FABIO YUKIO TORIUMI

Nonlinear control approaches for an underactuated
control moment gyroscope

São Paulo
2020

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Revised Version

(The original version can be found in the unit that hosts the Graduate Program)

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Concentration area:
Systems Engineering

Advisor:
Prof. Dr. Bruno Augusto Angélico

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2020

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Banca Examinadora

Prof. Dr. Bruno Augusto Angélico

Instituição: Escola Politécnica da Universidade de São Paulo (EP/USP)

Julgamento: Aprovado

Prof. Dr. Tiago Roux de Oliveira

Instituição: Universidade do Estado do Rio de Janeiro (UERJ)

Julgamento: Aprovado

Prof. Dr. Luiz Carlos Sandoval Góes

Instituição: Instituto Tecnológico de Aeronáutica (ITA)

Julgamento: Aprovado

Prof. Dr. Marco Henrique Terra

Instituição: Escola de Engenharia de São Carlos da Universidade de São Paulo (EESC/USP)

Julgamento: Aprovado

Prof. Dr. Paulo Sérgio Pereira da Silva

Instituição: Escola Politécnica da Universidade de São Paulo (EP/USP)

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Dedicated to my family and relatives, who always encouraged and supported me in all moments.

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“I have been impressed with the urgency of doing. Knowing is not enough; we must apply. Being willing is not enough; we must do” (SCHOLZ, 2007, p. 1).

-- Leonardo da Vinci

ABSTRACT

TORIUMI, F. Y. **Nonlinear control approaches for an underactuated control moment gyroscope.** 2020. Thesis (Doctor in Science) - Escola Politécnica, Universidade de São Paulo, São Paulo, 2020.

Control moment gyroscopes (CMG) are widely used as actuators in some important applications, as the attitude control of spacecraft and satellites, due to its interesting and advantageous characteristics, such as its efficient high output torque capability. Motivated by this, we explore here a CMG unit, model 750, from Educational Control Products (ECP), which is a complex and versatile system that can operate in both single-input/single-output (SISO) and multiple-input/multiple-output (MIMO) configurations. This underactuated electromechanical plant presents a highly coupled nonlinear dynamics with a wide operating range, posing additional challenges to the control design, due to the fact that the outputs to be controlled are unactuated states of the system and its tracking task involves time-varying trajectories. To deal with these challenges, three nonlinear control approaches, the feedback linearization (FL), sliding-mode (SM), and passivity-based (PB), are proposed to improve the performance of the ECP CMG tracking task and get a wider operating range. Since nonlinear control approaches are critically dependent on the system model, the CMG modeling process based on Gibbs-Appell method and Kane's dynamical equations is firstly developed, then its nonlinear model is explicitly presented and used for both controller designs and numerical simulations. Next, the application of the three nonlinear control approaches to the tracking task of a class of fully-actuated systems is briefly presented, followed by the specific issues inherent to the CMG underactuation. To overcome this problem, the CMG intrinsic reaction torques are employed in the three controller designs. Simulation and experiment results are considered to validate the effectiveness of the controllers and the final results are compared to each other for an overall discussion, where the maximum absolute error (MAE) and the integral absolute error (IAE) indexes are presented to corroborate the performance analysis.

Keywords: Control (Systems theory and control). Dynamic systems. Nonlinear models. Nonlinear control. Control moment gyroscope.

RESUMO

TORIUMI, Fabio Yukio **Técnicas de controle não linear aplicadas a um giroscópio de controle de momento sub-atuado**. 2020. Tese (Doutorado em Ciências) - Escola Politécnica, Universidade de São Paulo, São Paulo, 2020.

Giroscópios de controle de momento (ou do inglês, *control moment gyroscope* - CMG) são amplamente utilizados como atuadores em algumas aplicações importantes, como o controle de atitude de veículos espaciais e de satélites, devido às suas características interessantes e vantajosas, como seu eficiente e alto torque de saída. Motivado por isso, nós exploramos aqui uma unidade CMG, modelo 750, da *Educational Control Products* (ECP), que é um sistema complexo e versátil capaz de operar em ambas configurações *single-input/single-output* (SISO) e *multiple-input/multiple-output* (MIMO). Essa planta eletromecânica sub-atuada apresenta uma dinâmica não linear altamente acoplada com uma operação de amplo alcance, impondo desafios adicionais ao projeto de controle, devido ao fato das saídas a serem controladas serem os estados não atuados do sistema e a tarefa de seguimento de trajetória ser variante no tempo. Para lidar com esses desafios, três técnicas de controle não linear, a realimentação linearizante (*feedback linearization* - FL), os modos deslizantes (*sliding-mode* - SM) e o baseado em passividade (*passivity-based* - PB), são propostos para melhorar o desempenho da tarefa de seguimento de trajetória do CMG da ECP e aumentar o seu alcance de operação. Visto que as técnicas de controle não linear são criticamente dependentes do modelo do sistema, o processo de modelagem do CMG, baseado no método de Gibbs-Appell e nas equações dinâmicas de Kane é primeiramente desenvolvido, depois seu modelo não linear é explicitamente apresentado e usado para ambos os projetos dos controladores e as simulações numéricas. Em seguida, a aplicação das três técnicas de controle não linear à tarefa de seguimento de trajetória de uma classe de sistemas totalmente atuados é brevemente apresentada, seguido dos problemas específicos inerentes à sub-atuação do CMG. Para lidar com este problema, os torques reativos intrínsecos do CMG são empregados nos projetos dos três controladores. Resultados de simulações e experimentos são considerados para validar a eficácia dos controladores e os resultados finais são comparados entre si para uma discussão geral, onde os índices de erro absoluto máximo (*maximum absolute error* - MAE) e de integral do erro absoluto (*integral absolute error* - IAE) são apresentados para colaborar com a análise de desempenho.

Palavras-Chave: Controle (Teoria de sistemas e controle). Sistemas dinâmicos. Modelos não lineares. Controle não linear. Giroscópio de controle de momento.

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LIST OF ABBREVIATIONS

ASME	American Society of Mechanical Engineers
CMG	Control Moment Gyroscope
DGCMG	Double-Gimbal Control Moment Gyroscope
DGVSCMG	Double-Gimbal Variable-Speed Control Moment Gyroscope
DOF	Degree of Freedom
DSP	Digital Signal Processing
ECP	Educational Control Products
EP	Escola Politécnica
FL	Feedback Linearization
IAE	Integral Absolute Error
ICD	Individual Channel Design
IEEE	Institute of Electrical and Electronics Engineers
I/O	Input/Output
ISS	International Space Station
LCA	Laboratório de Controle Aplicado
LMI	Linear Matrix Inequality
LPV	Linear Parameter-Varying
LQR	Linear-Quadratic Regulator
MAE	Maximum Absolute Error
MCS	Momentum-Control System
MIMO	Multiple-Input/Multiple-Output
PB	Passivity-Based
PI	Proportional-Integral
PID	Proportional-Integral-Derivative
RHS	Right-Hand Side
SBA	Sociedade Brasileira de Automática
SGCMG	Single-Gimbal Control Moment Gyroscope
SGVSCMG	Single-Gimbal Variable-Speed Control Moment Gyroscope
SISO	Single-Input/Single-Output
SM	Sliding-Mode
SMI	Scalar Moment of Inertia
TST	Total Saturation Time
UMS	Underactuated Mechanical System
USP	Universidade de São Paulo
VFC	Viscous Friction Coefficient
VTOL	Vertical Take-Off and Landing

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1 INTRODUCTION

1.1 Motivation and literature review

A Control Moment Gyroscope (CMG) is a mechanical gyroscope type that exploits the reaction wheel characteristics, which can generate a higher output torque concurrently with lower energy consumption when compared to the conventional reaction wheels (LAPPAS, 2002; VOTEL; SINCLAIR, 2012).

It is generally composed of a flywheel mounted in a structure with one or more motorized gimbals that change its angular momentum direction, which creates a characteristic reaction torque called *gyroscopic torque* (WIE, 1998) that can be used to rotate any framework in which it is attached, e.g., a satellite or a spacecraft (ARMENISE et al., 2010). Figure 1 illustrates an example of a CMG structure with a flywheel spinning at a rate ω_w producing an angular momentum vector $\mathbf{h}_w$, which direction is changed when a torque T_g is applied to the gimbal, originating the gyroscopic torque T_r .

Figure 1: The gyroscopic torque acting on a CMG structure

Source: Author.

Related to its structure, Figure 2 depicts the two classes that they can be divided into: the single-gimbal CMG (SGCMG) and the double-gimbal CMG (DGCMD), both having their flywheels spinning at a constant rate during their operation (LEVE; HAMILTON; PECK, 2015), thus producing an angular momentum with constant magnitude. The SGCMG, presented in Figure 2a, has only one motorized gimbal that can tilt the angular momentum vector $\mathbf{h}_w$ within the plane normal to its spin axis $\mathbf{e}_g$, whereas Figure 2b illustrates the DGCMD that has an additional motorized outer gimbal that enables to orient $\mathbf{h}_w$ in any direction (when physical constraints are not considered).

Figure 2: The control moment gyroscope (CMG) classes

Source: Adapted from Yavuzoglu (2003, p. 22).

There also exist their version with a variable flywheel spinning rate: the single-gimbal variable-speed CMG (SGVSCMG) and double-gimbal variable-speed CMG (DGVSCMG) (SASAKI; SHIMOMURA; SCHAUB, 2018).

All these classes have their own advantages and disadvantages, e.g., although DGCMDs have an additional degree of freedom (DOF), their torque amplification advantage is effectively lost when compared to the SGCMGs, as well as the additional complications that this additional DOF yields, as a more complicated mechanical construction and a rise in its weight (YAVUZOGLU, 2003).

This requests the assembling of devices composed of a combination of them for attitude control in three axes. One that is extensively explored in the literature is the so called *CMG system*, whose some configuration examples are illustrated in Figure 3, that is composed only of SGCMGs, where three or more of them are necessary (KUROKAWA, 2007) to perform the attitude control, being typically four adopted for singularity avoidance and/or redundancy (LEVE, 2008). This singularity is a typical problem that appears in this system, which differs for each possible geometrical configuration (KUROKAWA, 2007). In this problem, the output torque cannot assume all the possible orientations within a specific combination of the gimbal angles, which must be avoided and taken into account during attitude control law designing.

Figure 3: Examples of CMG system assemblies

Source: Adapted from (a) Ohmura (2010, p. 2) and (b) Leve (2008, p. 48).

Due to all these characteristics, the CMG and the aforementioned assembly devices are widely used as actuators in some applications, such as:

a) system stabilization, e.g.:

- Gagne et al. (2012) uses a SGCMG to actively stabilize the epicardium in the context of beating-heart coronary artery bypass grafting;
- Yetkin et al. (2014), Park and Yi (2020) stabilize a unmanned bicycle, where the former use one SGCMG and the latter use a scissored-pair SGCMGs;
- Park and Cho (2018) presents a self-balancing robot by employing a CMG module composed of two SGVSCMG to maintain the balance of a two-wheeled mobile robot;

- and Li and Vallery (2012) investigates a wearable solution by employing a backpack containing a set of at least two SGVSCMGs to stabilize the human balance during gait, whereas Romtrairat et al. (2020) design and implement it with a scissored-pair SGCMGs to reduce the human fall risk, evaluating its practical performance with two healthy male subjects;
- b) and mainly for the attitude control of satellites and spacecraft, e.g.:
- Lappas (2002) focuses on the practical work on developing the hardware for a low cost pyramid configuration with four SGCMGs for agile small satellites;
 - Richie, Lappas and Prassinis (2009) designs and builds a prototype of a pyramid configuration with four SGVSCMGs for small satellites, testing its effectiveness on a three-axis air-bearing structure;
 - Zheng and Han (2013) presents an integrated velocity measurement and attitude control system of spacecraft using magnetically suspended DGCMG, validating numerically and experimentally its effectiveness;
 - Mumm et al. (2014) presents a box configuration of four SGCMGs for small spacecraft, testing its practical performance with a scissored-pair SGCMGs configuration;
 - and Evain et al. (2017) describes a experimental device of a pyramid configuration with six SGCMGs dedicated to the attitude control of nanosatellites.

Some illustrative examples of these applications can be found in Leve, Hamilton and Peck (2015), where the system stabilization (LEVE; HAMILTON; PECK, 2015, p. 31-32) is exemplified by the Lit Motors' C-1, a two-wheel automobile that has a scissored pair of SGCMGs to provide its roll stabilization, and the attitude control (LEVE; HAMILTON; PECK, 2015, p. 19-20) by the momentum-control system (MCS) of the International Space Station (ISS) that has four DGCMGs, each of them composed of two gimbals that enable the redirection of their momentum toward any point on a sphere.

These applications, and above all the attitude control ones, motivate herein the control application of a CMG unit, model 750, from Educational Control Products (ECP), illustrated in Figure 4, since the Laboratório de Controle Aplicado (LCA) of Escola Politécnica (EP) from Universidade de São Paulo (USP) has a physical unit of it, enabling the practical application and validation of designed controllers.

Figure 4: ECP model 750 CMG electromechanical unit from LCA

Source: Author.

It is worth mentioning that although this CMG is an educational system and its configuration is similar to a DGVSCMG ¹ that is commonly used in aerospace applications (AHMED; BERNSTEIN, 2002; STEVENSON; SCHAUB, 2012), it is yet a versatile underactuated system that presents a highly coupled nonlinear dynamics and may operate in both single-input/single-output (SISO) and multiple-input/multiple-output (MIMO) configurations with wide operating range, making it an interesting and challenging plant for control designing.

In view of it, some control techniques was proposed for it in the literature, specially for its MIMO configuration, such as the well established linear approaches, e.g.:

- a) Liceaga, Liceaga and Amezquita (2005) implements a multivariable control system in the framework of individual channel design (ICD) by decomposing the CMG coupled MIMO linear model into two SISO systems in series with a 2×2 system, where two internal control loops were also designed, being all controllers obtained by applying classical control theory techniques and validated with simulations and experiments;
- b) Durand et al. (2014) designs an event-based linear-quadratic regulator (LQR) control with integral action for its decoupled linear model, and validates it with experimental results;

¹it differs from the fact that the outer gimbal is not actuated.

- c) Boisseau et al. (2015) presents a discrete event-triggered LQR control with a discrete time linear disturbance observer for its decoupled linear model, validating it with experimental results;
- d) Mardan et al. (2015) implements a linear matrix inequality (LMI) based multi objective state feedback H_2/H_∞ controller for its decoupled linear model, in which an uncertain range for the flywheel rate is considered, and validates it with simulations;
- e) Angélico, Colón and Balthazar (2015) designs a multivariable H_∞ robust controller for its decoupled linear model, in which the flywheel rate variation is allowed, and is validated with experimental results;
- f) and Toriumi (2016) implements two discrete linear control approaches, a proportional-integral-derivative (PID) and a LQR control with integral action for its decoupled and, both minimum and non-minimum phase, coupled linear models, and validates them with simulations and experiments.

However, they effectively work only near the operating point considered for the controller design, restricting the CMG operating range.

There exist other control techniques that may cope with this problem, as the linear parameter-varying (LPV) approach (e.g., Abbas et al. (2013), Abbas et al. (2014), Theis, Radisch and Werner (2014), Hoffmann and Werner (2015), Saleem et al. (2019), and Calderón, Cisneros and Werner (2019)) and the nonlinear approaches (e.g., some papers presented in Appendix A).

Regarding the latter, there are some reasons for many control researchers and designers have been drawing more attention and interest in the development and application of nonlinear control approaches (SLOTINE; LI, 1991; SASTRY, 1999), as their ability to handle the nonlinearities in a wide operating range, enlarging the system workspace (e.g. for mechanical systems we can consider both position and speed variables) while providing a good performance (e.g. the stabilization or tracking accuracy). Some examples of nonlinear control approaches are the feedback linearization, sliding-mode, passivity-based, backstepping, Lyapunov-based, and nonlinear predictive control, to name a few.

Considering their abilities and the fact that attitude control applications generally require a tracking task in a wide operating range, this thesis relies its research upon the practical application of nonlinear control approaches, once the CMG hardware turns the continuous-time application possible with its reasonably high sampling frequency.

This decision is supported by the fact that physical systems present nonlinear behaviors, as stated by Iqbal et al. (2017, p. 301) “Real world systems are inherently nonlinear in nature at least when considered over wide operating range ...”, by Slotine and Li (1991, p. 4) “Physical systems are inherently nonlinear.”, by Dai and Jazar (2015, p. vii) “Nature is nonlinear in general as the responses of physics and engineering systems are nonlinear”, by Altinay (2012, p. ix) “All practical systems contain nonlinear dynamics.”, and by Vaidyanathan, Volos et al. (2016, p. v) “As most real-world control systems are nonlinear, nonlinear control theory has great applications for real-world control systems”.

