad (4.17)$$

$$\mathbb{Z}(k) = [\mathbb{P}(k) \otimes I]' \mathbb{D}(k) [\mathbb{P}(k) \otimes I] \geq 0. \quad (4.18)$$

$$\mathbb{L}(k) = [L_1(k) \cdots L_N(k)]. \quad (4.19)$$

$$a(t) = \mathbb{L}(t) \left[\prod_{l=0}^{t-1} \mathbb{A}(l) \right] \mu(0), \quad t = 1, \dots, T,$$

$$a = [a(1) \cdots a(T)]'. \quad (4.20)$$

Finally, for $t = 1, \dots, T$ and $1 \leq s \leq t$:

$$\begin{aligned} g(t, s) &= \mathbb{L}(t) \left[\prod_{l=s}^{t-1} \mathbb{A}(l) \right] \mathbb{Z}(s-1)^{\frac{1}{2}}, \\ b(t, s) &= \frac{1}{2} \sum_{l=0}^{s-1} g(t, l+1) g(s, l+1)', \\ \tilde{b}(i, j) &= \begin{cases} b(i, j) & i \geq j \\ b(j, i) & i < j \end{cases}, \\ \tilde{\mathbb{B}} &= \begin{bmatrix} \tilde{b}(1, 1) & \dots & \tilde{b}(1, T) \\ \vdots & \ddots & \vdots \\ \tilde{b}(T, 1) & \dots & \tilde{b}(T, T) \end{bmatrix}. \end{aligned} \quad (4.21)$$

4.6 Previous results

The following results represent the minimum previous knowledge we need to solve problem $PC(\nu, \beta, \alpha)$ using the optimal control law of problem $PU(\nu, \xi)$.

We basically set the conditions and explicit formulas for the optimal control strategy of problem $PU(\nu, \xi)$ and formulas for the expected system's output and total weighted variance. Once more, the reader is referred to (COSTA; OLIVEIRA, 2012) for further details and proofs.

Assumption 1 and Proposition 1 establish the conditions we need in order to state that the solution of the auxiliary problem $A(\nu, \lambda)$ will also be the solution of problem $PU(\nu, \xi)$.

Theorem 1 provides the optimal control law of $PU(\nu, \xi)$, which depends on λ as one can see from Equations (4.13) and (4.14), while Theorem 2 establishes the identity between the parameter λ and the input parameter ξ .

Proposition 2 gives an explicit formula for the expected value of the output and its total weighted variance when the optimal control policy (4.22) is applied.

Assumption 1: For each $i = 1, \dots, N$ and $k = 0, \dots, T-1$, we have that $H_i(k) \in$

$Im(\mathcal{R}_i(k, P(k+1)))$.

Proposition 1: If Assumption 1 holds then $H_i(k) \in Im(\mathcal{R}_i(k, P(k+1)))$ is satisfied for any $\lambda \in \mathbb{R}^T$.

Proof. See Proposition 5 in (COSTA; OLIVEIRA, 2012). $\square$

Theorem 1. If Proposition 1 holds then an optimal control strategy for problem $PU(\nu, \xi)$ is achieved by

$$u(k) = -\mathcal{R}_{\theta(k)}(k, P(k+1))^{\dagger} \left[\mathcal{G}_{\theta(k)}(k, P(k+1))x(k) - \frac{1}{2} \mathcal{H}_{\theta(k)}(k, V(k+1)) \right]. \quad (4.22)$$

Proof. See Theorem 1 in (COSTA; OLIVEIRA, 2012). $\square$

Theorem 2. Suppose that Assumption 1 holds. If $\tilde{\mathbb{B}} - 2\tilde{\mathbb{B}}\Gamma\tilde{\mathbb{B}} > 0$ then an optimal control strategy u^λ for problem $PU(\nu, \xi)$ is given as in (4.22) with

$$\lambda = (I - 2\Gamma\tilde{\mathbb{B}})^{-1}(\xi + 2\Gamma a). \quad (4.23)$$

Proof. See Theorem 2 in (COSTA; OLIVEIRA, 2012). $\square$

Proposition 2: If the control strategy (4.22) is applied to system (4.1) then

$$E[y^\mu(t)] = a(t) + \sum_{s=1}^T \lambda(s) \tilde{b}(t, s). \quad (4.24)$$

Moreover, under Assumption 1,

$$\begin{aligned} \sum_{t=1}^T \nu(t) Var[y^\mu(t)] &= \sum_{i=1}^N tr(P_i(0)Q_i(0)) - \sum_{t=1}^T \lambda(t)a(t) - \frac{1}{2} \sum_{j=1}^T \sum_{i=1}^T \lambda(i)\lambda(j)\tilde{b}(i, j) \\ &+ \sum_{t=1}^T \left\{ \lambda(t) - \nu(t) \left[a(t) + \sum_{s=1}^T \lambda(s)\tilde{b}(t, s) \right] \right\} \left[a(t) + \sum_{s=1}^T \lambda(s)\tilde{b}(t, s) \right]. \end{aligned} \quad (4.25)$$

Notice that the equation above can be rewritten using vectors as shown below.

$$\sum_{t=1}^T \nu(t) Var[y^\mu(t)] = \lambda' \left(\frac{1}{2} I - \tilde{\mathbb{C}}' \right) \tilde{\mathbb{B}} \lambda - 2\eta' \tilde{\mathbb{B}} \lambda + c - \eta' a, \quad (4.26)$$

where we define,

$$c = \sum_{i=1}^N \text{tr}(P_i(0)Q_i(0)), \quad (4.27)$$

$$\eta = [a(1)v(1) \cdots a(T)v(T)]' = \Gamma a, \quad (4.28)$$

$$\tilde{\mathbb{C}} = \begin{bmatrix} \tilde{b}(1,1)v(1) & \cdots & \tilde{b}(1,T)v(1) \\ \vdots & \cdots & \vdots \\ \tilde{b}(T,1)v(T) & \cdots & \tilde{b}(T,T)v(T) \end{bmatrix} = \Gamma \tilde{\mathbb{B}}. \quad (4.29)$$

Proof. See Proposition 7 in (COSTA; OLIVEIRA, 2012) and our appendix A for the vector formulation. $\square$

4.7 Optimal control law algorithm

In order to obtain an optimal control law for $PU(\nu, \xi)$, given the input risk parameters ν and ξ , one should follow the next steps:

1. Provide the set of assets, their returns and standard deviations in accordance with the notations defined in Chapter 3;
2. Set the initial conditions $x(0)$ and $\theta(0)$;
3. Establish convenient ν and ξ parameters according with the investor's risk preferences regarding the system's output and variance;
4. Compute the Riccati Equation (4.12) backwards with $P(T) = \nu(T)(L_1' L_1, \cdots, L_N' L_N)$ and $\nu(0) = 0$;
5. Obtain $\mathcal{K}$ along with $\mathcal{G}$ and $\mathcal{R}$ using Equations (4.9), (4.10), and (4.11);
6. Compute $\tilde{\mathbb{B}}$ using the set of equations from (4.15) to (4.21) in order to obtain λ with (4.23);
7. Compute λ using Equation (4.23);

8. Calculate $V(k)$ backwards using Equation (4.13) and $V(T) = \lambda(T)(L_1, \dots, L_N)$;
9. Compute $\mathcal{H}(k)$ using (4.14);
10. Finally, apply Equation (4.22) to get the optimal control law, $u(k)$, for each $k = 0, \dots, T - 1$.

In Chapter 5, we will show how to obtain the optimal control law for Problem 4.3, $PC(v, \beta, \alpha)$, using the algorithm above by properly fixing the parameter ξ in terms of β and α .

5 MAIN RESULTS

Our goal is to identify the explicit control strategy that maximizes the expected system's output while keeping the total weighted variance of the output restricted by a fixed value, α . This will be achieved here by proving that the optimal control law of Problem (4.3), $PC(\nu, \beta, \alpha)$, can be obtained through the optimal solution of the Unconstrained Problem (4.5), which in turn is achieved through the optimal solution of the Auxiliary Problem (4.6), $A(\nu, \lambda)$.

We start by considering the system as defined by Equations (4.1) and (4.2) and which are reproduced here.

$$x(k+1) = \left[\bar{A}_{\theta(k)}(k) + \sum_{s=1}^{\varepsilon^x} \tilde{A}_{\theta(k),s}(k) w_s^x(k) \right] x(k) + \left[\bar{B}_{\theta(k)}(k) + \sum_{s=1}^{\varepsilon^u} \tilde{B}_{\theta(k),s}(k) w_s^u(k) \right] u(k),$$

$$x(0) = x_0, \theta(0) = \theta_0, \text{ and}$$

$$y(t) = L_{\theta(t)}(t)x(t),$$

where $k = 0, 1, \dots, T-1$ and $t = 1, \dots, T$. See Section 4.1 for further details about the notations and definitions regarding our system.

In order to ease the reading, we also reproduce below the formal definitions of $PC(\nu, \beta, \alpha)$, $PU(\nu, \xi)$, and $A(\nu, \lambda)$ considering the system's output given by Equation (4.2).

$$PC(\nu, \beta, \alpha) : \max_{u \in \mathbb{U}} \sum_{t=1}^T \left[\beta(t) E[y^u(t)] \right], \text{ subject to } \sum_{t=1}^T \nu(t) \text{Var}[y^u(t)] \leq \alpha,$$

$$PU(\nu, \xi) : \min_{u \in \mathbb{U}} \sum_{t=1}^T \left[\nu(t) \text{Var}[y^u(t)] - \xi(t) E[y^u(t)] \right],$$

and

$$A(\nu, \lambda) := \min_{u \in \mathbb{U}} E \left\{ \sum_{t=1}^T \left[\nu(t) y^u(t)^2 - \lambda(t) y^u(t) \right] \right\},$$

where, $\nu = [\nu(1) \cdots \nu(T)]'$, $\nu(t) \geq 0$, is the input parameter associated with the variance of the system's output in all three problems, $\alpha \geq 0$ is the maximum total weighted

variance of the system's output in the $PC(\nu, \beta, \alpha)$ problem, and $\beta = [\beta(1) \cdots \beta(T)]'$, $\beta(t) \geq 0$, $\xi = [\xi(1) \cdots \xi(T)]'$, $\xi(t) \geq 0$, and $\lambda = [\lambda(1) \cdots \lambda(T)]'$, $\lambda(t) \geq 0$, are the input parameters associated with the system's output in the $PC(\nu, \beta, \alpha)$, $PU(\nu, \xi)$, and $A(\nu, \lambda)$ problems, respectively.