Moreover, the increasing advance of technology in the last years has been impulsing and demanding sophisticated control laws to meet exigent specifications of control designs (VUKIC, 2003; IQBAL et al., 2017). This raised the interest of many researchers on nonlinear control applications in such broad areas, e.g., automotive, mechatronic, robotics, aircraft and spacecraft control, process control, biomedical engineering, mechanical engineering, and electrical engineering (SLOTINE; LI, 1991; ALTINAY, 2012; VAIDYANATHAN; VOLOS et al., 2016).

It is corroborated by a number of developed practical nonlinear control systems in the literature, such as some applications in the chemical industry assessed by Ogunnaike and Wright (1997), the ones presented in Altinay (2012), Vaidyanathan, Volos et al. (2016), and the references cited within Iqbal et al. (2017). As consequence, a number of new approaches, ideas, and results has been emerging for it Altinay (2012).

However, some difficulties and challenges need to be surmounted, as the nonexistence of systematic procedures for designing nonlinear control systems (SLOTINE; LI, 1991, p. 12), as stated by Ortega et al. (1998, p. 5)

It is widely recognized, at least as far back as Poincaré, that nonlinear systems can exhibit extremely complex dynamic behaviour. This renders futile the quest of a monolithic control theory applicable for all systems, we must therefore specialize the class of systems under consideration in one way or another.

and by Slotine and Li (1991, p. 203)

As in the analysis of nonlinear control systems, there is no general method for designing nonlinear controllers. What we have is a rich collection of alternative and complementary techniques, each best applicable to particular classes of nonlinear control problems.

Moreover, they may also require rigorous hardware and tuning process, as stated by Iqbal et al. (2017, p. 310) “Nonlinear control schemes may need more expensive actuators

to survive longer. Most of the nonlinear controllers pose greater risk to damage expensive systems if not implemented with care. Also, complex and exhaustive tuning process may be involved”.

Therefore, the nonlinear control approaches needs to be developed and designed specifically for each class of systems and problems.

Considering the ECP CMG, it belongs to a class of systems called *underactuated mechanical systems* (UMS) that have fewer independent control actuators than DOF to be controlled, yielding to “challenging control problems since the control design must typically exploit some coupling between the unactuated states and the actuated states to achieve control objective” (DIXON et al., 2003, p. 269).

This fact also makes many traditional nonlinear control methods to be not directly applicable. Moreover, accordingly to Liu and Yu (2013, p. 930)

Highly non-linear characteristics and reduced dimension of input space are intractable issues for controlling UMS. The control methods were designed either to adapt to non-linearities or to reduce the system order to a low-dimensional model. However, not all of these methods are practically applicable because of system constraints (e.g. actuator power limitation).

what corroborates the difficult in applying nonlinear control approaches to UMS.

There exist few works in the literature that consider some nonlinear control approaches to UMS, e.g.:

- a) Ortega et al. (1998) presents the passivity-based control for robots with flexible joints and for some underactuated electrical machines, validating them with simulation results;
- b) Olfati-Saber (2001) is devoted to nonlinear control, reduction, and classification of UMS, applied to a variety of robotics and aerospace systems, where detailed control design based on non-collocated partial feedback linearization and simulation results are provided;
- c) Dixon et al. (2003) implements two nonlinear energy-based coupling control laws to underactuated overhead crane systems, validating it with experimental results, and also designs a Lyapunov-based controller to a underactuated planar vertical take-off and landing (VTOL) aircraft and validate it with simulation results;

- d) Aneke (2003) deals with the tracking and feedback stabilization problems for a class of UMS with second-order nonholonomic constraints, where a cascaded backstepping approach is proposed to the tracking problem and a proposed control method is developed, designed and implemented to an underactuated H-Drive manipulator, for which experimental tests are performed;
- e) Liu and Yu (2013) presents a survey of UMS with several well-known examples, some of its classifications based on different system properties, and a highlight of some control methods based on feedback linearization, energy-based, backstepping, sliding-mode, and fuzzy logic for its stabilization;
- f) and Lee and Tuan (2016) develops, designs and implements the feedback linearization to a underactuated 3D crane system, validating it with an experimental study.

Lastly, the control design becomes even more challenging for the ECP CMG tracking task since the main outputs to be controlled are its unactuated states.

Summarizing, the tracking task of the ECP CMG presents some issues and challenges that may be considered and tackled during the controller designs based on nonlinear control approaches. To the best of author's knowledge, there is no paper in the literature that deals with this specific problem besides from the author's itself.

1.2 Objectives and contributions

The main objective of this thesis is to improve the performance of the CMG tracking task to get a wider operating range. To accomplish this, the following specific objectives are aimed:

- a) develop the CMG modeling;
- b) study of nonlinear control approaches;
- c) design nonlinear controllers for the CMG tracking task based on these approaches;
- d) implement and validate the designed controllers via numerical simulations and real-time practical experiments;
- e) analyze and compare all obtained results.

This thesis thus provides its main contribution: the development, design and practical implementation of three nonlinear controllers for the CMG tracking task, based on feedback linearization (FL), sliding-mode (SM), and passivity-based (PB) approaches. Although they were specifically developed for the CMG tracking task problem, the author believes that their design procedures may be applied to the tracking task of other underactuated systems with their unactuated state(s) as the output(s) of the system, in which internal control signal(s) is(are) employed to deal with the underactuation, by choosing it(them) as the reaction force(s)/torque(s) of the system.

As these nonlinear controllers highly depend on a nonlinear model of the system, another contribution herein is the development of the CMG modeling process, in which dynamics stage employs the Gibbs-Appell equations in Kane's proposed dynamical equations form. Although the author has already developed the modeling process employing the Euler-Lagrange formulation in his master's dissertation (TORIUMI, 2016), it was opted to redo it in order to consolidate the CMG nonlinear model. In this case, both modeling processes reached the same results, being the nonlinear model identically to each other. Moreover, the nonlinear model was organized and explicitly shown as function of the CMG parameters in Appendix C, unlike Toriumi (2016) that presents only the numerical nonlinear model, in which the values of the CMG parameters were already considered and replaced in.

1.3 Publications

The publications derived from this thesis are listed in Appendix A, in which their information, together with their abstract, are presented. A summary and comments of them are presented below.

The developed CMG modeling process with the Gibbs-Appell equations were addressed and published in both Toriumi et al. (2017) and Toriumi and Angélico (2020a). In the latter, the modeling process were presented and used for the feedback linearization approach. And concerning the former, the only difference is that the kinematics stage were developed using the quaternion algebra, but the dynamics stage and the derived nonlinear model are the same presented here.

And the development of the three nonlinear controllers derived six papers: three of them address the CMG in its SISO configuration and were published in Toriumi, Angélico and Tannuri (2018) (FL), and in Toriumi and Angélico (2018) (SM), and the PB

was submitted and accepted by the SBA Journal of Control, Automation and Electrical Systems; and the other three address the CMG in its MIMO configuration and were published in Toriumi and Angélico (2020a) (FL), in Toriumi and Angélico (2019b) (SM), and in Toriumi and Angélico (2020b) (PB).

1.4 Structure of the text

First, the description of the ECP model 750 CMG and its modeling process are presented in Chapter 2, in which the derived nonlinear model, explicitly presented in Appendix C, is rewritten in two suitable forms to be used in the nonlinear controller designs for the CMG tracking task, described in Chapter 4. But before it, Chapter 3 introduces the three abovementioned nonlinear control approaches and briefly describes their application for a class of fully-actuated systems. Then, Chapter 4 presents the specific issues that arise from the CMG underactuation when applying these approaches and tackle them by employing the CMG intrinsic reaction torques, developing then the controller design for each approach. The CMG experimental setup and the designed controllers validation are presented in Chapter 5, where numerical simulations and real-time practical experiments are performed with each designed controller and their results are individually analyzed and compared with each other for an overall discussion. At last, Chapter 6 provides the final remarks and the proposals for future research.

2 ECP CMG DESCRIPTION AND MODELING

Before beginning the modeling process, the ECP model 750 CMG unit is described in Section 2.1. Then, the kinematics stage is developed in Section 2.2, where frames are attached to the CMG bodies in order to describe its relative orientation, followed by the dynamics stage in Section 2.3 that derives the CMG equations of motion, composed of four nonlinear equations. Lastly, Section 2.4 presents these equations rewritten in two suitable forms to be used for the nonlinear controllers design in Chapter 4.

2.1 Plant description

The ECP model 750 CMG unit, described in Fig. 5, consists of four rigid bodies: body A (the rotating base), body B (the outer gimbal), body C (the inner gimbal), and body D (an axisymmetric brass disc, called rotor), of which angular positions θ_k ($k = 1, 2, 3, 4$) are respectively measured by incremental encoders A , B , C , and D ¹.

Figure 5: CMG description

Source: Author.

¹This notation is different from the one presented in the ECP manual (ECP, 1999, p. 60) due to the adopted convention described in Section 2.2.

Its two DC motors (motors #1 and #2) apply torques T_1 and T_2 on axes #4 and #3, respectively, that rotate bodies D and C . On the other hand, bodies A and B do not have any actively applied torque and are only rotated by intrinsic reaction torques generated on the plant structure. There are also brakes installed on axes #1 and #2 that can be actuated via toggle switches to lock bodies A and B angular positions (θ_1 and θ_2).

With both brakes deactivated, when the rotor angular velocity varies, body B rotates by reaction around axis #2 in the opposite direction, obeying the principle of conservation of angular momentum, and when the inner gimbal is tilted while the rotor is rotating, body A is rotated around axis #1 by the generated gyroscopic torque.

Remark 1. *It is worth remarking that all CMG bodies do not have trasnlatational displacement (only rotational movement), are all considered symmetric and their center of mass coincide with the rotor center.*

Remark 2. *The rotor is responsible for generating the angular momentum that is essential for the CMG operation, thus its velocity must be nonzero during it.*

Remark 3. *Note that this system is underactuated and has two subsystems: an unactuated one, composed of bodies A and B dynamics, and an actuated one, composed of bodies C and D dynamics. The main goal of the controller design herein is to perform the tracking task of the unactuated states (θ_1 and θ_2) with the system control action composed of both motor torques (T_1 and T_2).*

Since there are incremental encoders, they need to be set to zero at the beginning of every procedure. Fig. 5 illustrates $\theta_k = 0$ rad ($k = 1, 2, 3, 4$), in which:

- θ_1 can be set to zero at any position of body A , as all CMG bodies are assumed to be symmetric,
- θ_2 is set to zero with body B oriented perpendicular to body A ,
- θ_3 is set to zero with body C oriented perpendicular to body B , and
- θ_4 can be set to zero at any position of body D , as it is axisymmetric.

This is important because the CMG operates respecting the following assumptions:

Assumption 1. *Due to the existence of a singularity at $\theta_2 = \pm 90^\circ$ ($\pm \pi/2$ rad), that is known as a gimbal lock (HEMINGWAY; O'REILLY, 2018), the body B angular position operates within the interval of $\pm 80^\circ$ ($\pm 4\pi/9$ rad), i.e.,*

$$-80^\circ < \theta_2(t) < 80^\circ, \quad \forall t \geq 0. \quad (2.1)$$

Assumption 2. *The body C angular position is always within the interval of $\pm 90^\circ$ ($\pm \pi/2$ rad) during its operation due to physical constraints, i.e.,*

$$-90^\circ < \theta_3(t) < 90^\circ, \quad \forall t \geq 0. \quad (2.2)$$

Assumption 3. *In accordance with Remark 2, the rotor velocity is always within the interval of 200 rpm ($20\pi/3$ rad/s) to 800 rpm ($80\pi/3$ rad/s) during its operation, i.e.,*

$$200 \text{ rpm} < \dot{\theta}_4(t) < 800 \text{ rpm}, \quad \forall t \geq 0. \quad (2.3)$$

Being acquainted with the plant characteristics, the modeling process can be initiated in the following.

2.2 Kinematics stage

Firstly, frames are attached to each CMG body in order to describe its relative orientation. It is executed based on the convention ² defined in Section 3.4 in Craig (2005, p. 67-69), in which frame $\{\mathbf{i}\}$ ($\mathbf{i} \in \mathbb{N}$), composed of dextral sets of orthogonal unit vectors $\hat{X}_{\mathbf{i}}$, $\hat{Y}_{\mathbf{i}}$ and $\hat{Z}_{\mathbf{i}}$, is attached rigidly to each body. This convention was adopted because of its convenient and methodical process to link the reference frames to the system bodies.

The resulting attached frames $\{i\}$ ($i = 1, 2, 3, 4$) on CMG bodies A , B , C , and D , respectively, are shown in Fig. 6, concurrently with the inertial reference frame $\{0\}$, attached to the CMG fixed base, and motors applied torques T_1 and T_2 . All these frames have their origins fixed at the rotor center (since all CMG bodies are assumed to be symmetric, as stated in Remark 1), although they are dislocated in Fig. 6 to improve their illustration.

One can see the detailed process in (TORIUMI, 2016, p. 73). This way, with the frames attached to the bodies, their motions can now be described by their velocity vectors.

Craig (2005, p. 144-146), in its Section 5.6, iteratively derives the linear and angular velocity vectors by computing the velocity of each link of a manipulator in order, starting from its base. Then, the velocity of link $\mathbf{i} + 1$ will be that of link $\mathbf{i}$ plus the new velocity components that were added by joint $\mathbf{i} + 1$.

²also known as the *modified Denavit-Hartenberg parameters* (CORKE, 2017, p. 218-219).

Figure 6: Resulting bodies attached frames

Source: Author.

In the CMG case, as their bodies do not have translational displacements, only its angular velocity vectors need to be computed. This is done by employing the following iterative formulation:

$${}^i\boldsymbol{\omega}_i = {}_{i-1}\mathbf{R} {}^{i-1}\boldsymbol{\omega}_{i-1} + \begin{bmatrix} 0 & 0 & \dot{\theta}_i \end{bmatrix}^\top, \quad (i = 1, 2, 3, 4), \quad (2.4)$$

composed of body angular velocities $\dot{\theta}_i \in \mathbb{R}$ in $\hat{Z}_i$ direction and rotation matrices ${}_{i-1}^i\mathbf{R} \in \mathbb{R}^{3 \times 3}$, as presented in (2.5), that describe the orientation of frame $\{i-1\}$ relative to $\{i\}$, composed of direction cosine elements (for which the notations $c_x = \cos(x)$ and $s_x = \sin(x)$ are defined):

$$\begin{aligned} {}_0^1\mathbf{R} &= \begin{bmatrix} c_{\theta_1} & s_{\theta_1} & 0 \\ -s_{\theta_1} & c_{\theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad {}_1^2\mathbf{R} = \begin{bmatrix} s_{\theta_2} & 0 & c_{\theta_2} \\ c_{\theta_2} & 0 & -s_{\theta_2} \\ 0 & 1 & 0 \end{bmatrix}, \\ {}_2^3\mathbf{R} &= \begin{bmatrix} c_{\theta_3} & 0 & -s_{\theta_3} \\ -s_{\theta_3} & 0 & -c_{\theta_3} \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad {}_3^4\mathbf{R} = \begin{bmatrix} c_{\theta_4} & 0 & s_{\theta_4} \\ -s_{\theta_4} & 0 & c_{\theta_4} \\ 0 & -1 & 0 \end{bmatrix}. \end{aligned} \quad (2.5)$$

As the CMG fixed base position does not change in time, ${}^0\boldsymbol{\omega}_0 = [0 \ 0 \ 0]^\top$, and the

following CMG bodies angular velocity vectors are derived by calculating (2.4):

$$\begin{aligned}
 {}^1\boldsymbol{\omega}_1 &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, {}^2\boldsymbol{\omega}_2 = \begin{bmatrix} c_{\theta_2}\dot{\theta}_1 \\ -s_{\theta_2}\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, {}^3\boldsymbol{\omega}_3 = \begin{bmatrix} c_{\theta_2}c_{\theta_3}\dot{\theta}_1 - s_{\theta_3}\dot{\theta}_2 \\ -c_{\theta_2}s_{\theta_3}\dot{\theta}_1 - c_{\theta_3}\dot{\theta}_2 \\ -s_{\theta_2}\dot{\theta}_1 + \dot{\theta}_3 \end{bmatrix}, \\
 \text{and } {}^4\boldsymbol{\omega}_4 &= \begin{bmatrix} -\alpha\dot{\theta}_1 - s_{\theta_3}c_{\theta_4}\dot{\theta}_2 + s_{\theta_4}\dot{\theta}_3 \\ -\beta\dot{\theta}_1 + s_{\theta_3}s_{\theta_4}\dot{\theta}_2 + c_{\theta_4}\dot{\theta}_3 \\ c_{\theta_2}s_{\theta_3}\dot{\theta}_1 + c_{\theta_3}\dot{\theta}_2 + \dot{\theta}_4 \end{bmatrix}, \text{ with } \begin{cases} \alpha = s_{\theta_2}s_{\theta_4} - c_{\theta_2}c_{\theta_3}c_{\theta_4}, \\ \beta = s_{\theta_2}c_{\theta_4} + c_{\theta_2}c_{\theta_3}s_{\theta_4}. \end{cases}
 \end{aligned} \tag{2.6}$$

This way, these angular velocity vectors are used in the next dynamics stage.

2.3 Dynamics stage

For this stage, the Gibbs-Appell equations are first considered (BARUH, 1999, p. 509):

$$U_k = \sum_{i=1}^N \left(\frac{\partial \mathbf{v}_{G_i}^\top}{\partial w_k} \mathbf{F}_i + \frac{\partial \boldsymbol{\omega}_i^\top}{\partial w_k} \mathbf{M}_{G_i} \right), \quad (k = 1, 2, \dots, d), \tag{2.7}$$

in which $N, d \in \mathbb{N}^*$ are the number of bodies of the plant and the number of DOF, respectively; additionally, the vectors $\mathbf{F}_i, \mathbf{M}_{G_i} \in \mathbb{R}^3$ denote the resultant vector of all forces external to the i^{th} body and the resultant external moment vector about the center of mass of the i^{th} body, the vectors $\mathbf{v}_{G_i}, \boldsymbol{\omega}_i \in \mathbb{R}^3$ denote the linear velocity vector of center of mass and the angular velocity vector of the i^{th} body, and $w_k \in \mathbb{R}$ is the k^{th} quasi-velocity.

This equation is recognized as the generalized force composed of two components, the first one is the translational component and the second one is the rotational component.

As some authors show equivalences between Gibbs-Appell and Kane's equations, as in (BARUH, 1999, p. 525) and (DESLOGE, 1987), Kane's dynamical equations (KANE; LIKINS; LEVINSON, 1983, p. 275) are also considered:

$$U_k + \tilde{U}_k = 0, \quad (k = 1, 2, \dots, d), \tag{2.8}$$

in which $\tilde{U}_k$ is the generalized inertia force, defined as (BARUH, 1999, p. 525):

$$\tilde{U}_k = \sum_{i=1}^N \left(\frac{\partial \mathbf{v}_{G_i}^\top}{\partial w_k} \tilde{\mathbf{F}}_i + \frac{\partial \boldsymbol{\omega}_i^\top}{\partial w_k} \tilde{\mathbf{M}}_{G_i} \right), \quad (k = 1, 2, \dots, d), \tag{2.9}$$

in which $\tilde{\mathbf{F}}_i = -m_i \mathbf{a}_{G_i}$ is the resultant inertia force vector acting through the center of

mass of the i^{th} body ($m_i \in \mathbb{R}$ is the i^{th} body mass and $\mathbf{a}_{G_i} \in \mathbb{R}^3$ is the acceleration of the center of mass of the i^{th} body) and $\widetilde{\mathbf{M}}_{G_i} = -\dot{\mathbf{H}}_{G_i}$ is the resultant inertia torque vector ($\mathbf{H}_{G_i} \in \mathbb{R}^3$ is the angular momentum about the center of mass of the i^{th} body).

As the ECP CMG bodies do not have translational displacement (Remark 1), the first component within the sum of (2.7) and (2.9) vanish, and (2.8) results in the following equations of motion:

$$\mathcal{E}_k = \sum_{i=1}^N \left[\frac{\partial \boldsymbol{\omega}_i^\top}{\partial w_k} \left(\mathbf{M}_{G_i} - \dot{\mathbf{H}}_{G_i} \right) \right] = 0, \quad (k = 1, 2, \dots, d), \quad (2.10)$$

where $N = 4$, as the CMG has four bodies (with $i = 1, 2, 3, 4$ related to bodies A, B, C , and D , respectively), $d = 4$, as it has four DOF ($k = 1, 2, 3, 4$). The quasi-velocities are $w_k = \dot{\theta}_k$, and the angular velocity vectors $\boldsymbol{\omega}_i = {}^i\boldsymbol{\omega}_i$ were obtained previously in (2.6).

Thus, the CMG equations of motion can be rewritten in its vector form:

$$\underbrace{\begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \\ \mathcal{E}_4 \end{bmatrix}}_{\mathcal{E} \in \mathbb{R}^4} = \underbrace{\begin{bmatrix} \frac{\partial \boldsymbol{\omega}_1^\top}{\partial \dot{\theta}_1} & \frac{\partial \boldsymbol{\omega}_2^\top}{\partial \dot{\theta}_1} & \frac{\partial \boldsymbol{\omega}_3^\top}{\partial \dot{\theta}_1} & \frac{\partial \boldsymbol{\omega}_4^\top}{\partial \dot{\theta}_1} \\ \frac{\partial \boldsymbol{\omega}_1^\top}{\partial \dot{\theta}_2} & \frac{\partial \boldsymbol{\omega}_2^\top}{\partial \dot{\theta}_2} & \frac{\partial \boldsymbol{\omega}_3^\top}{\partial \dot{\theta}_2} & \frac{\partial \boldsymbol{\omega}_4^\top}{\partial \dot{\theta}_2} \\ \frac{\partial \boldsymbol{\omega}_1^\top}{\partial \dot{\theta}_3} & \frac{\partial \boldsymbol{\omega}_2^\top}{\partial \dot{\theta}_3} & \frac{\partial \boldsymbol{\omega}_3^\top}{\partial \dot{\theta}_3} & \frac{\partial \boldsymbol{\omega}_4^\top}{\partial \dot{\theta}_3} \\ \frac{\partial \boldsymbol{\omega}_1^\top}{\partial \dot{\theta}_4} & \frac{\partial \boldsymbol{\omega}_2^\top}{\partial \dot{\theta}_4} & \frac{\partial \boldsymbol{\omega}_3^\top}{\partial \dot{\theta}_4} & \frac{\partial \boldsymbol{\omega}_4^\top}{\partial \dot{\theta}_4} \end{bmatrix}}_{\mathbf{D} \in \mathbb{R}^{4 \times 12}} \underbrace{\begin{bmatrix} \mathbf{M}_{G_1} - \dot{\mathbf{H}}_{G_1} \\ \mathbf{M}_{G_2} - \dot{\mathbf{H}}_{G_2} \\ \mathbf{M}_{G_3} - \dot{\mathbf{H}}_{G_3} \\ \mathbf{M}_{G_4} - \dot{\mathbf{H}}_{G_4} \end{bmatrix}}_{\mathbf{f} \in \mathbb{R}^{12}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0} \in \mathbb{R}^4}, \quad (2.11)$$

$$\text{with } \mathbf{D} = \begin{bmatrix} 0 & 0 & 1 & c_{\theta_2} & -s_{\theta_2} & 0 & c_{\theta_2}c_{\theta_3} & -c_{\theta_2}s_{\theta_3} & -s_{\theta_2} & -\alpha & -\beta & c_{\theta_2}s_{\theta_3} \\ 0 & 0 & 0 & 0 & 0 & 1 & -s_{\theta_3} & -c_{\theta_3} & 0 & -s_{\theta_3}c_{\theta_4} & s_{\theta_3}s_{\theta_4} & c_{\theta_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s_{\theta_4} & c_{\theta_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

being $\alpha, \beta \in \mathbb{R}$ described in (2.6),

The CMG external moment vector about the center of mass of the i^{th} body is given by:

$$\mathbf{M}_{G_i} = \begin{bmatrix} 0 & 0 & \eta_i \end{bmatrix}^\top, \quad (i = 1, 2, 3, 4), \quad \begin{cases} \eta_1 = -\eta_4 c_{\theta_2} s_{\theta_3} + \eta_3 s_{\theta_2} - \mu_1 \dot{\theta}_1, \\ \eta_2 = -\eta_4 c_{\theta_3} - \mu_2 \dot{\theta}_2, \\ \eta_3 = T_2 - \mu_3 \dot{\theta}_3, \\ \eta_4 = T_1 - \mu_4 \dot{\theta}_4, \end{cases} \quad (2.12)$$

with $\mu_i \in \mathbb{R}$ being the viscous friction coefficient (VFC) of axis $\#i$.

And the rate of change of CMG angular momentum about the center of mass of the i th body is given by the following equation (BARUH, 1999, p. 434-435):

$$\dot{\mathbf{H}}_{G_i} = \mathcal{I}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\Omega}_i \mathcal{I}_i \boldsymbol{\omega}_i, \quad (2.13)$$

in which $\boldsymbol{\Omega}_i \in \mathbb{R}^{3 \times 3}$ is the skew-symmetric matrix (BARUH, 1999, p. 91) associated with vector $\boldsymbol{\omega}_i$:

$$\boldsymbol{\Omega}_i = \begin{bmatrix} 0 & -\omega_i(3) & \omega_i(2) \\ \omega_i(3) & 0 & -\omega_i(1) \\ -\omega_i(2) & \omega_i(1) & 0 \end{bmatrix},$$

and the diagonal matrices $\mathcal{I}_i \in \mathbb{R}^{3 \times 3}$ are the CMG inertia matrices ³:

$$\begin{aligned} \mathcal{I}_1 &= \text{diag}(I_A, J_A, K_A), \quad \mathcal{I}_2 = \text{diag}(K_B, I_B, J_B), \\ \mathcal{I}_3 &= \text{diag}(K_C, J_C, I_C), \quad \text{and } \mathcal{I}_4 = \text{diag}(K_D, I_D, J_D), \end{aligned} \quad (2.14)$$

where $I_m, J_m, K_m \in \mathbb{R}$ ($m = A, B, C, D$) are the scalar moments of inertia (SMI) of the m^{th} CMG body, and $K_D = I_D$, since the rotor is axisymmetric.

Thus, the following $\dot{\mathbf{H}}_{G_i}$ are obtained by solving (2.13) for $i = 1, 2, 3, 4$:

$$\begin{aligned} \dot{\mathbf{H}}_{G_1} &= \begin{bmatrix} 0 \\ 0 \\ K_A \ddot{\theta}_1 \end{bmatrix}, \quad \dot{\mathbf{H}}_{G_2} = \begin{bmatrix} K_B c_{\theta_2} \ddot{\theta}_1 + (I_B - J_B - K_B) s_{\theta_2} \dot{\theta}_1 \dot{\theta}_2 \\ -I_B s_{\theta_2} \ddot{\theta}_1 - (I_B + J_B - K_B) c_{\theta_2} \dot{\theta}_1 \dot{\theta}_2 \\ J_B \ddot{\theta}_2 - (I_B - K_B) c_{\theta_2} s_{\theta_2} \dot{\theta}_1^2 \end{bmatrix}, \\ \dot{\mathbf{H}}_{G_3} &= \begin{bmatrix} K_C (c_{\theta_2} c_{\theta_3} \ddot{\theta}_1 - s_{\theta_3} \ddot{\theta}_2 - s_{\theta_2} c_{\theta_3} \dot{\theta}_1 \dot{\theta}_2 - \eta \dot{\theta}_3) + (I_C - J_C) (s_{\theta_2} \dot{\theta}_1 - \dot{\theta}_3) \eta \\ -J_C (c_{\theta_2} s_{\theta_3} \ddot{\theta}_1 + c_{\theta_3} \ddot{\theta}_2 - s_{\theta_2} s_{\theta_3} \dot{\theta}_1 \dot{\theta}_2 + \zeta \dot{\theta}_3) + (I_C - K_C) (s_{\theta_2} \dot{\theta}_1 - \dot{\theta}_3) \zeta \\ -I_C (s_{\theta_2} \ddot{\theta}_1 - \ddot{\theta}_3 + c_{\theta_2} \dot{\theta}_1 \dot{\theta}_2) - (J_C - K_C) \eta \zeta \end{bmatrix}, \\ \dot{\mathbf{H}}_{G_4} &= \begin{bmatrix} I_D (-\alpha \ddot{\theta}_1 - s_{\theta_3} c_{\theta_4} \ddot{\theta}_2 + s_{\theta_4} \ddot{\theta}_3 + \gamma) + J_D (\eta + \dot{\theta}_4) (-\beta \dot{\theta}_1 + s_{\theta_3} s_{\theta_4} \dot{\theta}_2 + c_{\theta_4} \dot{\theta}_3) \\ I_D (-\beta \ddot{\theta}_1 + s_{\theta_3} s_{\theta_4} \ddot{\theta}_2 + c_{\theta_4} \ddot{\theta}_3 - \delta) - J_D (\eta + \dot{\theta}_4) (-\alpha \dot{\theta}_1 - s_{\theta_3} c_{\theta_4} \dot{\theta}_2 + s_{\theta_4} \dot{\theta}_3) \\ J_D (c_{\theta_2} s_{\theta_3} \ddot{\theta}_1 + c_{\theta_3} \ddot{\theta}_2 + \ddot{\theta}_4 - s_{\theta_2} s_{\theta_3} \dot{\theta}_1 \dot{\theta}_2 + \zeta \dot{\theta}_3) \end{bmatrix}, \end{aligned}$$

being $\eta = c_{\theta_2} s_{\theta_3} \dot{\theta}_1 + c_{\theta_3} \dot{\theta}_2$, $\zeta = c_{\theta_2} c_{\theta_3} \dot{\theta}_1 - s_{\theta_3} \dot{\theta}_2$, and:

$$\begin{aligned} \gamma &= s_{\theta_3} \left(\beta c_{\theta_2} \dot{\theta}_1^2 - 2c_{\theta_2} s_{\theta_3} s_{\theta_4} \dot{\theta}_1 \dot{\theta}_2 - c_{\theta_3} s_{\theta_4} \dot{\theta}_2^2 \right) - 2\eta c_{\theta_4} \dot{\theta}_3, \\ \delta &= s_{\theta_3} \left(\alpha c_{\theta_2} \dot{\theta}_1^2 + 2c_{\theta_2} s_{\theta_3} c_{\theta_4} \dot{\theta}_1 \dot{\theta}_2 + c_{\theta_3} c_{\theta_4} \dot{\theta}_2^2 \right) - 2\eta s_{\theta_4} \dot{\theta}_3, \end{aligned}$$

³It is worth remarking that they differ from the ones from ECP manual (ECP, 1999, p. 61) due to the adopted convention to attach the frames to the CMG bodies in Section 2.2, as shown in Appendix B.

with α and β described in (2.6).

Finally, by solving (2.11), it results in the following four nonlinear equations:

$$\begin{aligned}\mathcal{E}_1 = & -\mu_1\dot{\theta}_1 - (K_A + K_B + K_C + I_D - \mathcal{Y}s_{\theta_2}^2 - \mathcal{X}c_{\theta_2}^2s_{\theta_3}^2)\ddot{\theta}_1 + \mathcal{X}c_{\theta_2}c_{\theta_3}s_{\theta_3}\ddot{\theta}_2 + \mathcal{Z}s_{\theta_2}\ddot{\theta}_3 - \\ & J_Dc_{\theta_2}s_{\theta_3}\ddot{\theta}_4 + 2(\mathcal{Y} - \mathcal{X}s_{\theta_3}^2)c_{\theta_2}s_{\theta_2}\dot{\theta}_1\dot{\theta}_2 + 2\mathcal{X}c_{\theta_2}^2c_{\theta_3}s_{\theta_3}\dot{\theta}_1\dot{\theta}_3 - \mathcal{X}s_{\theta_2}c_{\theta_3}s_{\theta_3}\dot{\theta}_2^2 + \\ & (\mathcal{X} + \mathcal{Z} - 2\mathcal{X}s_{\theta_3}^2)c_{\theta_2}\dot{\theta}_2\dot{\theta}_3 + (s_{\theta_2}s_{\theta_3}\dot{\theta}_2 - c_{\theta_2}c_{\theta_3}\dot{\theta}_3)J_D\dot{\theta}_4 \\ = & 0,\end{aligned}\tag{2.15}$$

$$\begin{aligned}\mathcal{E}_2 = & -\mu_2\dot{\theta}_2 + \mathcal{X}c_{\theta_2}c_{\theta_3}s_{\theta_3}\ddot{\theta}_1 - (J_B + J_C + J_D + \mathcal{X}s_{\theta_3}^2)\ddot{\theta}_2 - J_Dc_{\theta_3}\ddot{\theta}_4 - (\mathcal{Y} - \mathcal{X}s_{\theta_3}^2)c_{\theta_2}s_{\theta_2}\dot{\theta}_1^2 + \\ & (\mathcal{X} - \mathcal{Z} - 2\mathcal{X}s_{\theta_3}^2)c_{\theta_2}\dot{\theta}_1\dot{\theta}_3 - 2\mathcal{X}c_{\theta_3}s_{\theta_3}\dot{\theta}_2\dot{\theta}_3 + (\dot{\theta}_3 - s_{\theta_2}\dot{\theta}_1)J_Ds_{\theta_3}\dot{\theta}_4 \\ = & 0,\end{aligned}\tag{2.16}$$

$$\begin{aligned}\mathcal{E}_3 = & T_2 - \mu_3\dot{\theta}_3 + \mathcal{Z}s_{\theta_2}\ddot{\theta}_1 - \mathcal{Z}\ddot{\theta}_3 - \mathcal{X}c_{\theta_2}^2c_{\theta_3}s_{\theta_3}\dot{\theta}_1^2 - (\mathcal{X} - \mathcal{Z} - 2\mathcal{X}s_{\theta_3}^2)c_{\theta_2}\dot{\theta}_1\dot{\theta}_2 + \mathcal{X}c_{\theta_3}s_{\theta_3}\dot{\theta}_2^2 + \\ & (c_{\theta_2}c_{\theta_3}\dot{\theta}_1 - s_{\theta_3}\dot{\theta}_2)J_D\dot{\theta}_4 \\ = & 0,\end{aligned}\tag{2.17}$$

$$\begin{aligned}\mathcal{E}_4 = & T_1 - \mu_4\dot{\theta}_4 - J_D(c_{\theta_2}s_{\theta_3}\ddot{\theta}_1 + c_{\theta_3}\ddot{\theta}_2 + \ddot{\theta}_4 - s_{\theta_2}s_{\theta_3}\dot{\theta}_1\dot{\theta}_2 + c_{\theta_2}c_{\theta_3}\dot{\theta}_1\dot{\theta}_3 - s_{\theta_3}\dot{\theta}_2\dot{\theta}_3) \\ = & 0,\end{aligned}\tag{2.18}$$

with:

$$\mathcal{X} = K_C - J_C - J_D + I_D, \quad \mathcal{Y} = K_B - I_B + K_C - I_C, \quad \text{and} \quad \mathcal{Z} = I_C + I_D.\tag{2.19}$$

2.4 Nonlinear equation forms

To assist the nonlinear controllers design in Chapter 4, the equations (2.15)-(2.18) are rewritten in two suitable forms: the *general form* (Subsection 2.4.1) used for the PB approach, and the *companion form* (Subsection 2.4.2) used for both FL and SM approaches.