In order to use the optimal control formulation of $PU(\nu, \xi)$ that complies with the restriction of $PC(\nu, \beta, \alpha)$, we will fix the parameter ξ through an appropriate relation between ξ , β , and α . In this way, given ν , β , and α , one could compute ξ with the formulation we will develop here and, then, obtain λ with Equation (4.23) and the optimal control using the algorithm described in Section 4.7.

In the remaining paragraphs, we first prove in Proposition 3 that the optimal solution of problem $PU(\nu, \xi)$ will also be an optimal solution of problem $PC(\nu, \beta, \alpha)$ whenever $\xi = m\beta$, with m positive and $\sum_{t=1}^T \nu(t) \text{Var}[y^{u^*}(t)] = \alpha$. Then, in Theorem 3, we formalize how to obtain the explicit optimal control policy of problem $PC(\nu, \beta, \alpha)$.

Proposition 3: Suppose that in problem $PU(\nu, \xi)$ we have that $\xi = m\beta$ holds for some $m \in \mathbb{R}^+$. If u^* is an optimal solution of problem $PU(\nu, \xi)$ such that $\sum_{t=1}^T \nu(t) \text{Var}[y^{u^*}(t)] = \alpha$, then u^* is an optimal solution of problem $PC(\nu, \beta, \alpha)$.

Proof. Suppose by contradiction that u^* is *not* an optimal solution of $PC(\nu, \beta, \alpha)$. On the other hand, suppose u^* is an optimal solution of problem $PU(\nu, \xi)$, i.e.,

$$C(u^*) = \min_{u \in \mathcal{U}} C(u), \quad (5.1)$$

such that

$$\sum_{t=1}^T \nu(t) \text{Var}[y^{u^*}(t)] = \alpha. \quad (5.2)$$

Lets also assume that there exists an optimal solution of $PC(\nu, \beta, \alpha)$, named u , and

$u \in \mathbb{U}$. Hence, due to the optimality of u and from Equation (5.2), we can state that

$$\begin{aligned} \sum_{t=1}^T v(t) \text{Var}[y^u(t)] &\leq \sum_{t=1}^T v(t) \text{Var}[y^{u^*}(t)] \Leftrightarrow \\ \sum_{t=1}^T v(t) \text{Var}[y^u(t)] &\leq \alpha \end{aligned} \quad (5.3)$$

and

$$\sum_{t=1}^T \beta(t) E[y^u(t)] > \sum_{t=1}^T \beta(t) E[y^{u^*}(t)]. \quad (5.4)$$

Therefore, multiplying both sides of the Inequality (5.4) by $m > 0$, and recalling that $\xi = m\beta$, we obtain

$$\begin{aligned} \sum_{t=1}^T m\beta(t) E[y^u(t)] &> \sum_{t=1}^T m\beta(t) E[y^{u^*}(t)] \Leftrightarrow \\ \sum_{t=1}^T \xi(t) E[y^u(t)] &> \sum_{t=1}^T \xi(t) E[y^{u^*}(t)]. \end{aligned} \quad (5.5)$$

Finally, from (4.4) we have that the cost of the control law u for problem $PU(v, \xi)$ is given by

$$C(u) = \sum_{t=1}^T \left[v(t) \text{Var}[y^u(t)] - \xi(t) E[y^u(t)] \right]. \quad (5.6)$$

Substituting (5.3) and (5.5) into (5.6) we have that

$$\begin{aligned} C(u) &\leq \alpha - \sum_{t=1}^T \xi(t) E[y^u(t)] \Leftrightarrow \\ C(u) &< \alpha - \sum_{t=1}^T \xi(t) E[y^{u^*}(t)], \end{aligned} \quad (5.7)$$

and using (5.2) into (5.7) we obtain that

$$C(u) < \sum_{t=1}^T \left[v(t) \text{Var}[y^{u^*}(t)] - \xi(t) E[y^{u^*}(t)] \right] = C(u^*). \quad (5.8)$$

Thus, $C(u) < C(u^*)$ for some $u \in \mathbb{U}$, in contradiction to Hypothesis (5.1). Therefore, u^* is an optimal solution of problem $PC(v, \beta, \alpha)$. $\square$

We will make the following set of definitions in order to present Theorem 3:

$$\begin{aligned}
\mathbb{W}_1 &\in \mathbb{B}(\mathbb{R}^T) : \mathbb{W}_1 = [(I - 2\tilde{\mathbb{C}})^{-1}]' [\frac{1}{2}I - \tilde{\mathbb{C}}'] \tilde{\mathbb{B}}(I - 2\tilde{\mathbb{C}})^{-1}, \\
\mathbb{W}_2 &\in \mathbb{B}(\mathbb{R}, \mathbb{R}^T) : \mathbb{W}_2 = -2a'\tilde{\mathbb{C}}(I - 2\tilde{\mathbb{C}})^{-1}, \\
r_1 &\in \mathbb{R} : r_1 = \beta'\mathbb{W}_1\beta, \\
r_2 &\in \mathbb{R} : r_2 = -(2\eta'\mathbb{W}_1' + 2\eta'\mathbb{W}_1 + \mathbb{W}_2)\beta, \\
r_3 &\in \mathbb{R} : r_3 = 4\eta'\mathbb{W}_1\eta + 2\mathbb{W}_2\eta + c - \alpha - \eta'a, \\
f(\beta, \alpha) &\in \mathbb{R} : f(\beta, \alpha) = \frac{r_2 + \sqrt{r_2^2 - 4r_1r_3}}{2r_1}.
\end{aligned} \tag{5.9}$$

Theorem 3. Suppose Assumption 1 holds, $\tilde{\mathbb{B}} - 2\tilde{\mathbb{B}}\Gamma\tilde{\mathbb{B}} > 0$, and $f(\beta, \alpha) > 0$. Let

$$\xi = f(\beta, \alpha)\beta. \tag{5.10}$$

Then an optimal control strategy u^λ for problem $PC(v, \beta, \alpha)$ is given as in (4.22) with λ as in (4.23) and ξ as in (5.10).

Proof. Given that u^λ is an optimal solution of $PU(v, \xi)$, we have from Equation (4.26) that $\sum_{t=1}^T v(t)Var[y^{u^\lambda}(t)] = \alpha$ is equivalent to

$$\alpha = \lambda' \left(\frac{1}{2}I - \tilde{\mathbb{C}}' \right) \tilde{\mathbb{B}}\lambda - 2\eta'\tilde{\mathbb{B}}\lambda + c - \eta'a. \tag{5.11}$$

It follows that, applying λ from (4.23) into the previous Equation (5.11) we obtain the following quadratic equation with the unknown vector ξ :

$$\begin{aligned}
&\lambda' \left(\frac{1}{2}I - \tilde{\mathbb{C}}' \right) \tilde{\mathbb{B}}\lambda - 2\eta'\tilde{\mathbb{B}}\lambda + c - \alpha - \eta'a = 0 \Leftrightarrow \\
&(\xi' + 2a'\Gamma')[(I - 2\Gamma\tilde{\mathbb{B}})^{-1}]' \left(\frac{1}{2}I - \tilde{\mathbb{C}}' \right) \tilde{\mathbb{B}}(I - 2\Gamma\tilde{\mathbb{B}})^{-1}(\xi + 2\Gamma a) \\
&- 2\eta'\tilde{\mathbb{B}}(I - 2\Gamma\tilde{\mathbb{B}})^{-1}(\xi + 2\Gamma a) + c - \alpha - \eta'a = 0.
\end{aligned} \tag{5.12}$$

After some algebraic manipulation and recalling the definitions of η and $\tilde{\mathbb{C}}$ from

(4.28) and (4.29), respectively, Equation (5.12) becomes

$$\begin{aligned} & \xi'[(I - 2\tilde{C})^{-1}]' \left(\frac{1}{2}I - \tilde{C}' \right) \tilde{B}(I - 2\tilde{C})^{-1}\xi + \xi'[(I - 2\tilde{C})^{-1}]' \left(\frac{1}{2}I - \tilde{C}' \right) \tilde{B}(I - 2\tilde{C})^{-1}2\eta \\ & + 2\eta'[(I - 2\tilde{C})^{-1}]' \left(\frac{1}{2}I - \tilde{C}' \right) \tilde{B}(I - 2\tilde{C})^{-1}\xi + 2\eta'[(I - 2\tilde{C})^{-1}]' \left(\frac{1}{2}I - \tilde{C}' \right) \tilde{B}(I - 2\tilde{C})^{-1}2\eta \\ & - 2a'\tilde{C}(I - 2\tilde{C})^{-1}\xi - 4a'\tilde{C}(I - 2\tilde{C})^{-1}\eta + c - \alpha - \eta'a = 0. \end{aligned} \quad (5.13)$$

Substituting $\mathbb{W}_1$, $\mathbb{W}_2$, and r_3 from (5.9) into Equation (5.13) we obtain

$$\begin{aligned} & \xi'\mathbb{W}_1\xi + 2\xi'\mathbb{W}_1\eta + 2\eta'\mathbb{W}_1\xi + 4\eta'\mathbb{W}_1\eta + \mathbb{W}_2\xi + 2\mathbb{W}_2\eta + c - \alpha - \eta'a = 0 \Leftrightarrow \\ & \xi'\mathbb{W}_1\xi + 2\eta'\mathbb{W}_1'\xi + 2\eta'\mathbb{W}_1\xi + \mathbb{W}_2\xi + r_3 = 0. \end{aligned} \quad (5.14)$$

Assuming ξ is a positive multiple of β , that is $\xi = m\beta$, and substituting the definitions of r_1 and r_2 from (5.9) into (5.14) we have that

$$\begin{aligned} & \beta'\mathbb{W}_1\beta m^2 + 2\eta'\mathbb{W}_1'\beta m + 2\eta'\mathbb{W}_1\beta m + \mathbb{W}_2\beta m + r_3 = 0 \Leftrightarrow \\ & \beta'\mathbb{W}_1\beta m^2 + (2\eta'\mathbb{W}_1' + 2\eta'\mathbb{W}_1 + \mathbb{W}_2)\beta m + r_3 = 0 \Leftrightarrow \\ & r_1 m^2 - r_2 m + r_3 = 0, \end{aligned} \quad (5.15)$$

whose positive solution is $m = f(\beta, \alpha)$ as defined in (5.9).