They are composed of some plant parameters, e.g., the bodies SMI and the axes VFC, and their values are presented in Appendix E, which are used to perform the simulations and experiments in Chapter 5.

2.4.1 General form

The general form (SLOTINE; LI, 1991, p. 393) of an n -DOF MIMO system dynamics is given by the following Euler-Lagrange equations in matrix form:

$$\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u}, \quad (2.20)$$

in which vector $\mathbf{q} \in \mathbb{R}^n$ is composed of the generalized coordinates, $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^n$ is the vector of gravitational terms, $\mathbf{u} \in \mathbb{R}^n$ is the control input vector composed of the force-torque acting on the system, and $\mathbf{M}(\mathbf{q}), \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n \times n}$ are the symmetric inertia matrix and centripetal-Coriolis matrix, respectively.

For the CMG case, by rewriting its four nonlinear equations (2.15)-(2.18), yields to its following general form:

$$\mathbf{M}(\theta_2, \theta_3) \ddot{\mathbf{q}} + \mathbf{V}(\theta_2, \theta_3, \dot{\mathbf{q}}) \dot{\mathbf{q}} = \mathbf{u} + \boldsymbol{\tau}_f(\dot{\mathbf{q}}), \quad (2.21)$$

with $\mathbf{q}(t) = [\theta_1(t) \ \theta_2(t) \ \theta_3(t) \ \theta_4(t)]^\top \in \mathbb{R}^4$, $\mathbf{u}(t) = [0 \ 0 \ T_2(t) \ T_1(t)]^\top \in \mathbb{R}^4$,

$$\mathbf{M}(\theta_2, \theta_3) = \begin{bmatrix} m_u & -\mathcal{X}c_{\theta_2}c_{\theta_3}s_{\theta_3} & -\mathcal{Z}s_{\theta_2} & J_Dc_{\theta_2}s_{\theta_3} \\ -\mathcal{X}c_{\theta_2}c_{\theta_3}s_{\theta_3} & \mathcal{X}s_{\theta_3}^2 + J_B + J_C + J_D & 0 & J_Dc_{\theta_3} \\ -\mathcal{Z}s_{\theta_2} & 0 & \mathcal{Z} & 0 \\ J_Dc_{\theta_2}s_{\theta_3} & J_Dc_{\theta_3} & 0 & J_D \end{bmatrix}, \boldsymbol{\tau}_f(\dot{\mathbf{q}}) = \begin{bmatrix} -\mu_1\dot{\theta}_1 \\ -\mu_2\dot{\theta}_2 \\ -\mu_3\dot{\theta}_3 \\ -\mu_4\dot{\theta}_4 \end{bmatrix},$$

$$\mathbf{V}(\theta_2, \theta_3, \dot{\mathbf{q}}) = \begin{bmatrix} v_{u_1} & v_{u_2} + v_{u_3} + \mathcal{X}s_{\theta_2}c_{\theta_3}s_{\theta_3}\dot{\theta}_2 & v_{c_1} - \frac{\mathcal{Z}}{2}c_{\theta_2}\dot{\theta}_2 & v_{c_2} \\ v_{u_2} - v_{u_3} & \mathcal{X}c_{\theta_3}s_{\theta_3}\dot{\theta}_3 & v_{c_3} & -\frac{J_D}{2}s_{\theta_3}(\dot{\theta}_3 - s_{\theta_2}\dot{\theta}_1) \\ -v_{c_1} - \frac{\mathcal{Z}}{2}c_{\theta_2}\dot{\theta}_2 & -v_{c_3} & 0 & v_a \\ v_{c_2} & -\frac{J_D}{2}s_{\theta_3}(\dot{\theta}_3 + s_{\theta_2}\dot{\theta}_1) & -v_a & 0 \end{bmatrix},$$

$$m_u = K_A + K_B + K_C + I_D - \mathcal{Y}s_{\theta_2}^2 - \mathcal{X}c_{\theta_2}^2s_{\theta_3}^2,$$

$$v_{u_1} = -(\mathcal{Y} - \mathcal{X}s_{\theta_3}^2)c_{\theta_2}s_{\theta_2}\dot{\theta}_2 - \mathcal{X}c_{\theta_2}^2c_{\theta_3}s_{\theta_3}\dot{\theta}_3,$$

$$v_{u_2} = -\frac{1 - 2s_{\theta_3}^2}{2}\mathcal{X}c_{\theta_2}\dot{\theta}_3,$$

$$v_{u_3} = -(\mathcal{Y} - \mathcal{X}s_{\theta_3}^2)c_{\theta_2}s_{\theta_2}\dot{\theta}_1 - \frac{\mathcal{Z}}{2}c_{\theta_2}\dot{\theta}_3 - \frac{J_D}{2}s_{\theta_2}s_{\theta_3}\dot{\theta}_4,$$

$$v_{c_1} = -\mathcal{X}c_{\theta_2}^2c_{\theta_3}s_{\theta_3}\dot{\theta}_1 - \frac{1 - 2s_{\theta_3}^2}{2}\mathcal{X}c_{\theta_2}\dot{\theta}_2 + \frac{J_D}{2}c_{\theta_2}c_{\theta_3}\dot{\theta}_4,$$

$$v_{c_2} = \frac{J_D}{2}(c_{\theta_2}c_{\theta_3}\dot{\theta}_3 - s_{\theta_2}s_{\theta_3}\dot{\theta}_2),$$

$$v_{c_3} = -\frac{\mathcal{X} - \mathcal{Z} - 2\mathcal{X}s_{\theta_3}^2}{2}c_{\theta_2}\dot{\theta}_1 + \mathcal{X}c_{\theta_3}s_{\theta_3}\dot{\theta}_2 - \frac{J_D}{2}s_{\theta_3}\dot{\theta}_4,$$

$$v_a = \frac{J_D}{2}(s_{\theta_3}\dot{\theta}_2 - c_{\theta_2}c_{\theta_3}\dot{\theta}_1),$$

with $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ defined in (2.19).

Note that (2.21) does not present the vector $\mathbf{g}(\mathbf{q})$ due to the statement in Remark 1.

To aid the PB controller design process, (2.21) is organized in two $\mathbb{R}^2$ subsystems: the unactuated and actuated ones, yielding to the following form:

$$\underbrace{\begin{bmatrix} \mathbf{M}_u & \mathbf{M}_c \\ \mathbf{M}_c^\top & \mathbf{M}_a \end{bmatrix}}_{\mathbf{M}(\theta_2, \theta_3)} \underbrace{\begin{bmatrix} \ddot{\mathbf{q}}_u \\ \ddot{\mathbf{q}}_a \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} \mathbf{V}_u & \mathbf{V}_{cr} \\ \mathbf{V}_{cl} & \mathbf{V}_a \end{bmatrix}}_{\mathbf{V}(\theta_2, \theta_3, \dot{\mathbf{q}})} \underbrace{\begin{bmatrix} \dot{\mathbf{q}}_u \\ \dot{\mathbf{q}}_a \end{bmatrix}}_{\dot{\mathbf{q}}} = \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{T} \end{bmatrix}}_{\mathbf{u}} + \underbrace{\begin{bmatrix} \boldsymbol{\tau}_{f_u} \\ \boldsymbol{\tau}_{f_a} \end{bmatrix}}_{\boldsymbol{\tau}_f(\dot{\mathbf{q}})}, \text{ with:} \quad (2.22)$$

$$\mathbf{M}_u = \begin{bmatrix} m_u & -\mathcal{X}c_{\theta_2}c_{\theta_3}s_{\theta_3} \\ -\mathcal{X}c_{\theta_2}c_{\theta_3}s_{\theta_3} & \mathcal{X}s_{\theta_3}^2 + J_B + J_C + J_D \end{bmatrix}, \quad \mathbf{M}_a = \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & J_D \end{bmatrix}, \quad \mathbf{M}_c = \begin{bmatrix} -\mathcal{Z}s_{\theta_2} & J_Dc_{\theta_2}s_{\theta_3} \\ 0 & J_Dc_{\theta_3} \end{bmatrix},$$

$$\mathbf{V}_u = \begin{bmatrix} v_{u1} & v_{u2} + v_{u3} + \mathcal{X}s_{\theta_2}c_{\theta_3}s_{\theta_3}\dot{\theta}_2 \\ v_{u2} - v_{u3} & \mathcal{X}c_{\theta_3}s_{\theta_3}\dot{\theta}_3 \end{bmatrix}, \quad \mathbf{V}_a = \begin{bmatrix} 0 & v_a \\ -v_a & 0 \end{bmatrix}, \quad \boldsymbol{\tau}_{f_u} = \begin{bmatrix} -\mu_1\dot{\theta}_1 \\ -\mu_2\dot{\theta}_2 \end{bmatrix}, \quad \boldsymbol{\tau}_{f_a} = \begin{bmatrix} -\mu_3\dot{\theta}_3 \\ -\mu_4\dot{\theta}_4 \end{bmatrix},$$

$$\mathbf{V}_{cr} = \begin{bmatrix} v_{c1} - \frac{\mathcal{Z}}{2}c_{\theta_2}\dot{\theta}_2 & v_{c2} \\ v_{c3} & \frac{-J_D}{2}s_{\theta_3}(\dot{\theta}_3 - s_{\theta_2}\dot{\theta}_1) \end{bmatrix}, \quad \mathbf{V}_{cl} = \begin{bmatrix} -v_{c1} - \frac{\mathcal{Z}}{2}c_{\theta_2}\dot{\theta}_2 & -v_{c3} \\ v_{c2} & \frac{-J_D}{2}s_{\theta_3}(\dot{\theta}_3 + s_{\theta_2}\dot{\theta}_1) \end{bmatrix},$$

with the unactuated state vector $\mathbf{q}_u(t) = [\theta_1(t) \ \theta_2(t)]^\top \in \mathbb{R}^2$, the actuated state vector $\mathbf{q}_a(t) = [\theta_3(t) \ \theta_4(t)]^\top \in \mathbb{R}^2$, the control vector $\mathbf{T}(t) = [T_2(t) \ T_1(t)]^\top \in \mathbb{R}^2$ composed of both motor torques, the viscous friction torque vector $\boldsymbol{\tau}_f(\dot{\mathbf{q}}) \in \mathbb{R}^4$, the symmetric inertia matrix $\mathbf{M}(\theta_2, \theta_3) \in \mathbb{R}^{4 \times 4}$, and the centripetal-Coriolis matrix $\mathbf{V}(\theta_2, \theta_3, \dot{\mathbf{q}}) \in \mathbb{R}^{4 \times 4}$. The subindexes $(\cdot)_u$ and $(\cdot)_a$ stand for the unactuated and actuated dynamics, respectively, and $(\cdot)_c$ for the coupling off-diagonal blocks, being $(\cdot)_{cl}$ and $(\cdot)_{cr}$ the left and right blocks.

2.4.2 Companion form

Slotine and Li (1991, p. 209) presents the so-called *companion form* of a SISO system as:

$$x^{(n)} = f(\mathbf{x}) + b(\mathbf{x})u, \quad (2.23)$$

where u is the scalar control input, x is the scalar output of interest, $\mathbf{x}$ is the system state vector, and $f(\mathbf{x})$ and $b(\mathbf{x})$ are nonlinear function of the states.

Similarly, for a MIMO system, the companion form is given by the following generic form:

$$x_k^{(n)} = f_k(\mathbf{x}) + \sum_{l=1}^{\alpha} b_{kl}(\mathbf{x})u_l, \quad (k = 1, 2, \dots, d), \quad (2.24)$$

being $\alpha, d \in \mathbb{N}^*$ the total number of control inputs and DOFs, respectively.

Regarding the CMG case, firstly, the angular acceleration variables in (2.21) need to be isolated:

$$\ddot{\mathbf{q}} = \mathbf{M}(\theta_2, \theta_3)^{-1} (\mathbf{u} + \boldsymbol{\tau}_f(\dot{\mathbf{q}}) - \mathbf{V}(\theta_2, \theta_3, \dot{\mathbf{q}}) \dot{\mathbf{q}}), \quad (2.25)$$

to generate the nonlinear model (explicitly presented in Appendix C) that is used for simulations in Chapter 5.

Then, its companion form is obtained by rewriting it in the following form:

$$\ddot{\theta}_k = \frac{\mathfrak{N}_k}{\mathfrak{d}_k \mathfrak{D}}, \quad (k = 1, 2, 3, 4), \quad (2.26)$$

$$\mathfrak{N}_1 = \bar{f}_1 + g_1(\boldsymbol{\mu}) + b_{11}T_1 + b_{12}T_{\text{gyro}},$$

$$\mathfrak{N}_2 = \bar{f}_2 + g_2(\boldsymbol{\mu}) + b_{21}T_1 + b_{22}T_{\text{gyro}},$$

$$\mathfrak{N}_3 = \bar{f}_3 + g_3(\boldsymbol{\mu}) + b_3T_2,$$

$$\mathfrak{N}_4 = \bar{f}_4 + g_4(\boldsymbol{\mu}) + b_4T_1,$$

$$\mathfrak{d}_1 = \mathfrak{d}_2 = 1,$$

$$\mathfrak{d}_3 = \mathcal{Z},$$

$$\mathfrak{d}_4 = J_D, \text{ and}$$

$$\mathfrak{D} = \mathcal{V}(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) + \mathcal{F}\mathcal{G}c_{\theta_2}^2 + \mathcal{E}\mathcal{H}c_{\theta_2}^2c_{\theta_3}^2,$$

where:

$$\mathcal{X} = K_C - J_C - J_D + I_D, \quad \mathcal{Y} = K_B - I_B + K_C - I_C,$$

$$\mathcal{Z} = I_C + I_D, \quad \mathcal{W} = J_B + J_C, \quad \mathcal{V} = K_A + I_B,$$

$$\mathcal{E} = J_C - K_C - I_D, \quad \mathcal{F} = J_B + K_C + I_D, \quad \mathcal{G} = K_B - I_B + J_C, \quad \mathcal{H} = K_B - J_B - I_B,$$

and $\bar{f}_k$, $g_k(\boldsymbol{\mu})$, b_{11} , b_{12} , b_{21} , b_{22} , b_3 , and b_4 (explicitly given next) are nonlinear functions of the states (being $g_k(\boldsymbol{\mu})$ also function of the viscous friction coefficients μ_k) and $T_{\text{gyro}} = J_D\dot{\theta}_3\dot{\theta}_4$ is the CMG gyroscopic torque.

$$\begin{aligned} \bar{f}_1 = & (\mathcal{F} + \mathcal{E}c_{\theta_3}^2) T_2 s_{\theta_2} + [\mathcal{E}(\mathcal{Y} + \mathcal{W}) + J_D\mathcal{W}] \dot{\theta}_1^2 c_{\theta_2}^2 c_{\theta_3} s_{\theta_2} s_{\theta_3} + J_D\mathcal{W} \dot{\theta}_1 \dot{\theta}_4 c_{\theta_2} c_{\theta_3} s_{\theta_2} \\ & + [(\mathcal{Y} + \mathcal{G})(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) + J_D\mathcal{W}c_{\theta_3}^2] \dot{\theta}_1 \dot{\theta}_2 c_{\theta_2} s_{\theta_2} - \mathcal{F}(\mathcal{Y} - \mathcal{G}) \dot{\theta}_2 \dot{\theta}_3 c_{\theta_2} \\ & - [\mathcal{E}(2\mathcal{W} + \mathcal{Y} - \mathcal{G}) + J_D\mathcal{W}] (\dot{\theta}_1 c_{\theta_2} s_{\theta_3} + \dot{\theta}_2 c_{\theta_3}) \dot{\theta}_3 c_{\theta_2} c_{\theta_3}, \end{aligned}$$

$$\begin{aligned}
\bar{f}_2 = & -\mathcal{E}T_2c_{\theta_2}c_{\theta_3}s_{\theta_2}s_{\theta_3} - \mathcal{V}(\mathcal{Y} - \mathcal{X} + \mathcal{X}c_{\theta_3}^2)\dot{\theta}_1^2c_{\theta_2}s_{\theta_2} + [\mathcal{E}(\mathcal{Y} + \mathcal{G}) + J_D\mathcal{G}]\dot{\theta}_1^2c_{\theta_2}^3c_{\theta_3}^2s_{\theta_2} \\
& - \mathcal{G}(\mathcal{Y} - \mathcal{X})\dot{\theta}_1^2c_{\theta_2}^3s_{\theta_2} - [J_D(\mathcal{V} + \mathcal{G}c_{\theta_2}^2) + \mathcal{E}(\mathcal{Y} + \mathcal{G})c_{\theta_2}^2]\dot{\theta}_1\dot{\theta}_2c_{\theta_3}s_{\theta_2}s_{\theta_3} \\
& - \mathcal{V}[\mathcal{X} + \mathcal{Z} + (2\mathcal{E} + J_D)c_{\theta_3}^2]\dot{\theta}_1\dot{\theta}_3c_{\theta_2} - \mathcal{G}(\mathcal{X} + \mathcal{Z})\dot{\theta}_1\dot{\theta}_3c_{\theta_2}^3 + \mathcal{V}(2\mathcal{E} + J_D)\dot{\theta}_2\dot{\theta}_3c_{\theta_3}s_{\theta_3} \\
& - [\mathcal{E}(\mathcal{Y} + \mathcal{G}) + J_D\mathcal{G}](\dot{\theta}_1c_{\theta_2}c_{\theta_3} - \dot{\theta}_2s_{\theta_3})\dot{\theta}_3c_{\theta_2}^2c_{\theta_3} - J_D(\mathcal{V} + \mathcal{G}c_{\theta_2}^2)\dot{\theta}_1\dot{\theta}_4s_{\theta_2}s_{\theta_3}, \\
\bar{f}_3 = & -\mathcal{Z}\mathcal{F}T_1c_{\theta_2}s_{\theta_2}s_{\theta_3} + \mathcal{E}[\mathcal{Z}(\mathcal{Y} + \mathcal{F} + \mathcal{E}) + \mathcal{V}(\mathcal{F} - \mathcal{X}c_{\theta_3}^2)]\dot{\theta}_1^2c_{\theta_2}^2c_{\theta_3}s_{\theta_3} \\
& + J_D(\mathcal{V}\mathcal{F} + \mathcal{Z}\mathcal{W})\dot{\theta}_1^2c_{\theta_2}^2c_{\theta_3}s_{\theta_3} - \mathcal{E}[(\mathcal{F} - \mathcal{Z})(\mathcal{Z} - \mathcal{G}) + \mathcal{X}\mathcal{H}c_{\theta_3}^2]\dot{\theta}_1^2c_{\theta_2}^4c_{\theta_3}s_{\theta_3} \\
& + J_D[\mathcal{G}(J_B + K_C) - I_C\mathcal{W} + I_D\mathcal{H}]\dot{\theta}_1^2c_{\theta_2}^4c_{\theta_3}s_{\theta_3} \\
& + [\mathcal{Z}(\mathcal{V} + \mathcal{Y} + \mathcal{G})(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) + \mathcal{V}\mathcal{F}(\mathcal{X} + 2\mathcal{E}c_{\theta_3}^2) + J_D(\mathcal{Z}\mathcal{W} + 2\mathcal{V}\mathcal{F})c_{\theta_3}^2]\dot{\theta}_1\dot{\theta}_2c_{\theta_2} \\
& - [\mathcal{F}(\mathcal{Y} - \mathcal{G})(\mathcal{Z} - \mathcal{G}) + J_D\mathcal{F}\mathcal{G} + \mathcal{E}(\mathcal{Z}\mathcal{Y} + 2\mathcal{X}\mathcal{H}c_{\theta_3}^2)c_{\theta_3}^2]\dot{\theta}_1\dot{\theta}_2c_{\theta_2}^3 \\
& - \mathcal{V}\mathcal{X}\mathcal{E}\dot{\theta}_1\dot{\theta}_2c_{\theta_2}c_{2\theta_3}c_{\theta_3}^2 + (\mathcal{E} + J_D)[3\mathcal{F}\mathcal{G} - \mathcal{W}(2\mathcal{Z} + \mathcal{Y})]\dot{\theta}_1\dot{\theta}_2c_{\theta_2}^3c_{\theta_3}^2 \\
& - \mathcal{Z}[\mathcal{W}(\mathcal{E} + J_D) + \mathcal{E}(\mathcal{F} - \mathcal{Z})](\dot{\theta}_1c_{\theta_2}s_{\theta_3} + \dot{\theta}_2c_{\theta_3})\dot{\theta}_3c_{\theta_2}c_{\theta_3}s_{\theta_2} \\
& + \mathcal{X}[\mathcal{V}(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) + (\mathcal{F}\mathcal{G} + \mathcal{E}\mathcal{H}c_{\theta_3}^2)c_{\theta_2}^2]\dot{\theta}_2^2c_{\theta_3}s_{\theta_3} - \mathcal{Z}\mathcal{F}(\mathcal{Y} - \mathcal{G})\dot{\theta}_2\dot{\theta}_3c_{\theta_2}s_{\theta_2} \\
& + J_D[\mathcal{V}(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) + \mathcal{Z}\mathcal{W}]\dot{\theta}_1\dot{\theta}_4c_{\theta_2}c_{\theta_3} - J_D[\mathcal{V}(\mathcal{W} - \mathcal{E}s_{\theta_3}^2) + (\mathcal{F}\mathcal{G} + \mathcal{E}\mathcal{H}c_{\theta_3}^2)c_{\theta_2}^2]\dot{\theta}_2\dot{\theta}_4s_{\theta_3} \\
& + J_D[\mathcal{G}(J_B + K_C) - I_C\mathcal{W} + I_D\mathcal{H} + \mathcal{E}\mathcal{H}c_{\theta_3}^2]\dot{\theta}_1\dot{\theta}_4c_{\theta_2}^3c_{\theta_3} - J_D\mathcal{Z}\mathcal{W}\dot{\theta}_3\dot{\theta}_4c_{\theta_2}c_{\theta_3}s_{\theta_2}, \\
\bar{f}_4 = & -J_D\mathcal{F}T_2c_{\theta_2}s_{\theta_2}s_{\theta_3} + J_D[(\mathcal{F} - \mathcal{Z})(\mathcal{Y} + \mathcal{Z}) + \mathcal{H}(\mathcal{G} + J_D + \mathcal{X}c_{\theta_3}^2)]\dot{\theta}_1^2c_{\theta_2}^3c_{\theta_3}s_{\theta_2} \\
& + J_D\mathcal{V}(\mathcal{Y} - \mathcal{X} + \mathcal{X}c_{\theta_3}^2)\dot{\theta}_1^2c_{\theta_2}c_{\theta_3}s_{\theta_2} + J_D[\mathcal{V}(\mathcal{F} - \mathcal{X}c_{\theta_3}^2) - (\mathcal{Y}\mathcal{F} + \mathcal{X}\mathcal{H}c_{\theta_3}^2)c_{\theta_2}^2]\dot{\theta}_1\dot{\theta}_2s_{\theta_2}s_{\theta_3} \\
& - J_D[\mathcal{V}(\mathcal{F} - \mathcal{X} - \mathcal{Z} + \mathcal{X}c_{\theta_3}^2) + ((\mathcal{F} - \mathcal{Z})(\mathcal{Y} + \mathcal{Z}) - \mathcal{X}\mathcal{H}s_{\theta_3}^2)c_{\theta_2}^2]\dot{\theta}_1\dot{\theta}_3c_{\theta_2}c_{\theta_3} \\
& + J_D[\mathcal{V}(\mathcal{F} + \mathcal{X}c_{\theta_3}^2) + (\mathcal{Y}\mathcal{F} + \mathcal{X}\mathcal{H}c_{\theta_3}^2)c_{\theta_2}^2]\dot{\theta}_2\dot{\theta}_3s_{\theta_3} + J_D^2(\mathcal{V} + \mathcal{H}c_{\theta_2}^2)(\dot{\theta}_1s_{\theta_2} - \dot{\theta}_3)\dot{\theta}_4c_{\theta_3}s_{\theta_3}, \\
g_1(\boldsymbol{\mu}) = & (\mathcal{F}\mu_4\dot{\theta}_4 + \mathcal{E}\mu_2\dot{\theta}_2c_{\theta_3})c_{\theta_2}s_{\theta_3} - (\mathcal{F} + \mathcal{E}c_{\theta_3}^2)(\mu_3\dot{\theta}_3s_{\theta_2} + \mu_1\dot{\theta}_1), \\
g_2(\boldsymbol{\mu}) = & [\mathcal{V} + (\mathcal{Y} + \mathcal{Z})c_{\theta_2}^2]\mu_4\dot{\theta}_4c_{\theta_3} - [\mathcal{V} + (\mathcal{G} - \mathcal{E}c_{\theta_3}^2)c_{\theta_2}^2]\mu_2\dot{\theta}_2 + \mathcal{E}(\mu_3\dot{\theta}_3s_{\theta_2} + \mu_1\dot{\theta}_1)c_{\theta_2}c_{\theta_3}s_{\theta_3}, \\
g_3(\boldsymbol{\mu}) = & \mathcal{Z}[\mathcal{F}\mu_4\dot{\theta}_4c_{\theta_2}s_{\theta_2}s_{\theta_3} + \mathcal{E}\mu_2\dot{\theta}_2c_{\theta_2}c_{\theta_3}s_{\theta_2}s_{\theta_3} - (\mathcal{F} + \mathcal{E}c_{\theta_3}^2)\mu_1\dot{\theta}_1s_{\theta_2}] \\
& - [(\mathcal{V} + \mathcal{Z})(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) - \mathcal{F}(\mathcal{Z} - \mathcal{G})c_{\theta_2}^2 - \mathcal{E}(\mathcal{Z} - \mathcal{H})c_{\theta_2}^2c_{\theta_3}^2]\mu_3\dot{\theta}_3, \\
g_4(\boldsymbol{\mu}) = & -[\mathcal{V}(\mathcal{F} - \mathcal{X}c_{\theta_3}^2) + \mathcal{F}(\mathcal{G} + J_D)c_{\theta_2}^2 - \mathcal{X}\mathcal{H}c_{\theta_2}^2c_{\theta_3}^2]\mu_4\dot{\theta}_4 \\
& + J_D[(\mathcal{V} + (\mathcal{Y} + \mathcal{Z})c_{\theta_2}^2)\mu_2\dot{\theta}_2c_{\theta_3} + \mathcal{F}(\mu_3\dot{\theta}_3s_{\theta_2} + \mu_1\dot{\theta}_1)c_{\theta_2}s_{\theta_3}], \\
b_{11} = & -\mathcal{F}c_{\theta_2}s_{\theta_3}, \\
b_{12} = & -\mathcal{W}c_{\theta_2}c_{\theta_3}, \\
b_{21} = & -[\mathcal{V} + (\mathcal{Y} + \mathcal{Z})c_{\theta_2}^2]c_{\theta_3}, \\
b_{22} = & (\mathcal{V} + \mathcal{G}c_{\theta_2}^2)s_{\theta_3}, \\
b_3 = & (\mathcal{V} + \mathcal{Z})(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) - \mathcal{F}(\mathcal{Z} - \mathcal{G})c_{\theta_2}^2 - \mathcal{E}(\mathcal{Z} - \mathcal{H})c_{\theta_2}^2c_{\theta_3}^2, \\
b_4 = & \mathcal{V}(\mathcal{F} - \mathcal{X}c_{\theta_3}^2) + \mathcal{F}(\mathcal{G} + J_D)c_{\theta_2}^2 - \mathcal{X}\mathcal{H}c_{\theta_2}^2c_{\theta_3}^2.
\end{aligned}$$

3 NONLINEAR CONTROL APPROACHES

Before applying the three nonlinear control approaches (FL, SM, and PB) for the CMG tracking task, their straightforward application to a tracking problem of fully-actuated systems (systems that have an equal number of actuators and DOF) are briefly presented in this chapter. Their theory and main concepts (which are the basis for this thesis) are presented and developed in Slotine and Li (1991) (FL and SM) and Paden and Panja (1988), Ortega et al. (1998) (PB).

Additional noteworthy references about these and also other nonlinear control approaches are Isidori (1995), Heck (1991), Khalil and Grizzle (2002), Utkin, Guldner and Shi (2009), Shtessel et al. (2014), Khalil (2015).

Firstly, the FL is presented in Section 3.1, followed by the SM approach in Section 3.2, and the PB in Section 3.3. Then, Section 3.4 presents a discussion about these three approaches and Section 3.5 introduces the problem statement for the UMS case.

3.1 Feedback linearization (FL)

“Feedback linearization deals with techniques for transforming original system models into equivalent models of a simpler form” (SLOTINE; LI, 1991, p. 204).

With the above statement, the basic idea of the FL approach is to algebraically transform a nonlinear system dynamics into a (fully or partially) linear one, for which the powerful and well-known linear design techniques can be applied to complete the control design (SLOTINE; LI, 1991; KHALIL; GRIZZLE, 2002).

Thus, in the simplest form, FL is employed not only to cancel out the nonlinearities in a nonlinear system but also to impose a desired linear dynamics.

As a tracking control problem is being addressed, the input-output linearization approach¹ is briefly presented. Beginning with SISO systems, it can be applied by considering its nonlinear dynamics in the following companion form:

$$y^{(r)} = f(\mathbf{x}) + b(\mathbf{x})u, \quad (3.1)$$

¹which, for fully-actuated systems (more specifically, in robotics), is commonly referred to as *computed-torque* method (CRAIG, 2005, p. 290).

in which $y \in \mathbb{R}$ is the output of interest, $\mathbf{x} \in \mathbb{R}^n$ is the system state vector, $u \in \mathbb{R}$ is the control input, $r \in \mathbb{N}^*$ is the system relative degree (i.e., the smallest integer such that the control input appears), and $f(\mathbf{x}), b(\mathbf{x}) \in \mathbb{R}$ are nonlinear functions of the states.

Therefore, assuming that $b(\mathbf{x}) \neq 0, \forall \mathbf{x}(t)$, and by choosing u from (3.1) as:

$$u = u_{\text{FL}} = \frac{v - f(\mathbf{x})}{b(\mathbf{x})}, \quad (3.2)$$

the nonlinearities are canceled out, leading to a simple input-output relation:

$$y^{(r)} = v. \quad (3.3)$$

For tracking tasks of a desired output trajectory $y_d(t) \in \mathbb{R}$, the control law $v \in \mathbb{R}$ can be chosen as a linear pole-placement controller given by:

$$v = y_d^{(r)} - k_{r-1}\tilde{y}^{(r-1)} - \dots - k_1\dot{\tilde{y}} - k_0\tilde{y}, \quad (3.4)$$

in which $\tilde{y}(t) = y(t) - y_d(t)$ is the output tracking error and coefficients k_i ($i = 0, 1, \dots, r-1$) are chosen to impose a desired linear dynamics, so that the polynomial $p^r + k_{r-1}p^{(r-1)} + \dots + k_1p + k_0$ has all its roots strictly in the left half of the complex plane.

Therefore, an exponentially stable dynamics:

$$\tilde{y}^{(r)} + k_{r-1}\tilde{y}^{(r-1)} + \dots + k_1\dot{\tilde{y}} + k_0\tilde{y} = 0 \quad (3.5)$$

is obtained for the output tracking error $\tilde{y}(t)$, which implies that $\tilde{y}(t) \rightarrow 0$, i.e., the output of interest $y(t)$ will converge to the output desired trajectory $y_d(t)$.

Similarly, the same results can be obtained for fully-actuated MIMO square systems (i.e. systems with the same number of outputs and inputs), which companion form is given by:

$$y_i^{(r_i)} = f_i(\mathbf{x}) + \sum_{j=1}^m b_{ij}(\mathbf{x})u_j, \quad (i, j = 1, \dots, m), \quad (3.6)$$

where i and j are related to the i^{th} system output ($y_i \in \mathbb{R}$) and j^{th} control input ($u_j \in \mathbb{R}$), respectively, $m \in \mathbb{N}^*$ is the number of system outputs and control inputs, $r_i \in \mathbb{N}^*$ is the relative degree of y_i (i.e., the smallest integer such that at least one of the control inputs u_j appears in $y_i^{(r_i)}$), and $f_i(\mathbf{x}), b_{ij}(\mathbf{x}) \in \mathbb{R}$ are nonlinear functions of the states.