Therefore, following the algorithm of Section 4.7 by fixing ξ in terms of the input parameters β and α through $\xi = f(\beta, \alpha)\beta$, $f(\beta, \alpha) > 0$, and computing λ with Equation (4.23), we can obtain the optimal control law of problem $PU(v, \xi)$, u^λ , that yields $\sum_{t=1}^T v(t)Var[y^{u^\lambda}(t)] = \alpha$. Hence, from Proposition 3, we have that u^λ is also an optimal solution of $PC(v, \beta, \alpha)$, completing the proof. $\square$

6 NUMERICAL EXAMPLES

In this chapter we illustrate the application of our results in the management of a portfolio of financial assets. In Section 6.1 we describe the input parameters we will consider and the criteria used to calculate them. Section 6.2 shows the simulations' results when we impose different boundaries on the total variance of the portfolio's value and, we also illustrate how the portfolio's value, its variance and the optimal assets allocation change with variations in the risk parameters β and ν .

6.1 Input parameters estimation

In this section we will present the data used to estimate the operation mode transition matrix, $\mathbb{P}(k)$, and the matrices $\bar{A}_{\theta(k)}(k)$, $\tilde{A}_{\theta(k)}(k)$, $\bar{B}_{\theta(k)}(k)$, and $\tilde{B}_{\theta(k)}(k)$ as defined in Chapter 3.

6.1.1 Data series

We considered the following Brazilian securities to be part of our portfolio and a proper index to compute the stochastic operation modes of our market (Table 1).

Table 1 - List of assets.

Ticker ^{1,2}	Description	Sector
IBOV	Bovespa index	-
CDI	Interbank deposit rate	Fixed income
EMBR3	Embraer ON	Aviation OEM
ITUB4	Itau Unibanco PN	Banking
PETR4	Petrobras PN	Oil & Gas
VALE5	Cia. Vale do Rio Doce PN	Metal & Mining

The Bovespa Index (Ibovespa) is a total return index compiled as a weighted average of a theoretical portfolio of stocks and it is designed to gauge the stock market's

¹The CDI data source was the website <https://www.cetip.com.br>.

²The Ibovespa and risky assets' data source was the website <http://www.infomoney.com.br> which just make available data from the BMF & Bovespa.

average performance tracking changes in the prices of the more actively traded and better representative stocks of the Brazilian stock market.

The CDI is the overnight interbank deposit rate and it will be our proxy for the short term risk-free rate due to its high liquidity and low risk. This rate is settled daily and, therefore, it is possible to capture the short term variations in the market expectations about the cost of money.

We chose risky securities from different industries in an attempt to decrease their correlation and improve our diversification and options during optimization. Notwithstanding, all four risky assets figure among the most traded securities in the Bovespa stock exchange and, therefore, are very representative of that market. The prices of the risky assets chosen were adjusted by dividends payments, stock splits, etc, in order to consider their proper historical performance.

All data series range from Jan-08-2001 to May-29-2015, covering 3.564 working days. We contemplated a week as the standard interval of seven calendar days and reduced or increased this period to adjust for holidays. Given the long Brazilian holidays like Carnival and near holidays like Christmas and 1st of January, we reached a set of 740 weeks with a loss of around 12 weeks due to the accumulated adjustments over 14 years and 5 months.

We chose the weekly return and its smoothing, through a 12-week moving average, in order to reduce noise and improve the statistics results as suggested in (KOLLER; GOEDHART; WESSELS, 2005). All prices are in nominal Brazilian reais.

6.1.2 Market's operation modes

The market's operation modes were defined using the Bovespa index depicted in Figure 1 below.

There are some possible approaches to define modes and their probability transition matrix. For instance, one could use econometric models, distributions adjustments as in (OLIVEIRA, 2011), or even simpler approaches without distribution adjustments

Figure 1 - Ibovespa index and its weekly average returns.

Source: Author.

as used in (COSTA; OKIMURA, 2009).

We will follow a mixed but still similar methodology as suggested in (COSTA; OKIMURA, 2009) and (OLIVEIRA, 2011). The main difference of our methodology regards the criteria to split the market return's distribution, keeping the focus on other technical aspects of the resulting operation modes.

Our approach consists by first defining three regions around the market's 12-week average return, R_{avg} , each with width of one standard deviation, σ_R , multiplied by an adjustment factor, a_f . Once we have the three middle regions with the same width, we will automatically be left with two regions at both extremes with no boundaries. Thus, each region will have the following interval:

- Region 1: $] - \infty, R_{avg} - 1.5a_f\sigma_R]$
- Region 2: $]R_{avg} - 1.5a_f\sigma_R, R_{avg} - 0.5a_f\sigma_R]$
- Region 3: $]R_{avg} - 0.5a_f\sigma_R, R_{avg} + 0.5a_f\sigma_R]$
- Region 4: $]R_{avg} + 0.5a_f\sigma_R, R_{avg} + 1.5a_f\sigma_R]$

- Region 5: $]R_{avg} + 1.5a_f\sigma_R, +\infty[$

We then set the adjustment factor in order to obtain extremes regions with enough data points to be considered statistically significant as defined below. Thus, in our example, we proceeded with the Ibovespa's 12-week average returns distribution and defined each operation mode as described below.

- (i) it shall have five operation modes considering scenarios of very low returns ("1 - Stress"), low returns ("2 - Low"), stable returns ("3 - Stable"), high returns ("4 - High"), and very high returns ("5 - Boom"); and
- (ii) every operation mode shall have more than 20 data points or at least 5% of the total data points.

After computing the market's returns and their standard deviation, we set $a_f = 1$ to obtain the intervals of each region as defined above. This led to 55 and 41 data points in the two extreme scenarios respectively, providing them enough data points to be statistically significant as defined in item (ii) above.

The distribution and the operation modes' intervals obtained are shown in Figure 2 and Table 2 below.

Table 2 - Market operation modes.

Mode	Description	Returns' range		Occurrence	Frequency
		<i>Min</i>	<i>Max</i>		
1	Stress	$-\infty$	-1.5%	55	7.4%
2	Low	-1.5%	-0.4%	158	21.4%
3	Stable	-0.4%	0.7%	285	38.5%
4	High	0.7%	1.9%	201	27.2%
5	Boom	1.9%	$+\infty$	41	5.5%

Once established the operation modes, we proceeded with the calculation of the jumping frequencies to and from each mode. The resulting transition probabilities is depicted in Table 3.

Figure 2 - Ibovespa's operation modes distributions.

Source: Author.

Table 3 - Operation modes transition matrix.

Mode	Mode				
	1	2	3	4	5
1	70.9%	29.1%	0.0%	0.0%	0.0%
2	9.5%	67.1%	23.4%	0.0%	0.0%
3	0.4%	12.0%	70.0%	16.9%	0.7%
4	0.0%	0.5%	23.8%	69.7%	6.0%
5	0.0%	0.0%	2.4%	31.7%	65.9%

6.1.3 Expected returns

In Table 4 we show the annual average returns for each security given the operation modes obtained from the procedure described in the previous section.

Overall, the results followed the expected trend of higher returns for bullish markets and lower returns for bearish markets with the exception of the CDI asset in the Stress mode. There, the CDI provided a higher return probably due to a higher demand for safe assets and/or expectation that the fixed income benchmark rate would increase to tackle the potential money outflow from Brazil.

Table 4 - Annual expected returns per market operation mode.

	Mode				
Security	1	2	3	4	5
Ibov	-68.6%	-36.0%	10.7%	81.8%	220.4%
CDI	14.6%	12.4%	13.0%	13.9%	18.8%
EMBR3	-49.8%	-12.3%	9.9%	21.2%	188.3%
ITUB4	-56.3%	-26.1%	22.6%	86.5%	143.3%
PETR4	-70.3%	-43.0%	16.1%	101.5%	136.5%
VALE5	-53.5%	-29.5%	23.5%	72.9%	137.1%

6.1.4 Expected covariances

The expected covariances were calculated using the same set of modes as described previously and their resulting upper triangular matrices are depicted in Tables 5, 6, 7, 8, and 9 below. Overall, we can notice that on average there is an increase in volatility from the Stable/High modes towards the Stress and Boom modes.

Table 5 - Annual expected covariances for mode 1.

	Security					
Security	Ibov	CDI	EMBR3	ITUB4	PETR4	VALE5
Ibov	0.002609	-0.000006	0.002365	0.001177	0.002028	0.002984
CDI		0.000016	-0.000201	0.000016	0.000174	0.000264
EMBR3			0.031981	-0.000757	-0.004895	0.001228
ITUB4				0.003384	0.000300	-0.000247
PETR4					0.012779	0.007728
VALE5						0.013388

Table 6 - Annual expected covariances for mode 2.

	Security					
Security	Ibov	CDI	EMBR3	ITUB4	PETR4	VALE5
Ibov	0.000444	-0.000007	-0.000038	0.000432	0.000463	0.000116
CDI		0.000024	-0.000188	0.000002	0.000116	0.000128
EMBR3			0.010090	0.000527	-0.001179	0.000204
ITUB4				0.002344	0.000159	-0.000476
PETR4					0.009060	0.002828
VALE5						0.004521

Table 7 - Annual expected covariances for mode 3.

Security	Security					
	Ibov	CDI	EMBR3	ITUB4	PETR4	VALE5
Ibov	0.000549	-0.000001	0.000335	0.000684	0.000477	0.000506
CDI		0.000030	-0.000093	0.000053	0.000074	0.000065
EMBR3			0.007809	0.000559	-0.000724	0.000862
ITUB4				0.002799	0.000296	0.000643
PETR4					0.004692	0.000215
VALE5						0.003980

Table 8 - Annual expected covariances for mode 4.