Its vector form can be obtained by rewriting (3.6) in the following form:

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \underbrace{\begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{bmatrix} b_{11}(\mathbf{x}) & \dots & b_{1m}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ b_{m1}(\mathbf{x}) & \dots & b_{mm}(\mathbf{x}) \end{bmatrix}}_{\mathbf{B}(\mathbf{x})} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}}_{\mathbf{u}}, \quad (3.7)$$

composed of the control input vector $\mathbf{u} \in \mathbb{R}^m$, the vector $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$, and the so called *decoupling matrix* $\mathbf{B}(\mathbf{x}) \in \mathbb{R}^{m \times m}$. It is worth to recall the system output vector $\mathbf{y} \in \mathbb{R}^m$ (with components y_i) and the *total relative degree* of the system, given by $r = \sum_{i=1}^m r_i$.

If $\mathbf{B}(\mathbf{x})$ is nonsingular for all $\mathbf{x}(t)$, then $\mathbf{u}$ from (3.7) can be chosen as:

$$\mathbf{u} = \mathbf{u}_{\text{FL}} = \mathbf{B}(\mathbf{x})^{-1} (\mathbf{v} - \mathbf{f}(\mathbf{x})), \quad (3.8)$$

where each component of $\mathbf{v} \in \mathbb{R}^m$ is chosen as (3.4), leading to exponentially convergent tracking for all components of $\mathbf{y}$.

Summarizing, for SISO/MIMO systems, the control input u_{FL} (3.2)/ $\mathbf{u}_{\text{FL}}$ (3.8) will lead to an exponentially stable dynamics (3.5) of the i^{th} output tracking error $\tilde{y}_i$, i.e., the i^{th} output of interest y_i will track a desired time varying trajectory y_{i_d} .

3.2 Sliding-mode (SM)

The idea of the SM approach is to replace n^{th} order problems by equivalent first order ones² that are easier to control, achieving a “perfect” performance in the presence of arbitrary parameter inaccuracies, which is another interesting feature of the SM approach, its potential to deal with modeling inaccuracies that can have unfavorable effects on nonlinear control systems (SLOTINE; LI, 1991; UTKIN; GULDNER; SHI, 2009).

However, a high control activity is needed to achieve such a performance, yielding to a phenomenon called *chattering* (UTKIN; GULDNER; SHI, 2009), which is undesired in practical applications since it not only causes physical damages to the system (as high wear of moving mechanical parts and high heat losses in electrical power circuits), but also may lead it to unpredictable instability (PERRUQUETTI; BARBOT, 2002).

There are some methods in the literature that can be implemented to reduce it, e.g., smoothing out the control discontinuity, as presented in Slotine and Li (1991),

²its worth remarking that there exists an advanced strategy of sliding-mode that explores its higher order time derivatives, the so called *higher-order sliding-mode*.

Perruquetti and Barbot (2002), Khalil (2015), Hamayun et al. (2016), and the higher-order sliding-mode approaches (LEVANT, 1998; LEVANT, 2003; LAGHROUCHE; PLESTAN; GLUMINEAU, 2005; OLIVEIRA; ESTRADA; FRIDMAN, 2017), as the second-order sliding-mode (NUNES; HSU; LIZARRALDE, 2006; CAPISANI; FERRARA, 2012) and one of its most popular algorithm called *super-twisting* (VIDAL; NUNES; HSU, 2017; OLIVEIRA et al., 2018), being the former one implemented herein. Naturally, this reduction yields to a trade-off between tracking performance and parametric uncertainties, which requires a suitable choice to enable the practical application.

Likewise made in FL, consider the fully-actuated MIMO ³ system in its companion form (3.6), but now the nonlinear functions $f_i(\mathbf{x})$ and $b_{ij}(\mathbf{x})$ may be not exactly known, being at least their imprecision upper bounded by known continuous functions of $\mathbf{x}$ and the sign of $b_{ij}(\mathbf{x})$ known. Consider yet that $f_i(\mathbf{x})$ may be not exactly known due to inaccuracy on parameter $\mu_k \in \mathbb{R}$ ($k = 1, \dots, p$), which maximum ($\mu_{k_{\max}}$) and minimum ($\mu_{k_{\min}}$) values are known. Therefore, it is convenient to separate $f_i(\mathbf{x})$ into two parts:

$$f_i(\mathbf{x}) = \bar{f}_i(\mathbf{x}) + g_i(\boldsymbol{\mu}, \mathbf{x}),$$

where $\bar{f}_i(\mathbf{x})$ is composed of the known terms and $g_i(\boldsymbol{\mu}, \mathbf{x})$ is function of the uncertain parameter μ_k , being $\boldsymbol{\mu} \in \mathbb{R}^p$ the parameter inaccuracy vector (with components μ_k).

Hence, the companion form (3.6) is now given by:

$$y_i^{(r_i)} = \bar{f}_i(\mathbf{x}) + g_i(\boldsymbol{\mu}, \mathbf{x}) + \sum_{j=1}^m b_{ij}(\mathbf{x})u_j, \quad (i, j = 1, \dots, m). \quad (3.9)$$

The tracking control problem consists in making the output vector $\mathbf{y}_i = \begin{bmatrix} y_i & \dots & y_i^{(r_i-1)} \end{bmatrix}^\top$ to track a desired trajectory $\mathbf{y}_{i_d} = \begin{bmatrix} y_{i_d} & \dots & y_{i_d}^{(r_i-1)} \end{bmatrix}^\top$ in the presence of model imprecision on $g_i(\boldsymbol{\mu}, \mathbf{x})$ and/or $b_{ij}(\mathbf{x})$. For this, the SM makes $\mathbf{y}_i$ to converge towards sliding surfaces s_i . Its structure is similar to a feedback linearizing, composed of a nominal part, but it has additional terms to deal with model uncertainties. It is based on a defined time-varying surface $S(t)$ in the state-space $\mathcal{R}^{(r_i)}$ by the scalar equation $s_i(\mathbf{y}_i, t) = 0$, which may be given by (SLOTINE; LI, 1991, p. 278):

$$s_i(\mathbf{y}_i, t) = \left(\frac{d}{dt} + \lambda_i \right)^{r_i-1} \tilde{y}_i, \quad (3.10)$$

being $\lambda_i \in \mathbb{R}_+^*$ a strictly positive constant (which choice, for mechanical systems, is

³the equivalent SISO problem can be derived from the MIMO application by defining $m = 1$ and is thus omitted herein for brevity.

typically limited by three factors stated in Slotine and Li (1991, p. 301-303)) and $\tilde{y}_i = y_i - y_{i_d}$ the tracking error of i^{th} output ($y_{i_d} \in \mathbb{R}$ is the desired position of y_i).

Remark 4. *If both initial conditions of any output and its desired trajectory are not equal ($y_i(0) \neq y_{i_d}(0)$), then the tracking is achieved only after a transient, known as the reaching phase, followed by the sliding phase, which is the ideal sliding mode according to $s_i = 0$ and $\dot{s}_i = 0$.*

By differentiating (3.10) once with respect to time and replacing (3.9) in it, leads to:

$$\begin{aligned} \dot{s}_i &= \left(\frac{d}{dt} + \lambda_i \right)^{r_i} \tilde{y}_i = \underbrace{y_i^{(r_i)} - y_{i_d}^{(r_i)} + r_i \lambda_i \tilde{y}_i^{(r_i-1)} + \dots + r_i \lambda_i^{r_i-1} \tilde{y}_i + \lambda_i^{r_i} \tilde{y}_i}_{p_i(\mathbf{y}_i)} \\ &= \underbrace{\bar{f}_i(\mathbf{x}) + g_i(\boldsymbol{\mu}, \mathbf{x}) + \sum_{j=1}^m b_{ij}(\mathbf{x}) u_j}_{y_i^{(r_i)}}, \end{aligned} \quad (3.11)$$

where the control inputs u_j appear. It can also be rewritten in its vector form:

$$\underbrace{\begin{bmatrix} \dot{s}_1 \\ \vdots \\ \dot{s}_m \end{bmatrix}}_{\dot{\mathbf{s}}} = \underbrace{\begin{bmatrix} \bar{f}_1(\mathbf{x}) \\ \vdots \\ \bar{f}_m(\mathbf{x}) \end{bmatrix}}_{\bar{\mathbf{f}}(\mathbf{x})} + \underbrace{\begin{bmatrix} g_1(\boldsymbol{\mu}, \mathbf{x}) \\ \vdots \\ g_m(\boldsymbol{\mu}, \mathbf{x}) \end{bmatrix}}_{\mathbf{g}(\boldsymbol{\mu}, \mathbf{x})} + \underbrace{\begin{bmatrix} p_1(\mathbf{y}_1) \\ \vdots \\ p_m(\mathbf{y}_m) \end{bmatrix}}_{\mathbf{p}(\mathbf{y})} + \underbrace{\begin{bmatrix} b_{11}(\mathbf{x}) & \dots & b_{1m}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ b_{m1}(\mathbf{x}) & \dots & b_{mm}(\mathbf{x}) \end{bmatrix}}_{\mathbf{B}(\mathbf{x})} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}}_{\mathbf{u}}, \quad (3.12)$$

composed of the control input vector $\mathbf{u} \in \mathbb{R}^m$, the vectors $\bar{\mathbf{f}}(\mathbf{x}), \mathbf{g}(\boldsymbol{\mu}, \mathbf{x}), \mathbf{p}(\mathbf{y}) \in \mathbb{R}^m$, and the decoupling matrix $\mathbf{B}(\mathbf{x}) \in \mathbb{R}^{m \times m}$.

To deal with the model uncertainty, if $\mathbf{B}(\mathbf{x})$ is nonsingular for all $\mathbf{x}(t)$, $\mathbf{u}$ from (3.12) can be expressed as:

$$\mathbf{u} = \mathbf{u}_{\text{SM}} = \mathbf{B}(\mathbf{x})^{-1} (\hat{\mathbf{u}}_{\text{eq}} + \mathbf{u}_d), \quad (3.13)$$

in which $\hat{\mathbf{u}}_{\text{eq}} \in \mathbb{R}^m$ is the estimated equivalent control input vector and $\mathbf{u}_d \in \mathbb{R}^m$ the vector of discontinuous control actions.

Vector $\hat{\mathbf{u}}_{\text{eq}}$ is an estimation of the *equivalent control* input vector $\mathbf{u}_{\text{eq}}$ (SLOTINE; LI, 1991, p. 283) that corresponds to the ideal motion of the system during sliding phase by canceling out the nonlinearities of (3.12), similarly to the FL procedure. Its estimation is needed due to inaccuracy on term $\mathbf{g}(\boldsymbol{\mu}, \mathbf{x})$ in (3.12). Thus, by considering its estimated vector $\hat{\mathbf{g}}(\mathbf{x})$ instead, $\hat{\mathbf{u}}_{\text{eq}}$ is given by:

$$\hat{\mathbf{u}}_{\text{eq}} = -\hat{\mathbf{g}}(\mathbf{x}) - \bar{\mathbf{f}}(\mathbf{x}) - \mathbf{p}(\mathbf{y}). \quad (3.14)$$

The estimated vector $\widehat{\mathbf{g}}(\mathbf{x}) \in \mathbb{R}^m$ has an estimation error bounded by function $\widetilde{\mathbf{g}}(\boldsymbol{\mu}, \mathbf{x}) \geq |\widehat{\mathbf{g}}(\mathbf{x}) - \mathbf{g}(\boldsymbol{\mu}, \mathbf{x})|$. As a simplification, its equality may be considered, which is obtained when $\widehat{\mathbf{g}}(\mathbf{x})$ is defined by replacing the mean estimation values of the h uncertain parameters in $\mathbf{g}(\boldsymbol{\mu}, \mathbf{x})$, i.e.:

$$\widehat{\mathbf{g}}(\mathbf{x}) = \mathbf{g}(\boldsymbol{\mu}, \mathbf{x})|_{\mu_k = \frac{\mu_{k_{\max}} + \mu_{k_{\min}}}{2}}, \quad (k = 1, \dots, h). \quad (3.15)$$

This yields to the maximum estimation function $\widetilde{\mathbf{g}}_{\max}(\mathbf{x}) \in \mathbb{R}^m$ that is given by the maximum estimation error of each parameter, i.e.,

$$\widetilde{\mathbf{g}}_{\max}(\mathbf{x}) = \mathbf{g}(\boldsymbol{\mu}, \mathbf{x})|_{\mu_k = \frac{\mu_{k_{\max}} - \mu_{k_{\min}}}{2}}. \quad (3.16)$$

On the other hand, vector $\mathbf{u}_d$ is composed of discontinuous control action that provides finite-time convergence to the sliding surfaces during the reaching phase. This convergence is shown by considering quadratic Lyapunov functions $V_i(s_i) = s_i^2/2$. To guarantee asymptotic stability, its time derivative $\dot{V}_i(s_i, \dot{s}_i)$ must be negative-definite and satisfy the *sliding condition* (SLOTINE; LI, 1991, p. 280), given by:

$$\dot{V}_i(s_i, \dot{s}_i) = s_i \dot{s}_i \leq -\eta_i |s_i| < 0, \quad (3.17)$$

to achieve finite-time convergence.

Its switching height parameter $\eta_i \in \mathbb{R}_+^*$ that is a strictly positive constant guarantees that $\dot{V}_i(s_i, \dot{s}_i)$ becomes negative-definite, and is directly related to the reaching time $t_{\text{reach}_i} \in \mathbb{R}_+^*$ required to hit the surface $s_i = 0$. This relation is obtained by integrating (3.17) between $t = 0$ and $t = t_{\text{reach}_i}$ (SLOTINE; LI, 1991, p. 281-282):

$$0 - s_i(0) = s_i(t_{\text{reach}_i}) - s_i(0) \leq -\eta_i(t_{\text{reach}_i} - 0), \quad (3.18)$$

implying that:

$$t_{\text{reach}_i} \leq \frac{|s_i(0)|}{\eta_i} \implies \eta_i \leq \frac{|s_i(0)|}{t_{\text{reach}_i}}, \quad (3.19)$$

in which, for the sake of simplification, the equality may be considered to determine η_i .

As an ideal sliding mode does not exist in practice due to physical limitations in real world systems, the discontinuous control action ⁴ leads to *chattering* that causes oscillations in the vicinity of the sliding surface during the sliding phase (PERRUQUETTI; BARBOT, 2002).

⁴e.g., that are commonly chosen at first as a weighted sign function $-k_i \text{sgn}(s_i)$.

To reduce it, some methods (that was exposed in the beginning of this subsection) can be implemented, as smoothing out the control discontinuity in a thin boundary layer neighboring the sliding surface, as an example for $r_i = 2$ illustrated in Figure 7, in which Φ_i is the boundary layer thickness for sliding surface s_i and $\delta_i = \Phi_i / \lambda_i^{r_i-1}$ is the boundary layer width. It can be done by defining $\mathbf{u}_d$ in (3.13) as:

$$\mathbf{u}_d = - \begin{bmatrix} k_1 \text{sat}(s_1/\Phi_1) \\ \vdots \\ k_m \text{sat}(s_m/\Phi_m) \end{bmatrix}, \quad (3.20)$$

in which sat is the saturation function, and $k_i \in \mathbb{R}_+^*$ is a weighting function chosen to be large enough in order to guarantee the sliding condition (3.17). It can be achieved by defining it as the sum of the i^{th} element of $\tilde{\mathbf{g}}_{\max}(\mathbf{x})$ and the switching height parameter η_i , i.e., $k_i = \tilde{g}_{\max_i}(\mathbf{x}) + \eta_i$ (SLOTINE; LI, 1991, p. 286).

Figure 7: Boundary layer neighboring the sliding surface $s_i(t)$

Source: Adapted from Slotine and Li (1991, p. 291).

The boundary layer thickness Φ_i is directly related with the maximum acceptable error for the i^{th} output of the system ($\tilde{y}_{i_{\max}}$) as it is limited by Φ_i 's width $\delta_i = \Phi_i / \lambda_i^{r_i-1}$:

$$\tilde{y}_{i_{\max}} = \frac{\Phi_i}{\lambda_i^{r_i-1}} \implies \Phi_i = \tilde{y}_{i_{\max}} \lambda_i^{r_i-1}. \quad (3.21)$$

Summarizing, the control input $\mathbf{u}_{\text{SM}}$ (3.13), which is composed of (3.14) and (3.20), will bring and keep the states into the vicinity of sliding surfaces $\dot{s}_i = s_i = 0$, leading to $\tilde{y}_i \rightarrow 0$, i.e., the i^{th} output of interest will track a desired time varying trajectory y_{i_d} with a maximum error $\tilde{y}_{i_{\max}}$, despite the inaccuracy on parameters μ_k .

Remark 5. *The case when $b_{ij}(\mathbf{x})$ is not exactly known was not developed herein since it did not happen for the CMG case in Section 4.3, but one can find solutions in Slotine and Li (1991, p. 287-288) and in Oliveira et al. (2007) for SISO systems, and in Oliveira, Peixoto and Hsu (2010) for MIMO systems.*

3.3 Passivity-based (PB)

The PB approach aims to render the closed-loop passive by using some passivity properties, seeing that passive systems cannot store more energy than is exogenously supplied to it (ORTEGA et al., 1998, p. 20). This feature yields to employ the system energy in the controller design, in which kinetic and potential energies play a role.

This approach is viewed as an extension of the energy-shaping plus damping injection technique (ORTEGA et al., 1998, p. 10). The energy-shaping stage modifies the energy function of the system to accomplish a passivation objective with a desired storage function composed of the new desired energy function, which, for tracking problems, will typically be selected by considering an error dynamics. Subsequently, the damping injection stage modifies the dissipation function to ensure asymptotic stability.

Because of this, the PB provides a direct physical interpretation of its control action, relevantly impacting on the controller tuning stage. This main asset explains its great success in applications, which rendered the commissioning of a nonlinear controller (that is a difficult task) quite transparent to the practicing engineer (ORTEGA et al., 1998, p. 47).

This way, consider the dynamics of an n -DOF fully-actuated MIMO system in its general form (2.20):

$$\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u}, \quad (3.22)$$

which has some fundamental properties that is taken into account in the control design (ORTEGA; SPONG, 1989; ZHANG et al., 2018):

Property 1. *The matrices $\dot{\mathbf{M}}(\mathbf{q})$ (the time derivative of $\mathbf{M}(\mathbf{q})$) and $\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}})$ present the following skew-symmetric relationship:*

$$\xi^\top \underbrace{\left(\frac{1}{2} \dot{\mathbf{M}}(\mathbf{q}) - \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \right)}_{\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})} \xi = 0, \quad \forall \xi \in \mathbb{R}^n, \quad (3.23)$$

i.e., if matrix $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$ satisfies the condition $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})^\top = -\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$.

Property 2. *The inertia matrix $\mathbf{M}(\mathbf{q})$ is symmetric, positive definite, and holds the following inequalities:*

$$h_{\min}\|\boldsymbol{\xi}\|^2 \leq \boldsymbol{\xi}^\top \mathbf{M}(\mathbf{q})\boldsymbol{\xi} \leq h_{\max}\|\boldsymbol{\xi}\|^2, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n, \quad (3.24)$$

with positive constants $h_{\min}, h_{\max} \in \mathbb{R}_+$.

With this, the energy-shaping stage begins by considering at first a desired storage function:

$$\mathcal{V}_q = \frac{1}{2} \dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \mathbf{e}^\top \mathbf{K}_p \mathbf{e}, \quad (3.25)$$

in which $\mathbf{e} = \mathbf{q} - \mathbf{q}_d \in \mathbb{R}^n$ is the tracking error vector (being $\mathbf{q}_d \in \mathbb{R}^n$ the desired trajectory of $\mathbf{q}$) and the proportional gain matrix $\mathbf{K}_p \in \mathbb{R}_+^{n \times n}$ is chosen as symmetric positive definite.

However, as a tracking control problem is being addressed, a desired behavior is imposed not only to $\mathbf{q}$ but also to $\dot{\mathbf{q}}$, which requires the modification of (3.25) by replacing $\dot{\mathbf{q}}$ by $\dot{\mathbf{e}}$ (ORTEGA et al., 1998, p. 95):

$$\mathcal{V}_e = \frac{1}{2} \dot{\mathbf{e}}^\top \mathbf{M}(\mathbf{q}) \dot{\mathbf{e}} + \frac{1}{2} \mathbf{e}^\top \mathbf{K}_p \mathbf{e}. \quad (3.26)$$

Note that both (3.25) and (3.26) are positive definite since their first quadratic form term is positive definite by Property 2, as well as the second quadratic form term since $\mathbf{K}_p$ is also positive definite.

Next, by differentiating (3.26) once with respect to time and by using Property 1 and (3.22):

$$\begin{aligned} \dot{\mathcal{V}}_e &= \dot{\mathbf{e}}^\top \mathbf{M}(\mathbf{q}) \ddot{\mathbf{e}} + \overbrace{\frac{1}{2} \dot{\mathbf{e}}^\top \dot{\mathbf{M}}(\mathbf{q}) \dot{\mathbf{e}}}^{=\dot{\mathbf{e}}^\top \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{e}}} + \dot{\mathbf{e}}^\top \mathbf{K}_p \dot{\mathbf{e}} = \dot{\mathbf{e}}^\top (\mathbf{M}(\mathbf{q}) \ddot{\mathbf{e}} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{e}} + \mathbf{K}_p \dot{\mathbf{e}}) \\ &= \dot{\mathbf{e}}^\top \left(\overbrace{\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}}^{=\mathbf{u} - \mathbf{g}(\mathbf{q})} - \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}}_d - \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_d + \mathbf{K}_p \dot{\mathbf{e}} \right) \\ &= \dot{\mathbf{e}}^\top (\mathbf{u} - \mathbf{g}(\mathbf{q}) - \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}}_d - \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_d + \mathbf{K}_p \dot{\mathbf{e}}), \end{aligned} \quad (3.27)$$

the control input vector $\mathbf{u}$ appears. Similarly to the FL procedure, it may be defined as:

$$\mathbf{u} = \mathbf{u}_{PB} = \mathbf{g}(\mathbf{q}) + \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}}_d + \mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_d - \mathbf{K}_p \mathbf{e} + \mathbf{u}_{\text{damp}}, \quad (3.28)$$

yielding to:

$$\dot{\mathcal{V}}_e = \dot{\mathbf{e}}^\top \mathbf{u}_{\text{damp}}. \quad (3.29)$$

The next step is the damping injection stage that renders (3.29) negative semi-definite by choosing the damping term $\mathbf{u}_{\text{damp}}$ as:

$$\mathbf{u}_{\text{damp}} = -\mathbf{K}_d \dot{\mathbf{e}}, \quad (3.30)$$

with the derivative gain matrix $\mathbf{K}_d \in \mathbb{R}_+^{n \times n}$ chosen as symmetric positive definite.

To conclude, an analysis of the closed-loop system stability is presented in Theorem 1, based on Paden and Panja (1988) and Maghenem, Loría and Panteley (2017), in which the Lyapunov stability and Matrosov's theorem (Theorem 3 presented in Appendix D) are invoked. But before it, as the tracking problem is being considered, the desired trajectory $\mathbf{q}_d(t)$ may be time-varying. Hence, the following assumption is made:

Assumption 4. *The time-varying desired trajectory $\mathbf{q}_d(t)$ and its first derivative $\dot{\mathbf{q}}_d(t)$ are continuous and bounded, i.e.*

$$\begin{cases} \mathbf{q}_{\text{dmin}} \leq \mathbf{q}_d(t) \leq \mathbf{q}_{\text{dmax}} \\ \dot{\mathbf{q}}_{\text{dmin}} \leq \dot{\mathbf{q}}_d(t) \leq \dot{\mathbf{q}}_{\text{dmax}} \end{cases}, \quad \forall t \geq 0, \quad (3.31)$$

where the subindexes $(\cdot)_{\text{min}}$ and $(\cdot)_{\text{max}}$ stand for the lower and upper boundaries, respectively.

Theorem 1. *If Assumption 4 holds, then the control law (3.28), together with (3.30), ensures the asymptotic convergence of $\mathbf{e}(t)$ and $\dot{\mathbf{e}}(t)$ to zero, i.e.*

$$\lim_{t \rightarrow \infty} \begin{bmatrix} \mathbf{q}(t) & \dot{\mathbf{q}}(t) \end{bmatrix}^\top = \begin{bmatrix} \mathbf{q}_d(t) & \dot{\mathbf{q}}_d(t) \end{bmatrix}^\top, \quad (3.32)$$

dealing with the tracking control problem.

Proof. Consider the non-autonomous tracking error dynamics, derived from (3.22), (3.28), and (3.30):

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{e}} = -(\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_d)\dot{\mathbf{e}} - \mathbf{K}_p\mathbf{e}, \quad (3.33)$$

and $\dot{\mathbf{x}} = f(t, \mathbf{x})$ defined implicitly by (3.33), with the state vector $\mathbf{x} = [\mathbf{e}^\top \ \dot{\mathbf{e}}^\top]^\top \in \mathbb{R}^{2n}$.

Let $\mathfrak{B}_\rho \triangleq \{\mathbf{x} \in \mathbb{R}^{2n} : \|\mathbf{x}\| < \rho\}$, then for each $\rho > 0$ there exist continuously differentiable functions $\mathcal{V}_e : \mathfrak{B}_\rho \rightarrow \mathbb{R}$ and $\mathcal{W} : \mathfrak{B}_\rho \rightarrow \mathbb{R}$ for all $t \geq 0$, and $\mathcal{V}_e^* : \mathfrak{B}_\rho \rightarrow \mathbb{R}$.

Next, consider the positive definite storage function $\mathcal{V}_e$ in (3.26) as a Lyapunov function candidate. It is time-varying, positive definite, and satisfies

$$\alpha_{\min}(\|\mathbf{x}\|) \leq \mathcal{V}_e(t, \mathbf{x}) \leq \alpha_{\max}(\|\mathbf{x}\|), \quad \forall t \geq 0, \forall \mathbf{x} \in \bar{\mathfrak{B}}_\rho, \quad (3.34)$$

where the continuous functions $\alpha_{\min}(\cdot)$, $\alpha_{\max}(\cdot)$ are strictly increasing and satisfy $\alpha_{\min}(0) = \alpha_{\max}(0) = 0$, and $\bar{\mathfrak{B}}_\rho$ denote a closed ball of radius ρ centered at the origin.

Then, substituting (3.28) and (3.30) into its derivative (3.27), leads to:

$$\dot{\mathcal{V}}_e(t, \mathbf{x}) = -\dot{\mathbf{e}}^\top \mathbf{K}_d \dot{\mathbf{e}} \leq \mathcal{V}_e^*(\mathbf{x}) \leq 0, \quad \forall t \geq 0, \forall \mathbf{x} \in \bar{\mathfrak{B}}_\rho. \quad (3.35)$$

This way, the conditions (1) and (2) of Theorem 3 are satisfied.

Remark 6. *It follows from these two conditions that $\mathbf{e}(t)$ and $\dot{\mathbf{e}}(t)$ are bounded. Additionally, as $\mathbf{q}_d(t)$ and $\dot{\mathbf{q}}_d(t)$ were assumed to be bounded on Assumption 4, $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ will be bounded as well, and consequently the right-hand side (RHS) of (3.33) will also be bounded. Therefore, by considering Property 2, which states that the minimum singular value of $\mathbf{M}(\mathbf{q})$ is bounded away from zero, it follows that $\ddot{\mathbf{e}}(t)$ is bounded.*

Define $\mathcal{M} \triangleq \{\mathbf{x} \in \bar{\mathfrak{B}}_\mathcal{R} \mid \mathcal{V}_e^*(\mathbf{x}) = 0\}$, $\mathcal{R} \in (0, \rho)$, and function $\mathcal{W}(t, \mathbf{x}) \triangleq \ddot{\mathcal{V}}_e(t, \mathbf{x})$, obtained by differentiating (3.35):

$$\mathcal{W}(t, \mathbf{x}) = -2\dot{\mathbf{e}}^\top \mathbf{K}_d \ddot{\mathbf{e}}, \quad \forall t \geq 0, \forall \mathbf{x} \in \mathcal{M}. \quad (3.36)$$

Then, it follows from the bounds on Remark 6 that for each $\mathcal{R}$ there exist $\gamma > 0$ such that

$$\max\{|\mathcal{W}(t, \mathbf{x})|, \|f(t, \mathbf{x})\|\} \leq \gamma, \quad \forall t \geq 0, \forall \mathbf{x} \in \bar{\mathfrak{B}}_\mathcal{R}. \quad (3.37)$$

This way, $|\mathcal{W}(t, \mathbf{x})|$ is bounded, thus satisfying condition (3) of Theorem 3. Moreover, the function $\dot{\mathcal{W}}(t, \mathbf{x}) = \mathcal{V}_e^{(3)}(t, \mathbf{x})$ is nonzero definite on $\mathcal{M}$, i.e., there exists a strictly increasing continuous function $\beta(\cdot)$ such that $\beta(0) = 0$ and

$$|\dot{\mathcal{W}}(t, \mathbf{x})| \geq \beta(\|\mathbf{x}\|), \quad \forall t \geq 0, \forall \mathbf{x} \in \mathcal{M}. \quad (3.38)$$

By the lemma given in Paden and Panja (1988), it follows that condition (4) of Theorem 3 is satisfied since $\dot{\mathcal{W}}(t, \mathbf{x})$ depends on time continuously through a bounded function.

Finally, as stated in (3.37), $\|f(t, \mathbf{x})\|$ is bounded for all $t \geq 0$ and for all $\mathbf{x} \in \bar{\mathfrak{B}}_\mathcal{R}$, satisfying the last condition (5) of Theorem 3. Hence, the origin of the closed-loop equilibrium point is asymptotically stable.

□

Similarly to what was presented in Paden and Panja (1988, p. 1703), the key fact to be checked is if $\mathcal{V}_e^{(3)}$ is negative definite on $\mathfrak{M} \triangleq \{(\mathbf{e}, \dot{\mathbf{e}}) | \dot{\mathcal{V}}_e = 0\}$, in which $\dot{\mathbf{e}} = \mathbf{0} \rightarrow \ddot{\mathbf{e}} = \mathbf{e}^{(3)} = \mathbf{0}$. Taking this into account, by differentiating (3.35) twice and by replacing in it the expression of $\ddot{\mathbf{e}}$, calculated in (3.33), yields to:

$$\begin{aligned} \mathcal{V}_e^{(3)} &= -2 (\mathbf{M}(\mathbf{q})^{-1} \mathbf{K}_p \mathbf{e})^\top \mathbf{K}_d (\mathbf{M}(\mathbf{q})^{-1} \mathbf{K}_p \mathbf{e}) \\ &= -2 \mathbf{e}^\top \left(\underbrace{\mathbf{K}_p^\top \mathbf{M}(\mathbf{q})^{-1} \mathbf{K}_d \mathbf{M}(\mathbf{q})^{-1} \mathbf{K}_p}_{\mathbf{\Xi}} \right) \mathbf{e}, \quad \forall (\mathbf{e}, \dot{\mathbf{e}}) \in \mathfrak{M}. \end{aligned} \quad (3.39)$$

where $\mathbf{\Xi}$ is a symmetric positive definite matrix, since $\mathbf{K}_p$ and $\mathbf{K}_d$ were chosen as symmetric positive definite matrices in (3.25) and (3.30), respectively, and $\mathbf{M}(\mathbf{q})$ obeys Property 2.

Therefore, (3.39) is a negative definite function of the tracking error vector, i.e., it is equal to zero if and only if $\mathbf{e} = \mathbf{0} \iff \mathbf{q} = \mathbf{q}_d$.

Summarizing, the control input $\mathbf{u}_{PB}$ (3.28), together with (3.30), accomplishes the tracking task of each generalized coordinate of the system by asymptotically converging $\mathbf{q}$ to its desired trajectory $\mathbf{q}_d$.

3.4 Discussion about the approaches

A shared characteristic of these three nonlinear control approaches is their dependency on the system model, as their control law are designed based on a nominal model of the physical system, making the modeling process an essential step for the success of their application.

The FL approach has been successfully applied in some practical control problems in the literature, although it typically requires full state measurement (requiring some additional solutions, e.g., an observer design to estimate its non-measured states or filtered derivatives to mitigate high frequency noises to obtain the velocities and accelerations, as adopted herein in Chapter 5) and does not guarantee robustness to parameter uncertainties or unmodeled dynamics, which may be severe when the linearizing transformation is poorly conditioned (SLOTINE; LI, 1991).

Despite these issues, it is a noteworthy approach for nonlinear control design as it can also be used as model-simplifying in other nonlinear control techniques, such as the SM and PB approaches in this chapter, of which was used in both control law definitions.

On the other hand, one of the main assets of the SM approach is its capacity to deal with model parameter uncertainties and unmodeled dynamics, where its control law is designed based on both a nominal model and some characterization of the model uncertainties, e.g., the previous knowledge of the uncertainty boundaries of a parameter value. Because of this, it has been very effective in a variety of practical control problems, although it also generally requires state measurements. Unfortunately, one of its drawbacks is the chattering phenomenon that arise to deal with the model imprecision, for which a trade-off between modeling and performance must be quantified and defined during the design process to reduce its effects (SLOTINE; LI, 1991).

Finally, the PB approach relies on passivity properties and provides a stability analysis of the closed-loop system even for tracking control problems, unlike the FL and SM approaches. Furthermore, its energy interpretation of passivity is intimately related with the physics of the system, which has a relevant impact on the controller tuning stage (ORTEGA et al., 1998).