Security	Security					
	Ibov	CDI	EMBR3	ITUB4	PETR4	VALE5
Ibov	0.000463	0.000004	0.000339	0.000388	0.000484	0.000218
CDI		0.000032	0.000115	0.000008	-0.000015	-0.000006
EMBR3			0.007030	0.000209	-0.000507	-0.000134
ITUB4				0.001542	0.000374	-0.000042
PETR4					0.004203	0.000030
VALE5						0.005064

Table 9 - Annual expected covariances for mode 5.

Security	Security					
	Ibov	CDI	EMBR3	ITUB4	PETR4	VALE5
Ibov	0.000523	-0.000037	0.000444	0.000798	0.000460	0.000280
CDI		0.000030	-0.000132	-0.000137	-0.000085	-0.000185
EMBR3			0.010195	-0.000587	-0.002001	-0.000020
ITUB4				0.004025	0.002561	0.000753
PETR4					0.005077	0.001845
VALE5						0.006882

6.2 Simulations' results

The objective in this section is to apply our results to a portfolio of financial securities, described by the System (4.1), and analyze its behavior when we impose a restriction on its total variance. We first simulate a portfolio under different levels of α and then we provide a sensitivity analysis for different risk parameters β and ν .

6.2.1 Simulations for different levels of α

The investments are allocated among one reference asset and four risky assets. The reference security will be the CDI and the risky assets will be EMBR3, ITUB4, PETR4, and VALE5.

We will use the parameters estimated previously to compute the matrices $\bar{A}$, $\tilde{A}$, $\bar{B}$, and $\tilde{B}$ as defined in Chapter 3. The matrix $\bar{A}$ represents the reference asset's return and $\bar{B}$ represents the risky assets' returns. The reference and risky assets' standard deviations matrices are associated with $\tilde{A}$ and $\tilde{B}$ respectively, where $\tilde{B}$ is obtained through the Cholesky decomposition of the covariance matrices.

Note that, while the matrices $\bar{A}$, $\tilde{A}$, $\bar{B}$, and $\tilde{B}$ are given in percentage terms, the system's output, $y(t)$, the portfolio's value, $x(k)$, and the control policy, $u(k)$, are all defined in monetary terms, $R\$$ in our case.

Our investment horizon is set at $T = 20$ weeks and the initial wealth as $x(0) = 1$ monetary unity. The Markov chain will have $N = 5$ operation modes starting at $\theta(0) = 3$ and its transition matrix is defined as in Table 3. The mutual correlation matrix between our multiplicative noises is defined as $\rho_{s1,s2}(k) = I$.

The weekly expected returns and standard deviations are then obtained through Tables 4, 5, 6, 7, 8, and 9 depending on the mode provided by the Markov chain on every $k = 0, \dots, T - 1$.

We then define four scenarios to run our model according to Table 10 below.

Table 10 - Scenarios definition.

Scenario	Problem applied	Risk parameters $t = 1, 2, \dots, 20$			Restriction ($R\$$)
		$\nu(t)$	$\xi(t)$	$\beta(t)$	α
A	$PU(\nu, \xi)$	1.0	1.0	-	-
B	$PC(\nu, \beta, \alpha)$	1.0	-	1.0	50.0
C	$PC(\nu, \beta, \alpha)$	1.0	-	1.0	20.0
D	$PC(\nu, \beta, \alpha)$	1.0	-	1.0	0.1

In the first scenario we solve the unconstrained problem, $PU(\nu, \xi)$, while in the other

three scenarios we solve the $PC(\nu, \beta, \alpha)$ problem for different restrictions on the total variance, α . In this way we can observe how the output, variance, and control policy change from an unconstrained problem to a constrained one with a decreasing total variance boundary.

The first scenario problem, $PU(\nu, \xi)$, is solved by following the steps as described in Section 4.7.

The $PC(\nu, \beta, \alpha)$ problem is solved following the same steps described in Section 4.7, however, we now take the restriction α into consideration by computing ξ using Equation (5.10) and β as defined in Table 10. With this calculated ξ , we obtain λ from Equation (4.23) and the optimal control policy of $PC(\nu, \beta, \alpha)$ using Equation (4.22).

The risk parameters $\nu(t)$, $\xi(t)$, and $\beta(t)$ were assumed to be equal to one in every time step. Thereby, by attributing the same risk relevance among all time steps related to both parameters, we are assuming that the investor has no risk bias towards neither the system's output nor its variance over time. Different risk aversion characteristics would lead to different relative weights among the elements of the risk parameters. In order to show how the results may change with changes in risk perceptions, we provided a sensitivity analysis for our $PC(\nu, \beta, \alpha)$ problem by changing β and ν in Section 6.2.2 below.

Finally, after 10,000 simulations, we obtain the results shown in the following charts and Table 11.

Table 11 - Resulting total variance and other parameters for $PC(\nu, \beta, \alpha)$ problem.

Scenario	r_1	r_2	r_3	$f(\beta, \alpha)$	$\xi(t)$	$\sum_{t=1}^T \nu(t)Var[y^u(t)]$
A	-	-	-	-	1.000	118.2
B	118.2	-3.6e-15	-50	0.651	0.651	50
C	118.2	-3.6e-15	-20	0.411	0.411	20
D	118.2	-3.6e-15	-0.1	0.029	0.029	0.1

As expected, for all last three scenarios, the solution of $PC(\nu, \beta, \alpha)$ problem led to $\sum_{t=1}^T \nu(t)Var[y^u(t)] = \alpha$ by finding a proper $f(\beta, \alpha)$.

The output of the system is shown in Figure 3 below where we can see that an

imposed lower total variance led to lower returns in every time step.

Figure 3 - System's output for all scenarios.

Source: Author.

The variance over time for all scenarios is shown in Figure 4. In all cases we can see that the shape of the curve smooths gradually as the restriction α decreases, which means that although our restriction targets the total variance, we are able to lower the variance accordingly in each step providing that we have the same risk input parameters in each scenario.

The control policies shown in Figures 5, 6, 7, and 8 reveal that the introduction of a lower restriction on the variance forces the solution to distribute a lower volume in all risky assets while keeping almost the same relative allocation among them.

On the other hand, the lower volume in the risky assets is now counter-balanced by a reduction in the borrowings using the CDI. Notice that the CDI changes from an instrument of leverage to an investment in the later periods in scenario D, (Figure 8.

The results suggest that the proportional optimal allocation is barely affected by the total variance restriction and that the minimization of the total variance is done by managing the allocation in the risk-free asset.

Figure 4 - System's output variance for all scenarios.

Source: author.

In the limit, we will obtain a portfolio almost fully allocated in the CDI as our restriction α approaches zero. The only reason it does not reach 100% allocation in the CDI is because we have considered the correlations between the CDI and the other assets different than zero, which implies there will be no control policy for $\alpha = 0$ in our example.

The numerical data analyzed in this chapter can be found in the appendix B

Figure 5 - Control policy for scenario A.

Source: author.

Figure 6 - Control policy for scenario B.

Source: Author.

Figure 7 - Control policy for scenario C.

Source: Author.

Figure 8 - Control policy for scenario D.

Source: Author.

6.2.2 Sensitivity analysis for β and ν

In the previous sections we described how to obtain the optimal allocation of assets in a portfolio using our results and also provided a sensitivity analysis regarding the restriction α . Now, our goal is to illustrate how the portfolio's value, its variance and optimal assets allocation changes with variations in one single element of both risk parameters β and ν .

In this section, we simulate an arbitrary scenario where we say the investor wishes to increase the portfolio's value around step $t = t_a$ and decrease the portfolio's volatility around another step $t = t_b$, which is achieved by adjusting the values of $\beta(t_a)$ and $\nu(t_b)$, respectively. The other way around or any other combination of possible scenarios would lead to the same conclusions regarding the results' sensitivity and, therefore, were omitted in this analysis. Nonetheless, in the Appendix C, we provide further sensitivity analysis for a diverse set of combinations of possible variations between the elements of β and ν in order to illustrate the generalization of the conclusions laid out here and summarized in the next section.

The portfolio's assets and their characteristics are the same as described earlier in Section 6.1. In order to facilitated the comparison among all scenarios, we will solve the $PC(\nu, \beta, \alpha)$ problem for the same arbitrary restriction, $\alpha = 100$, in all scenarios and run 10,000 simulations to compute the results for each of them.

We start with a base scenario, *cc*, where the vectors β and ν have all unitary elements and then we apply a variation in a single element of each vector to assess their effects on the portfolio's results. In order to ease the graphical observation of the variations in the results, we arbitrarily pick two specific time steps that are "sufficiently" distant from each other and, in this case, we chose $t_a = 4$ and $t_b = 9$. Then we set "big" enough new elements for $\beta(4)$ and $\nu(9)$ in order to get visible changes in our charts. After some try and error, we choose $\{3, 5, 7, 9, 11\}$ as the set of possible new coefficients' values for illustration purposes and label the following scenarios using the above rational, see Table 12.

We will focus the analysis based on four scenarios, *cc*, *beta7*, *nu7*, and

Table 12 - Scenarios that combine specific changes in the coefficients of β and ν and a base scenario cc .

Scenario	Problem applied	Input parameter			Resulting $f(\beta, \alpha)$
		$\beta(4)$	$\nu(9)$	α	
cc	$PC(\nu, \beta, \alpha)$	1	1	100	0.920
beta7	$PC(\nu, \beta, \alpha)$	7	1	100	0.819
nu7	$PC(\nu, \beta, \alpha)$	1	7	100	1.050
COMBINED-3	$PC(\nu, \beta, \alpha)$	3	3	100	0.939
COMBINED-5	$PC(\nu, \beta, \alpha)$	5	5	100	0.937
COMBINED-7	$PC(\nu, \beta, \alpha)$	7	7	100	0.925
COMBINED-9	$PC(\nu, \beta, \alpha)$	9	9	100	0.907
COMBINED-11	$PC(\nu, \beta, \alpha)$	11	11	100	0.887

Note: $\beta(t) = 1$ and $\nu(t) = 1$ for all the remaining t , $t \in \{1, \dots, 20\} - \{4, 9\}$.

COMBINED – 7. This will allow us to see the specific effects around the scenario cc by first changing $\beta(4)$ to 7 in scenario $beta7$, by changing $\nu(9)$ to 7 in scenario $nu7$, and then combining the effects of both changes in scenario *COMBINED – 7*. The remaining scenarios will only be used to illustrate the sensitivity of the results of scenario *COMBINED – 7* for different levels of $\beta(4)$ and $\nu(9)$.

Figures 9 and 10 show the portfolio's value and its variance for the four scenarios described above. As one can easily observe in the curve of $beta7$ versus cc , an increase in $\beta(4)$ leads to an increase in the portfolio's value from $t = 1$ up to $t = 4$ followed by a correspondent increase in the variance. After $t = 4$, β decreases back to 1 and the portfolio's value and its variance decreases.

In scenario $nu7$, we can see that an increase in $\nu(9)$ leads to a decrease in the variance from $t = 3$ up to $t = 9$ followed by a correspondent decrease in the portfolio's value. Then, at $t = 10$, ν decreases back to 1 which leads to the opposite effect just described.

Note that, given we imposed a restriction on the total variance for all scenarios, $\alpha = 100$, the higher variances obtained until $t = 4$ in scenario $beta7$ will be compensated with lower variances in the later periods when compared with those from the base scenario cc . The same reasoning applies for scenario $nu7$, where the restriction $\alpha = 100$

will force a compensation in the later periods leading them to show higher variances when compared with those from the base scenario *cc*. The correspondent returns in the later periods will follow their variances with the same trend of higher or lower values. However, the accumulated returns will also influence the final results and it explains why scenario *beta7* showed a higher return than the one of scenario *nu7* after $t = 10$.

As we can see in scenario *COMBINED* – 7, the combined effect of $\beta(4) = 7$ and $\nu(9) = 7$ leads to the same trends regarding the output and its variance as stated above. However, now there are some compensation on the final results due to the opposite effects of each variation.

Note that the parameters $\alpha = 100$, $\beta(4) = 7$, and $\nu(9) = 7$ will affect all the values of the portfolio and their volatility over the whole time period, $T = 20$. However, the strongest effect of $\beta(4)$ and $\nu(9)$ are around $t = 4$ and $t = 9$, respectively, which means that it is possible to be specific about the desired relevance of the portfolio's value and its volatility on each time step even though we can not be specific about the combined result a priori.

It is also worth to mention that by setting a coefficient seven times higher than any other coefficient does not mean achieving a portfolio's value seven times higher or a volatility seven times lower than otherwise. In our example, the portfolio's value increased from around 5 to 6 by choosing $\beta(4) = 7$, see curves *cc* and *beta7* in Figure 9 at $t = 4$.

Figure 9 - System's output for scenario *COMBINED* – 7, *beta7*, *nu7*, and base scenario *cc*.

Source: Author.

Figure 10 - System's output variance for scenario *COMBINED* – 7, *beta7*, *nu7*, and base scenario *cc*.

Source: Author.

Looking at the wealth allocation for scenarios *COMBINED* – 7 versus *cc* in Figure 11, we can see that setting a higher expected portfolio's value through $\beta(4) = 7$ leads to an increase in the wealth allocated in the risk assets and to a lower asset allocation in the risk-free asset up to $t = 4$. In the same way, setting a lower volatility at $t = 9$ through $\nu(9) = 7$ forces the opposite effect on the wealth allocation.

Note that, after the point $t = 9$, the asset allocation tends to the one of scenario *cc* because the values $\beta(4) = 7$ and $\nu(9) = 7$ coincidentally counter-balanced each other in this short period of time. However, this is not necessarily true if we choose other values for α , $\beta(t)$, and $\nu(t)$.

Figure 11 - Control policy for scenario *COMBINED* – 7 versus the base scenario *cc*.

Source: Author.

The results for the remaining scenarios of Table 12 will lead to an identical conclusion to the one described above and, therefore, we will only show their results in Figures 12 and 13 to illustrate the sensitivity of the output and the variance around scenario *cc* for different values of $\beta(4)$ and $\nu(9)$. Note that the variations in the portfolio's value and in its variance are not proportional to the changes in the coefficients, illustrating our comments above.

Figure 12 - System's output for scenario COMBINED-5,6,7,8,9 and base scenario *cc*.

Source: Author.

Figure 13 - System's output variance for scenario COMBINED-5, 6, 7, 8, 9 and base scenario *cc*.

Source: Author.

6.2.3 Sensitivity analysis summary and final comments

Following the above discussion and recalling the effects of α from the analysis carried out in Section 6.2.1, we can summarize the effects that β , ν , and α will have on the expected system's output, its variance, and on the wealth allocation policy, see Table 13.

Table 13 - Effects of changes in β , ν , and α on the expected portfolio's value, its variance, and on the wealth allocation policy.

Risk parameter	$E[y^u(t)]$	$Var[y^u(t)]$	Wealth allocation on	
			risk assets	risk-free asset
$\beta(t) \uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$
$\beta(t) \downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$
$\nu(t) \uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$
$\nu(t) \downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$
$\alpha \uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$
$\alpha \downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$

Note that these general effects are valid for variations within the elements of β and ν as well as variations between the cross elements of those vectors.

Nonetheless, there is no standard or scientific procedure to determine the exactly coefficients that someone should use as inputs for our risk parameters. In order to understand and recognize the investor's risk-aversion/risk-taking sentiment and the trade-offs between β , ν , and α , we suggest a similar analysis to the one developed in this chapter, where through modeling and testing different scenarios, someone would be able to understand the potential gains and losses involved and what would be the proper parameters inputs that represent his/her boundary conditions regarding returns and variances over time.

7 CONCLUSION

In this work we have considered the stochastic multi-period optimal control problem of discrete-time linear systems subject to Markov jumps and multiplicative noises. The constrained problem analyzed, $PC(\nu, \beta, \alpha)$, consisted in identifying the system's control law that maximizes the system's output while restricting its total weighted variance by the input parameter, α .

We provided an analytical solution of the $PC(\nu, \beta, \alpha)$ problem using the optimal control policy of the unconstrained problem, $PU(\nu, \xi)$. This was achieved through the variance equation, which allowed us to establish an identity between ξ and α by defining ξ as a multiple of β . Thus, the solution of $PC(\nu, \beta, \alpha)$ extends previous results in the literature regarding our system by allowing it to be controlled with a maximum total weighted variance, what was not possible before. Explicit conditions for the existence of an optimal control strategy for the $PC(\nu, \beta, \alpha)$ problem and an explicit way of calculating it were presented in Chapter 5.

To illustrate the application of our results, we simulated a portfolio selection problem with four risk assets and one risk-free security under the restriction of an arbitrary total weighted variance limit. We also simulated the unconstrained problem, $PU(\nu, \xi)$, for comparison purposes, and provided a sensitivity analysis for different risk parameters α , ν , and β .

Regarding future developments we would consider as relevant the following topics:

1. Restriction on the variance in each time period. Providing an analytical solution to the control law that takes into consideration a restriction on the maximum variance per period of time would be a significant improvement over our results and it would also make the model more suitable for practical applications.
2. Restriction on the maximum and minimum limits allocated per asset. For a portfolio manager, imposing such restrictions seems vital to the proper use of the our system given that he/she usually has strict investment policies that limit the

portfolio exposure to specific assets, countries, currencies, leverage, etc.

3. Consideration of infinite time horizon. Pension funds is the classical example of a type of investor which has an infinity time horizon and, therefore, a formulation that takes this characteristic into consideration would be relevant to broaden the kind of problem our model can handle and to spread its use with major investors.
4. Improvements on the methodology to estimate the operation modes and input parameters. As we have cited, there are different approaches on how to estimate the operation modes and input parameters, however, a blind application of them may lead to undesired results. It would be of interest to develop and consolidate the conditions and hypothesis that must guide anyone on the proper calculation of our input parameters. For instance, parameters with non-stationary statistics invalidate their use and, therefore, identifying these conditions seems to be very relevant for any practical application of our model.

APPENDIX A – DEMONSTRATION OF THE VECTOR FORMULATION IN THEOREM 2

In this section we want to prove that the Equation (4.25) can be rewritten in the following useful vector form:

$$\sum_{t=1}^T v(t) \text{Var}[y''(t)] = \lambda' \left(\frac{1}{2} I - \tilde{\mathbf{C}}' \right) \tilde{\mathbf{B}} \lambda - 2\eta' \tilde{\mathbf{B}} \lambda + c - \eta' a. \quad (\text{A.1})$$

where

$$c = \sum_{i=1}^N \text{tr}(P_i(0)Q_i(0)), \quad (\text{A.2})$$

$$\eta = [a(1)v(1) \cdots a(T)v(T)]' = \Gamma a, \text{ and} \quad (\text{A.3})$$

$$\tilde{\mathbf{C}} = \begin{bmatrix} \tilde{b}(1,1)v(1) & \cdots & \tilde{b}(1,T)v(1) \\ \vdots & \cdots & \vdots \\ \tilde{b}(T,1)v(T) & \cdots & \tilde{b}(T,T)v(T) \end{bmatrix} = \Gamma \tilde{\mathbf{B}}. \quad (\text{A.4})$$

Proof. The demonstration consists in expanding each of the three terms of Equation (4.25) and rewriting each of them using the necessary definitions.

Expanding term 1 and applying the definitions of a and λ , we obtain that

$$\sum_{t=1}^T \lambda(t)a(t) = \lambda(1)a(1) + \cdots + \lambda(T)a(T) = a' \lambda. \quad (\text{A.5})$$

Expanding term 2 and applying the definitions of $\tilde{\mathbb{B}}$ and λ , we obtain that

$$\begin{aligned}
 \sum_{j=1}^T \sum_{i=1}^T \lambda(j) \lambda(i) \tilde{b}(i, j) &= \left\{ \lambda(1) [\lambda(1) \tilde{b}(1, 1) + \dots + \lambda(T) \tilde{b}(T, 1)] + \right. \\
 &\quad + \lambda(2) [\lambda(1) \tilde{b}(1, 2) + \dots + \lambda(T) \tilde{b}(T, 2)] + \\
 &\quad \left. \dots + \lambda(t) [\lambda(1) \tilde{b}(1, T) + \dots + \lambda(T) \tilde{b}(T, T)] \right\} = \\
 &= \lambda' \tilde{\mathbb{B}} \lambda.
 \end{aligned} \tag{A.6}$$

Expanding term 3, we have that

$$\begin{aligned}
 \sum_{t=1}^T \left\{ \lambda(t) - \nu(t) \left[a(t) + \sum_{s=1}^T \lambda(s) \tilde{b}(t, s) \right] \right\} \left[a(t) + \sum_{s=1}^T \lambda(s) \tilde{b}(t, s) \right] \\
 = \left\{ \lambda(1) - \nu(1) [a(1) + (\lambda(1) \tilde{b}(1, 1) + \dots + \lambda(T) \tilde{b}(1, T))] \right\} \times \\
 \times [a(1) + (\lambda(1) \tilde{b}(1, 1) + \dots + \lambda(T) \tilde{b}(1, T))] \\
 + \left\{ \lambda(2) - \nu(2) [a(2) + (\lambda(1) \tilde{b}(2, 1) + \dots + \lambda(T) \tilde{b}(2, T))] \right\} \times \\
 \times [a(2) + (\lambda(1) \tilde{b}(2, 1) + \dots + \lambda(T) \tilde{b}(2, T))] \\
 + \dots + \\
 + \left\{ \lambda(T) - \nu(T) [a(T) + (\lambda(1) \tilde{b}(T, 1) + \dots + \lambda(T) \tilde{b}(T, T))] \right\} \times \\
 \times [a(T) + (\lambda(1) \tilde{b}(T, 1) + \dots + \lambda(T) \tilde{b}(T, T))].
 \end{aligned}$$

Applying the definitions of η and $\tilde{\mathbb{C}}$ into the above equation, we obtain that

$$\begin{aligned}
 \sum_{t=1}^T \left\{ \lambda(t) - \nu(t) \left[a(t) + \sum_{s=1}^T \lambda(s) \tilde{b}(t, s) \right] \right\} \left[a(t) + \sum_{s=1}^T \lambda(s) \tilde{b}(t, s) \right] &= \\
 = [\lambda - \eta - \tilde{\mathbb{C}} \lambda]' [a + \tilde{\mathbb{B}} \lambda] \\
 = [\lambda' - \eta' - \lambda' \tilde{\mathbb{C}}'] [a + \tilde{\mathbb{B}} \lambda] \\
 = \lambda' a - \eta' a - \lambda' \tilde{\mathbb{C}}' a + \lambda' \tilde{\mathbb{B}} \lambda - \eta' \tilde{\mathbb{B}} \lambda - \lambda' \tilde{\mathbb{C}}' \tilde{\mathbb{B}} \lambda \\
 = -\eta' a + (a' - a' \tilde{\mathbb{C}} - \eta' \tilde{\mathbb{B}}) \lambda + \lambda' (\tilde{\mathbb{B}} - \tilde{\mathbb{C}}' \tilde{\mathbb{B}}) \lambda.
 \end{aligned} \tag{A.7}$$

Noticing that

$$\begin{aligned}
 a' \tilde{\mathbf{C}} &= \begin{bmatrix} \tilde{b}(1, 1)a(1)v(1) + \dots + \tilde{b}(T, 1)a(T)v(T) \\ \vdots \\ \tilde{b}(1, T)a(1)v(1) + \dots + \tilde{b}(T, T)a(T)v(T) \end{bmatrix} \\
 &= \eta' \tilde{\mathbf{B}},
 \end{aligned} \tag{A.8}$$

Equation (A.7) can be rewritten as

$$\begin{aligned}
 \sum_{t=1}^T \left\{ \lambda(t) - v(t) \left[a(t) + \sum_{s=1}^T \lambda(s) \tilde{b}(t, s) \right] \right\} \left[a(t) + \sum_{s=1}^T \lambda(s) \tilde{b}(t, s) \right] \\
 = -\eta' a + (a' - a' \tilde{\mathbf{C}} - \eta' \tilde{\mathbf{B}}) \lambda + \lambda' (\tilde{\mathbf{B}} - \tilde{\mathbf{C}}' \tilde{\mathbf{B}}) \lambda \\
 = -\eta' a + a' \lambda - 2\eta' \tilde{\mathbf{B}} \lambda + \lambda' (\tilde{\mathbf{B}} - \tilde{\mathbf{C}}' \tilde{\mathbf{B}}) \lambda.
 \end{aligned} \tag{A.9}$$

Substituting (A.5), (A.6) and (A.9) into (4.25), we have that

$$\begin{aligned}
 \sum_{t=1}^T v(t) \text{Var}[y''(t)] &= c - a' \lambda - \frac{1}{2} \lambda' \tilde{\mathbf{B}} \lambda - \eta' a + a' \lambda - 2\eta' \tilde{\mathbf{B}} \lambda + \lambda' (\tilde{\mathbf{B}} - \tilde{\mathbf{C}}' \tilde{\mathbf{B}}) \lambda \\
 \Leftrightarrow \sum_{t=1}^T v(t) \text{Var}[y''(t)] &= \lambda' \left(\frac{1}{2} I - \tilde{\mathbf{C}}' \right) \tilde{\mathbf{B}} \lambda - 2\eta' \tilde{\mathbf{B}} \lambda + c - \eta' a,
 \end{aligned} \tag{A.10}$$

completing the demonstration. $\square$

APPENDIX B – NUMERICAL DATA OF SIMULATIONS OF SECTION 6.2.1

Table 14 - System's output for all scenarios.

Time	Scenario 1	Scenario 2	Scenario 3	Scenario 4
1	1.3	1.2	1.1	1.0
2	2.4	1.9	1.6	1.0
3	3.7	2.8	2.1	1.1
4	5.2	3.7	2.7	1.2
5	6.7	4.7	3.4	1.2
6	8.2	5.7	4.0	1.3
7	9.7	6.7	4.6	1.3
8	11.1	7.6	5.2	1.4
9	12.4	8.4	5.7	1.4
10	13.6	9.2	6.2	1.4
11	14.7	9.9	6.7	1.5
12	15.7	10.6	7.1	1.5
13	16.7	11.2	7.5	1.5
14	17.5	11.7	7.8	1.5
15	18.3	12.2	8.1	1.6
16	18.9	12.7	8.4	1.6
17	19.5	13.1	8.6	1.6
18	20.0	13.4	8.9	1.6
19	20.5	13.7	9.0	1.6
20	20.9	13.9	9.2	1.6

Table 15 - System's output variance for all scenarios.

Time	Scenario 1	Scenario 2	Scenario 3	Scenario 4
1	0.86	0.37	0.15	0.0007
2	3.54	1.50	0.60	0.003
3	6.19	2.62	1.05	0.005
4	8.25	3.49	1.40	0.007
5	9.58	4.05	1.62	0.008
6	10.21	4.32	1.73	0.009
7	10.30	4.36	1.74	0.009
8	9.97	4.22	1.69	0.008
9	9.37	3.96	1.58	0.008
10	8.58	3.63	1.45	0.007
11	7.70	3.25	1.30	0.007
12	6.78	2.87	1.15	0.006
13	5.87	2.48	0.99	0.005
14	5.01	2.12	0.85	0.004
15	4.20	1.78	0.71	0.004
16	3.48	1.47	0.59	0.003
17	2.84	1.20	0.48	0.002
18	2.28	0.97	0.39	0.002
19	1.81	0.77	0.31	0.002
20	1.42	0.60	0.24	0.001

Table 16 - Control policy for scenario A.

Time	CDI	EMBR3	ITUB4	PETR4	VALE5
0	-137.9	-24.9	87.1	9.8	66.9
1	-138.7	-21.8	88.5	15.5	57.8
2	-136.0	-20.0	87.3	17.4	53.6
3	-126.9	-18.2	82.3	17.4	49.1
4	-116.2	-16.5	76.4	17.0	44.5
5	-109.6	-15.5	72.9	16.3	42.6
6	-100.9	-14.3	68.3	15.6	39.6
7	-94.8	-13.3	65.1	14.8	37.8
8	-87.0	-12.0	60.9	14.2	35.0
9	-73.2	-11.0	53.5	12.5	30.6
10	-65.9	-10.0	49.4	11.5	28.5
11	-51.4	-9.2	41.5	9.6	24.2
12	-43.2	-8.4	37.0	8.3	22.0
13	-37.8	-7.2	34.1	7.9	19.7
14	-29.9	-6.4	29.7	6.9	17.2
15	-23.2	-5.4	25.9	6.1	14.8
16	-16.6	-4.6	22.2	5.2	12.8
17	-12.7	-3.9	20.0	4.7	11.4
18	-7.0	-3.3	16.8	3.9	9.7
19	-1.6	-2.7	13.7	3.3	7.7

Table 17 - Control policy for scenario B.

Time	CDI	EMBR3	ITUB4	PETR4	VALE5
0	-89.3	-16.2	56.6	6.4	43.5
1	-89.3	-14.0	57.2	10.2	37.1
2	-85.8	-12.9	55.4	11.0	34.2
3	-80.5	-11.8	52.6	11.1	31.5
4	-76.7	-11.1	50.5	11.0	30.0
5	-74.3	-10.3	49.4	11.0	28.9
6	-63.1	-9.4	43.2	9.7	25.4
7	-57.4	-8.6	40.2	9.2	23.3
8	-50.3	-7.9	36.3	8.4	21.1
9	-45.8	-7.2	33.9	7.9	19.5
10	-38.0	-6.6	29.6	6.7	17.4
11	-35.3	-5.8	28.2	6.6	16.3
12	-29.3	-5.3	25.0	5.8	14.4
13	-24.1	-4.6	22.0	5.2	12.7
14	-19.0	-4.2	19.2	4.3	11.4
15	-14.9	-3.6	17.0	3.9	9.9
16	-10.5	-3.1	14.4	3.3	8.5
17	-7.1	-2.6	12.6	3.0	7.2
18	-4.4	-2.1	11.0	2.6	6.2
19	-0.3	-1.7	8.7	2.1	4.9

Table 18 - Control policy for scenario C.

Time	CDI	EMBR3	ITUB4	PETR4	VALE5
0	-56.1	-10.2	35.8	4.0	27.5
1	-57.5	-9.0	36.9	6.5	24.1
2	-54.8	-8.3	35.5	7.0	22.0
3	-51.8	-7.5	34.0	7.2	20.3
4	-48.8	-7.0	32.4	7.1	19.1
5	-44.2	-6.3	29.9	6.8	17.2
6	-40.5	-5.8	27.9	6.4	16.1
7	-36.3	-5.3	25.6	5.9	14.6
8	-32.6	-4.9	23.6	5.5	13.6
9	-28.4	-4.6	21.4	4.9	12.4
10	-26.0	-4.2	20.1	4.6	11.6
11	-21.7	-3.7	17.8	4.1	10.2
12	-18.1	-3.4	15.8	3.6	9.1
13	-15.7	-2.9	14.4	3.4	8.2
14	-12.1	-2.6	12.4	2.9	7.1
15	-9.2	-2.2	10.8	2.5	6.3
16	-5.5	-1.9	8.7	2.1	5.0
17	-3.8	-1.6	7.7	1.8	4.5
18	-1.8	-1.3	6.6	1.6	3.8
19	0.2	-1.1	5.5	1.3	3.1

Table 19 - Control policy for scenario D.

Time	CDI	EMBR3	ITUB4	PETR4	VALE5
0	-3.0	-0.7	2.5	0.3	1.9
1	-3.1	-0.6	2.6	0.5	1.7
2	-2.9	-0.6	2.5	0.5	1.5
3	-2.7	-0.5	2.4	0.5	1.4
4	-2.5	-0.5	2.3	0.5	1.3
5	-2.2	-0.5	2.1	0.5	1.3
6	-2.0	-0.4	2.0	0.5	1.2
7	-1.8	-0.4	1.9	0.4	1.1
8	-1.5	-0.3	1.7	0.4	1.0
9	-1.1	-0.3	1.5	0.5	0.9
10	-0.9	-0.3	1.4	0.3	0.8
11	-0.6	-0.3	1.2	0.3	0.7
12	-0.3	-0.2	1.1	0.3	0.6
13	-0.1	-0.2	1.0	0.2	0.6
14	0.2	-0.2	0.9	0.2	0.5
15	0.3	-0.2	0.8	0.2	0.4
16	0.5	-0.1	0.7	0.2	0.4
17	0.7	-0.1	0.6	0.1	0.3
18	0.9	-0.9	0.5	0.1	0.3
19	1.0	-0.1	0.4	0.1	0.2

APPENDIX C – FURTHER SENSITIVITY ANALYSIS OF β AND ν

In this appendix we extend the sensitivity analysis of the risk parameters β and ν we provided in Chapter 6. Now, we define other scenarios depending on the different relative values of β and ν illustrating the results generalized in Table 13.

As previously stated, this type of sensitivity should help the investor to understand how the portfolio's variables may behave given his/her risk preferences and to assist him/her on the task of choosing the coefficients of β and ν .

The portfolio's assets and their characteristics are the same as described earlier in Section 6.1. In order to facilitated the comparison among all scenarios, we will solve the $PC(\nu, \beta, \alpha)$ problem for the same arbitrary restriction, α , in all scenarios and run 10,000 simulations to compute the results for each of them. As described in Chapter 5, solving this problem implies that, given the risk parameters β and ν and a maximum total variance, α , one could obtain the optimal control, $u(k)$, following the steps given in Section 4.7 and computing ξ using Equation (5.10).

We also arbitrarily define our risk parameters in order to keep the different results in a scale that would ease the observation of trends in the charts below. Notwithstanding, the analysis made here will be valid for different boundaries and restrictions that the investor may choose regarding the risk parameters as long as they do not violate the limits of our hypothesis laid out in the theorems and propositions of Chapters 4 and 5.

Thus, we set $\alpha = 100.0$ for all scenarios as before and define different scenarios by considering that the elements of β and ν may assume three possible behaviors over time: i) stay at a constant level equal to 1 or to a multiple of 1, ii) go up linearly from 1

to 10, or iii) go down linearly from 10 to 1.

We will split the analysis into the following sections. Section C.1 shows the portfolio's behavior when we change the vectors β and ν by multiples and, in Section C.2, we simulate the portfolio's behavior for different combinations of β and ν assuming they can either have a constant shape, an upward trend shape, or a downward trend shape over time.

C.1 Simulations for different constant levels of β and ν

In this section, our goal is to analyze what happens with the portfolio's behavior when we change a given vector β or ν by multiples of themselves. We will consider four scenarios where all elements of β and ν have constant and identical elements equal to 1, 2, 4, or 8. Then we solve the $PC(\nu, \beta, \alpha)$ problem to obtain the optimal control law in each scenario.

The scenarios labels and inputs are shown in Table 20 while in Figure 14 we show an illustration of the values that the elements of β and ν take over time in each scenario.

Table 20 - Scenarios when β and ν are constants and resulting $f(\beta, \alpha)$ of problem $PC(\nu, \beta, \alpha)$.

Scenario	Problem applied	Risk parameters' behavior for $t = 1, 2, \dots, T$			Resulting $f(\beta, \alpha)$
		$\beta(t)$	$\nu(t)$	α	
cc	$PC(\nu, \beta, \alpha)$	1	1	100	0.920
cc2	$PC(\nu, \beta, \alpha)$	2	2	100	0.650
cc4	$PC(\nu, \beta, \alpha)$	4	4	100	0.460
cc8	$PC(\nu, \beta, \alpha)$	8	8	100	0.325

Figure 14 - Illustrative diagram for the scenarios of Table 20.

Source: Author.

We provide below a short analysis of the expected effects of different levels of both β and ν to solve the $PC(\nu, \beta, \alpha)$ problem for the same α .

Given that solving $PC(\nu, \beta, \alpha)$ problem leads to a unique ξ , solving it to a multiple of β , for instance, $m\beta$, would just lead to an adjustment of $f(\beta, \alpha)$ by m in order keep the same ξ . It happens because we assumed ξ as a multiple of our input vector β , $\xi = f(\beta, \alpha)\beta$. Hence,

$$\begin{aligned}\xi \text{ of } PC(\nu, \beta, \alpha) &= \xi \text{ of } PC(\nu, m\beta, \alpha) \\ \Leftrightarrow \xi &= f(\beta, \alpha)\beta = f_{new}(\beta, \alpha)m\beta,\end{aligned}$$

and in this case, the resulting $f_{new}(\beta, \alpha)$ is $f(\beta, \alpha)/m$.

Therefore, in the scenarios of Table 20, changing the level of β by a multiple will have no effect on the results because it will lead to the same ξ and control policy.

Regarding the portfolio's volatility, provided that $\sum_{t=1}^T \nu(t)Var[y''(t)] = 100$ for all scenarios, by increasing the level of ν , we are forcing an optimal control strategy that

achieves a lower total variance. Note that it is the total **weighted** variance that has been kept constant, not the total variance alone. Thus, setting $\alpha = 100$ and $\nu(t) = 1, \forall t$, leads to the same results of setting $\alpha = 50$ and $\nu(t) = 2$, or $\alpha = 25$ and $\nu(t) = 4$, etc.

Therefore, given that any multiple of β does not change the results and that higher multiples of ν imply in solving an equivalent problem where α decreases, the resulting portfolio's behavior of the scenarios in Table 20 will follow the same behavior as described in Section 6.2.

Thereby, as one can see in Figures 15 and 16, the portfolio's value gets lower as ν increases. Once more, a lower total variance is achieved indirectly in these particular scenarios because $\sum_{t=1}^T \nu(t) \text{Var}[y^u(t)] = 100$ is kept constant while ν increases.

Figure 15 - System's output when β and ν are constants.

Source: Author.

Figure 16 - System's output variance when β and ν are constants.

Source: Author.

In the same way as in Section 6.2, the imposition of a lower total variance will lead to an increase in the wealth allocation on the risk-free asset, CDI, while decreasing proportionally the wealth allocation on the risk assets as we can see in Figures 17, 18, 19, and 20.

Figure 17 - Control policy for scenario cc.

Source: Author.

Figure 18 - Control policy for scenario cc2.

Source: Author.

Figure 19 - Control policy for scenario cc4.

Source: Author.

Figure 20 - Control policy for scenario cc8.

Source: Author.

C.2 Simulations for different combinations of β and ν

In this section we will expand the sensitivity analysis of the risk parameters β and ν for different combinations of them over time. Our goal is to understand the different trade-offs between the portfolio's value, variance, and asset allocation when we impose different weights to the elements of β and ν .

Now, we will consider variations within the elements of β and ν , assuming they either do not change or change linearly over time in an upward or downward trend. In the timeline, the elements of β and ν will form a curve, or shape, that we will refer to as "behavior" of β and ν .

The equations that describe these behaviors, see Table 21, were chosen arbitrarily in order to illustrate the potential relative trends between β and ν , and within their elements. Thus, we will verify the portfolio results when the elements of β increases while ν is kept constant, or when the elements of β decreases while the elements of ν increases, etc.

Once again, the boundaries of each behavior were chosen in order to keep the different results in a scale that would ease the observation of the trends.

Table 21 - Behaviors definitions for β and ν .

Input parameter's equations for $t = 1, 2, \dots, 20$		
Behavior	$\beta(t)$	$\nu(t)$
constant	1	1
up	$(9/19)*(t-1) + 1$	$(9/19)*t + 1$
down	$(9/19)*(20-t) + 1$	$(9/19)*(20-t) + 1$

We then grouped the nine potential combinations of the three behaviors (constant, up, and down) in scenarios as described in Table 22 and illustrated in Figure 21.

Table 22 - Scenarios that combine different β and ν behaviors and resulting $f(\beta, \alpha)$ of problem $PC(\nu, \beta, \alpha)$.

Scenario	Problem applied	Risk parameter's behavior			Resulting $f(\beta, \alpha)$
		β	ν	α	
uc	$PC(\nu, \beta, \alpha)$	up	constant	100	0.132
cc	$PC(\nu, \beta, \alpha)$	constant	constant	100	0.920
dc	$PC(\nu, \beta, \alpha)$	down	constant	100	0.220
ud	$PC(\nu, \beta, \alpha)$	up	down	100	0.285
cd	$PC(\nu, \beta, \alpha)$	constant	down	100	2.128
dd	$PC(\nu, \beta, \alpha)$	down	down	100	0.559
uu	$PC(\nu, \beta, \alpha)$	up	up	100	0.283
cu	$PC(\nu, \beta, \alpha)$	constant	up	100	1.878
du	$PC(\nu, \beta, \alpha)$	down	up	100	0.418

Figure 21 - Illustrative diagram for the scenarios of Table 22.

Source: Author.

Finally, Figures 22 and 23 illustrate the portfolio's value and its volatility for all nine scenarios. Their analysis will be split in the next 5 sections in order to make it easy to see the effects that each change is causing in the overall results.

Figure 22 - System's output for the scenarios of Table 22.

Source: Author.

Figure 23 - System's output variance for the scenarios of Table 22.

Source: Author.

C.2.1 Sensitivity analysis when α is not applied

The scenarios above, Table 22, were established considering we are solving $PC(v, \beta, \alpha)$ to obtain the optimal wealth allocation.

However, in order to compare the effects of imposing α , we also obtained the simulations of each of the nine scenarios from Table 22 considering no restriction α . This is achieved by solving $PU(v, \xi)$, where ξ will take the same behavior as β in each scenario. In Table 23, we show the scenarios as before with the symbol "*" to indicate that it is being applied $PU(v, \xi)$ instead of $PC(v, \beta, \alpha)$. There, we can also see the resulting total weighted variance for each scenario.

Table 23 - Resulting total weighted variance for the scenarios of Table 22 when there is no restriction α .

		Risk parameter's behavior		Resulting total weighted variance
Scenario	Problem applied	ξ	v	$\sum_{t=1}^T v(t) Var[y^u(t)]$
uc*	$PU(v, \xi)$	up	constant	5784.5
cc*	$PU(v, \xi)$	constant	constant	118.2
dc*	$PU(v, \xi)$	down	constant	2070.2
ud*	$PU(v, \xi)$	up	down	1227.2
cd*	$PU(v, \xi)$	constant	down	22.1
dd*	$PU(v, \xi)$	down	down	319.6
uu*	$PU(v, \xi)$	up	up	1251.3
cu*	$PU(v, \xi)$	constant	up	28.4
du*	$PU(v, \xi)$	down	up	574.4

The resulting portfolio's value and its variance using $PU(v, \xi)$ were depicted in Figures 24 and 26, respectively, and their zooms in Figures 25, 27, and 28.

Figure 24 - System's output for the scenarios of Table 23.

Source: Author.

Figure 25 - Zoom of Figure 24.

Source: Author.

Figure 26 - System's output variance for the scenarios of Table 23.

Source: Author.

Figure 27 - Zoom of Figure 26.

Source: Author.

Figure 28 - Zoom of Figure 27.

Source: Author.

The analysis laid out in the following paragraphs compares the returns and variances with the restriction $\alpha = 100$, Figures 22 and 23, with their respective scenarios when there is no restriction, Figures 24 and 26. When necessary, one should consider Figures 24 and 26's zooms in Figures 25, 27, and 28 for a better view.

In this way, we can see that when we solve $PC(\nu, \beta, \alpha)$ with a restriction α that is higher than the total weighted variance that we would have had without this restriction, then we get higher variances and returns over the whole period of time, see scenarios cd^* and cu^* in Table 23, where $\sum_{t=1}^T \nu(t) Var[y^u(t)] < 100$.

On the other hand, in the remaining scenarios of Table 23, $\sum_{t=1}^T \nu(t) Var[y^u(t)] > 100$ and, therefore, we obtain lower returns and variances over the whole period of time when we impose them $\alpha = 100$, see Figures 22 and 23 versus Figures 24 and 26.

The effects on the wealth allocation for the scenarios without the restriction α were omitted because they will follow the same pattern as extensively illustrated in the following analysis for the $PC(\nu, \beta, \alpha)$ problem, which is a higher allocation on the risk assets

versus the risk-free asset, whenever a higher return or variance are demanded and conversely, a lower allocation on the risk assets versus the risk-free asset, whenever a lower return or variance are demanded.

Given we only needed to solve $PU(\nu, \xi)$ here to show the sensitivity of our results when there is no α , we will proceed with the analysis in the remaining sections by focusing only in the scenarios of Table 22, where only $PC(\nu, \beta, \alpha)$ is applied.

C.2.2 Analysis for scenarios uc and dc

We first evaluate the effects of β when ν is kept constant, scenarios uc and dc , around the base scenario where both are constants, scenario cc .

Analyzing Figures 22 and 23, we can see that choosing β that increases over time leads to a higher return and variance on the later steps and to a lower return and variance in the earlier steps in relation to the base scenario, cc . Therefore, the upward β has the effect of rotating both the expected return and variance curves counter-clockwise around scenario cc . This is achieved by decreasing the wealth allocation in the risk assets in the earlier periods and increasing the wealth allocation in the same assets in the later periods when compared with the base scenario cc , see Figure 29 with the asset's allocation for scenarios uc and cc .

Conversely, choosing a β that decreases over time leads to a lower return and variance on the later steps and to a higher return and variance in the earlier steps in relation to the base scenario, cc . Therefore, the downward β has the effect of rotating both the expected return and variance curves clockwise around scenario cc . This behavior is achieved by increasing the wealth allocation in the risk assets in the earlier periods and decreasing the wealth allocation in the same assets in the later periods when compared with the base scenario cc , see Figure 30 for scenarios dc and cc .

Figure 29 - Control policy for scenario *uc* versus scenario *cc*.

Source: Author.

Figure 30 - Control policy for scenario *dc* versus scenario *cc*.

Source: Author.

The fact that a higher β in the early steps of scenario uc did not lead to higher returns relative to the cc scenario is due to the restriction α that will impose a lower return over all steps. Without the restriction α , the returns would have been higher for all steps as well as their variances, see Figures 24 and 26 for simulations of the scenarios when α is not applied. Therefore, it is the restriction α that creates the impression of a counter-clockwise shift around scenario cc when β increases over time.

The same rationale is true for scenario dc , but now the relative smallest coefficients appear in the second half of our timeline and, therefore, the expected returns and variances are more penalized from $t = 10$ to $t = 20$, creating the impression of a clockwise shift around scenario cc .

C.2.3 Analysis for scenarios cd and cu

The assessment of the scenarios where β is kept constant while ν varies, cd and cu , will also be analyzed around the base scenario where both are constants, cc .

In Figure 23, we can see that choosing a ν that increases over time leads to a lower variance in the later steps in comparison with the earlier steps. Conversely, a ν that decreases over time leads to a higher variance in the later steps when compared with the variance in the earlier steps. It means that the higher the risk aversion $\nu(t)$, the lower will be the variance at t .

Provided that we are solving the $PC(\nu, \beta, \alpha)$ problem, the overall variance were reduced to comply with the imposed restriction, $\alpha = 100$, when compared with the base scenario cc . Therefore, setting higher values for ν in scenarios cd and cu than in cc will lead to lower variances in those scenarios than in scenario cc , see in Figure 23.

Regarding the portfolio's return for scenarios cd , cu and cc , we can observe in Figure 22 that the imposition of a lower variance led to lower returns and conversely, the allowance of higher variance led to higher returns.

This behavior is achieved by decreasing the wealth allocation on the risk assets over the periods with a higher ν , lower variances allowed, and by increasing the wealth

allocation on the risk assets over the periods with a lower ν , higher variances allowed, see Figures 31 and 32 for a comparison of the assets' allocation in scenario *cd* and *cu* versus *cc*.

Figure 31 - Control policy for scenario *cd* versus scenario *cc*.

Source: Author.

Figure 32 - Control policy for scenario *cu* versus scenario *cc*.

Source: Author.

C.2.4 Analysis for scenarios ud and dd

In order to analyze the effects of changing β when ν has a downward trend, we will compare the results around the scenario where β is constant and ν has a downward trend, cd .

Therefore, observing the curves ud and dd around the curve cd in Figures 22 and 23, we can conclude that the increase of $\beta(t)$ leads to a higher return and variance at t and, a decrease of $\beta(t)$ leads to a lower return and variance at t .

As explained above, the restriction α leads to the impression of shifting the portfolio's value in a counter-clockwise direction around the scenario cd when β has an upward trend, as in scenario ud , and shifting it in a clockwise direction when β has a downward trend, as in scenario dd .

Once more, the downward trend of ν led to lower variances and returns over the earlier steps than in the later steps when compared with the curves uu , cu , and du for the reasons already exposed before.

When we compare all curves where ν has a downward trend (ud , cd , dd) with all curves where ν is constant (uc , cc , dc), see Figure 23, we can observe that the average higher values of ν in the first block leads to lower variances over time. Once more, this effect is explained by the imposition of the restriction α that "forces" a lower variance even when we set a lower ν . Note how a lower ν leads to higher variances when α is not applied in Figure 27 for the same scenarios mentioned here.

The control policy, shown in Figures 33 and 34, confirm the same effect as observed in Section C.2.2, which is that higher elements of β lead to a higher exposure to risk assets and conversely, lower elements of β lead to a lower exposure to risk assets

Figure 33 - Control policy for scenario *ud* versus scenario *cd*.

Source: Author.

Figure 34 - Control policy for scenario *dd* versus scenario *cd*.

Source: Author.

C.2.5 Analysis for scenarios uu and du

In our last scenarios, uu and du , we consider the variation of β when ν keeps an upward trend and we will compare these scenarios with the base scenario where β is constant, cu .

As in the previous Section C.2.4, by observing Figures 22 and 23, we can conclude that the increase of β over time leads to a higher return and variance, and a decrease of β leads to a lower return and variance. However, now the increase in the returns is not so strong in the later steps because it has been counterbalanced by a higher ν in those periods. Recall that higher elements of ν imposes a lower return and variance as we can verify when we compare the curves for ud , cd , and dd versus uu , cu , and du in Figures 22 and 23.

The control policy, shown in Figures 35 and 36, once more confirms the same effect already observed, which is that higher elements of β lead to a higher exposure to risk assets and conversely, lower elements of β lead to a lower exposure to risk assets.

Note that, even though the comparison of the allocation curves is hard after the step 12, it is still possible to observe where the curves of uu and du cross the curves of cu . It is also clear the shift caused by a higher β in scenario uu , and a lower β in scenario du in the early steps.

Figure 35 - Control policy for scenario uu versus scenario cu .

Source: Author.

Figure 36 - Control policy for scenario du versus scenario cu .

Source: Author.

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