3.5 Problem statement for the UMS case

These three approaches have a straightforward application for fully-actuated systems as presented in this chapter, what is not true for UMS, since its control input vector $\mathbf{u}$ has zeros in the unactuated subsystem, e.g., in the CMG case $\mathbf{u}(t) = [0 \ 0 \ T_2(t) \ T_1(t)]^\top$ in (2.21). This poses additional challenges to control problems as the control design must exploit some coupling between the unactuated and actuated states to achieve the control objective (LIU; YU, 2013).

Regarding the FL approach, a global tracking may be ensured by performing an input-to-state linearization, for which the total relative degree must be equal to the order of the system (KHALIL; GRIZZLE, 2002), but unfortunately it is not true for most underactuated systems, for which it is only possible to perform a partial linearization with the input-output linearization, whereby the effectiveness of the control hinges upon the stability of the states that were not linearized, which may be either stable or unstable depending on the resulting control inputs and the desired tracking trajectories. These nonlinearized states present the so called *internal dynamics* (SLOTINE; LI, 1991), which also appear in the SM approach.

Generally, it is very difficult to determine the stability of the internal dynamics since it is nonlinear, non-autonomous, and coupled to the "external" closed-loop dynamics.

Even though a Lyapunov or Lyapunov-like analysis may be useful for some systems, its general applicability is limited by the difficulty in finding a Lyapunov function candidate (SLOTINE; LI, 1991).

Considering the tracking problem, even studying the zero-dynamics (defined to be the internal dynamics of the system when its output is kept at zero by the input), no results on wide range stability can be drawn for the internal dynamics of nonlinear systems, i.e., only local stability is guaranteed for it, even if the zero-dynamics is globally exponentially stable (SLOTINE; LI, 1991, p. 227,258).

Concerning the PB approach, the underactuation also hinders its application, as the control input vector in (3.28) is not fully-actuated, precluding the damping injection stage.

Although it was not possible to prove the stability of the closed-loop system with the FL and SM controllers because of the difficulty in determining the stability of the internal dynamics for the tracking problem, the proposed idea to employ the intrinsic reaction forces/torques of the system overcame the system underactuation, enabling the application of the three approaches to the CMG MIMO tracking problem, developed in the next chapter, and accomplishing the tracking task of its unactuated states in a wide operating range, as presented in Chapter 5.

4 CMG CONTROL DESIGN

In this section, it is applied the aforementioned nonlinear control approaches to the tracking task of the ECP CMG.

Formerly, a problem statement concerning this process and its research progress during the thesis development are detailed and discussed. Then, the succeeding sections develop the nonlinear controllers design based on the three nonlinear control approaches.

4.1 Problem statement and its research progress

As stated in Chapter 2, the ECP CMG is an underactuated system and its tracking task involves the unactuated states.

Some complications arise from this, as the preclusion of the straightforward application of the nonlinear controllers (aforementioned in Section 3.5), in addition to the tracking task problem that introduces time-varying tracking signals, posing more challenges than the regulation problem that requires only a constant reference signal.

Initially, the CMG in its SISO configuration was explored before its MIMO one. It was convenient to start this manner once the problem is reduced from a 2×2 coupled MIMO system to a SISO system, which somewhat facilitates the controller feasibility analysis. It is worth noting that although the problem is simplified to a SISO system, its underactuation still remains, where the goal continues to be the tracking of the unactuated state.

This way, regarding the CMG SISO tracking problem, the application of both FL and SM approaches were straightforward, as can be seen in Toriumi, Angélico and Tannuri (2018) and Toriumi and Angélico (2018), respectively. On the other hand, the PB approach presented the first hindrance: the underactuation precludes the accomplishment of the damping assignment stage, as presented in article g in Appendix A ¹.

¹in this paper, it is presented a Lyapunov-based nonlinear controller development, but the author believes that it is actually the PBC approach. It happened due to author's lack of literature research at the time, but actually this paper's controller development presents both energy-shaping and damping assignment stages, as stated previously in the PBC approach.

To overcome this problem, this thesis proposes to employ an actuated state as an internal control signal (article g in Appendix A). This idea can be physically interpreted as employing the reaction forces/torques generated in the system to overcome the underactuation. This way, these reaction forces/torques, which are composed of the actuated states, are determined to accomplish the tracking of the unactuated states.

This idea was extended for the CMG MIMO tracking problem, in which the three nonlinear controller approaches also presented the underactuation hindrance. For this, two CMG reaction torques were employed: the gyroscopic torque T_{gyro} (for the three cases) and the reaction wheel torque T_{wheel} (only for the PB case), given by the following expressions (ECP, 1999):

$$T_{\text{gyro}} = J_D \dot{\theta}_3 \dot{\theta}_4, \quad (4.1)$$

$$T_{\text{wheel}} = J_D \ddot{\theta}_4. \quad (4.2)$$

Regarding the FL and SM approaches, the underactuation yields to a singularity in the control signal that arises in the decoupling matrix B , as will be shown and detailed in the next section. This problem was overcome by employing the CMG gyroscopic torque T_{gyro} ², from which one of the actuated states is chosen as an internal control signal. This change required an internal velocity control, yielding to a cascade form for the controller.

Although these two nonlinear controllers satisfactorily accomplished the CMG MIMO tracking task, they lack a closed-loop stability proof, since the stability analysis of a nonlinear system is a difficult process, as mentioned in Section 3.5.

An option found to overcome this problem was the PB approach, since the resulting closed-loop system stability can be analyzed by invoking both Lyapunov stability and Matrosov's theorem.

Regarding the PB approach for the CMG MIMO tracking task, the underactuation issue was overcome by rewriting the CMG nonlinear dynamics equations in order to separate its coupling terms between the unactuated and actuated subsystems, and then employing the internal reaction torques to accomplish the tracking of the unactuated states. Similarly to the SISO case, this idea can be physically interpreted as the reaction torques being determined to accomplish the tracking of the unactuated states, where the desired actuated states are extracted from the reaction torques.

²before the idea to employ the CMG gyroscopic torque, the author employed the term $\dot{\theta}_3$ instead, as developed in papers (TORIUMI; ANGÉLICO, 2020a; TORIUMI; ANGÉLICO, 2019b), which also satisfactorily fulfilled the tracking task.

All of these solutions for the CMG MIMO tracking task are developed next ³.

4.2 FL approach

For the CMG MIMO tracking problem that comprises two inputs (the torques of both DC motors: $\mathbf{u} = [T_1 \ T_2]^\top$) and two outputs (bodies A and B angular positions: $\mathbf{y} = [\theta_1 \ \theta_2]^\top$), it is instinctive to apply the input-output linearization approach by defining the control input vector as $\mathbf{u} = [T_1 \ T_2]^\top$. Considering this, the resulting decoupling matrix $\bar{\mathbf{B}}$ (derived from $\ddot{\theta}_1$ and $\ddot{\theta}_2$ described in Appendix C) is given by:

$$\bar{\mathbf{B}} = \frac{1}{\mathfrak{D}} \begin{bmatrix} -\mathcal{F}c_{\theta_2}s_{\theta_3} & (\mathcal{F} + \mathcal{E}c_{\theta_3}^2)s_{\theta_2} \\ -[\mathcal{V} + (\mathcal{Y} + \mathcal{Z})c_{\theta_2}^2]c_{\theta_3} & -\mathcal{E}c_{\theta_2}c_{\theta_3}s_{\theta_2}s_{\theta_3} \end{bmatrix}, \quad (4.3)$$

where all variables are also defined in Appendix C, and its determinant is equal to:

$$\det(\bar{\mathbf{B}}) = \frac{c_{\theta_3}s_{\theta_2}}{\mathfrak{D}}.$$

It (together with Remark 13) indicates that (4.3) is singular when $\theta_3(t) = 90^\circ + k180^\circ$ ($\pi/2 + k\pi$ rad) or $\theta_2(t) = 0^\circ + k180^\circ$ ($k \in \mathbb{Z}$). By interpreting it in a physical point of view:

- the singularity of the first condition arises as body B angular position (θ_2) cannot be directly controlled by neither the torque reaction generated by motor #1 torque (T_1) nor motor #2 torque (T_2), i.e., the second row of (4.3) becomes null,
- and the singularity of the second condition arises as torque T_2 does not directly affect neither bodies A nor B angular positions (θ_1 and θ_2), i.e., the second column of (4.3) becomes null.

Summarizing, it is not possible to apply the input-output linearization approach for the CMG MIMO tracking problem by defining the control input vector as $\mathbf{u} = [T_1 \ T_2]^\top$, since (4.3) is not invertible within the CMG operating range (Assumptions 1 and 2).

A possible way to avoid this problem is to redefine the control input vector. This can be done by employing the reaction torques generated in the plant, which is one of this thesis contribution. Thus, by considering the gyroscopic torque T_{gyro} (4.1) that was

³for the SISO tracking problem, one can refer to the author's publications Toriumi, Angélico and Tannuri (2018) (FL), Toriumi and Angélico (2018) (SM), and article g in Appendix A, once the main goal herein is the MIMO tracking task.

introduced in the previous section, its component $\dot{\theta}_3$ can be employed as a control signal for the FL, i.e., by redefining the control input as $\mathbf{u} = [T_1 \ T_{\text{gyro}}]^\top$, in which T_2 was replaced by T_{gyro} .

This redefinition leads to a decoupling matrix $\mathbf{B}$ (see (4.4)) with determinant:

$$\det(\mathbf{B}) = -\frac{c_{\theta_2}}{\mathfrak{D}}.$$

Since CMG operates with $\theta_2(t)$ bounded within range $|\theta_2| < 90^\circ$ ($\pi/2$ rad), as stated in Assumption 1, then $\mathbf{B}$ is nonsingular within the CMG operating range.

This way, the desired body C angular velocity $\dot{\theta}_{3_d}$ to achieve the necessary gyroscopic torque can be determined by extracting $\dot{\theta}_3$ from (4.1) and, consequently, it needs to be controlled separately with T_2 . For this, the input-output linearization approach can be applied to this SISO tracking problem by defining the output as $y = \dot{\theta}_3$ and the control input as $u = T_2$.

Figure 8 summarizes the FL controller for the CMG MIMO tracking task, which is divided into two parts: the outer tracking control and the inner velocity control, for both of which input-output linearization approach is proposed. The first part deals with the CMG tracking task, in which T_1 and $\dot{\theta}_{3_d}$ are calculated to guarantee that bodies A and B angular positions (θ_1 and θ_2) track their desired trajectories (θ_{1_d} and θ_{2_d}), while the second part guarantee the control of body C angular velocity by T_2 so that it is faster than the outer tracking control dynamics.

Figure 8: Block diagram of the FL controller for the CMG MIMO tracking task

Source: Author.

4.2.1 Outer tracking controller

This part concerns a second-order MIMO system with two inputs and two outputs ($m = 2$), being bodies A and B angular positions the outputs of interest ($\mathbf{y} = [\theta_1 \ \theta_2]^\top$), and motor #1 and gyroscopic torques the control inputs ($\mathbf{u} = [T_1 \ T_{\text{gyro}}]^\top$).

Its companion form (3.7) is given by:

$$\underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{\ddot{y}}} = \underbrace{\begin{bmatrix} (\bar{f}_1 + g_1(\boldsymbol{\mu}))/\mathfrak{D} \\ (\bar{f}_2 + g_2(\boldsymbol{\mu}))/\mathfrak{D} \end{bmatrix}}_{\mathbf{f}} + \underbrace{\begin{bmatrix} b_{11}/\mathfrak{D} & b_{12}/\mathfrak{D} \\ b_{21}/\mathfrak{D} & b_{22}/\mathfrak{D} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} T_1 \\ T_{\text{gyro}} \end{bmatrix}}_{\mathbf{u}}, \quad (4.4)$$

in which $\bar{f}_1$, $\bar{f}_2$, $g_1(\boldsymbol{\mu})$, $g_2(\boldsymbol{\mu})$, b_{11} , b_{12} , b_{21} , and b_{22} are presented in (2.26).

Since matrix $\mathbf{B}$ is invertible (as stated previously), the control law $\mathbf{u}_{\text{FL}}$, derived from (3.8), is given by:

$$\mathbf{u} = \mathbf{u}_{\text{FL}} = \mathbf{B}^{-1} (\mathbf{v} - \mathbf{f}), \quad (4.5)$$

with vector $\mathbf{v}$ (composed of elements derived from (3.4)):

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}} = \begin{bmatrix} \ddot{\theta}_{1d} - {}^1k_1\tilde{\theta}_1 - {}^1k_0\tilde{\theta}_1 \\ \ddot{\theta}_{2d} - {}^2k_1\tilde{\theta}_2 - {}^2k_0\tilde{\theta}_2 \end{bmatrix}, \quad (4.6)$$

in which $\tilde{\theta} = \theta - \theta_d$ is the position tracking error, θ_d is the desired trajectory, and the strictly positive constant coefficients ${}^j k_1$ and ${}^j k_0$ ($j = 1, 2$) are the control gains related to output θ_j .

To choose them, it is opted to employ the denominator polynomial of standard-form of second-order systems (OGATA, 2002), for which two performance specifications are considered: the maximum percent overshoot M_p and the rise time t_r . Their chosen values, as well as the resulting polynomial roots and control gains, are shown in Table 1.

Table 1: Values of performance specifications, resulting roots, and control gains for FL

Output of interest θ_j	M_p	t_r	Resulting roots	Control gains	
				${}^j k_1$	${}^j k_0$
θ_1	1.5 %	0.6 s	$-5.5513 \pm j4.1635$	11.1026	48.1517
θ_2	1.5 %	0.3 s	$-11.1026 \pm j8.3270$	22.2053	192.6068

Source: Author.

Thus, the control law $\mathbf{u}_{\text{FL}}$ (4.5) provides the necessary T_1 and T_{gyro} to keep bodies A and B angular positions (θ_1 and θ_2) tracking their desired trajectories (θ_{1d} and θ_{2d}) and obeying performance specifications M_p and t_r .

This way, $\dot{\theta}_{3d}$ can be derived by extracting $\dot{\theta}_3$ from (4.1):

$$T_{\text{gyro}} = J_D \dot{\theta}_3 \dot{\theta}_4 \implies \dot{\theta}_{3d} = \frac{T_{\text{gyro}}}{J_D \dot{\theta}_4}, \quad (4.7)$$

which is nonsingular since $\dot{\theta}_4(t) > 0$ during the CMG operation, as stated in Assumption 3.

Consequently, $\dot{\theta}_3$ needs to be controlled apart and it is done by the inner velocity controller described in the following subsection.

4.2.2 Inner velocity controller

This part concerns a first-order SISO system, whose body C angular velocity is the output of interest ($y = \dot{\theta}_3$) and motor #2 torque is the control input ($u = T_2$).

Its companion form (3.1) is given by:

$$\underbrace{\ddot{\theta}_3}_{\ddot{y}} = \underbrace{\frac{\bar{f}_3 + g_3(\boldsymbol{\mu})}{\mathcal{Z}\mathfrak{D}}}_{f} + \underbrace{\frac{b_3}{\mathcal{Z}\mathfrak{D}}}_{b} \underbrace{T_2}_{u}, \quad (4.8)$$

in which $\bar{f}_3$, $g_3(\boldsymbol{\mu})$, b_3 , and $\mathfrak{D}$ are also presented in (2.26).

By analyzing the term b and considering the SMI values in Table 5:

$$\begin{aligned} b &= \frac{(\mathcal{V} + \mathcal{Z})(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) - \mathcal{F}(\mathcal{Z} - \mathcal{G})c_{\theta_2}^2 - \mathcal{E}(\mathcal{Z} - \mathcal{H})c_{\theta_2}^2 c_{\theta_3}^2}{\mathcal{Z}(\mathcal{V}(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) + \mathcal{F}\mathcal{G}c_{\theta_2}^2 + \mathcal{E}\mathcal{H}c_{\theta_2}^2 c_{\theta_3}^2)} \\ &= \frac{\mathfrak{D} + \mathcal{Z}(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) - \mathcal{F}\mathcal{Z}c_{\theta_2}^2 - \mathcal{E}\mathcal{Z}c_{\theta_2}^2 c_{\theta_3}^2}{\mathcal{Z}\mathfrak{D}} = \frac{1}{\mathcal{Z}} + \frac{\mathcal{F} + \mathcal{E}c_{\theta_3}^2 - \mathcal{F}c_{\theta_2}^2 - \mathcal{E}c_{\theta_2}^2 c_{\theta_3}^2}{\mathfrak{D}} \\ &= \frac{1}{I_C + I_D} + \frac{(\mathcal{F} + \mathcal{E}c_{\theta_3}^2)s_{\theta_2}^2}{\mathfrak{D}} = 36.7647 + \frac{(0.0514 - 0.0058c_{\theta_3}^2)s_{\theta_2}^2}{\mathfrak{D}}, \end{aligned}$$

it is concluded that $b \geq \mathcal{Z}^{-1}$, $\forall \theta_2(t), \theta_3(t)$, (i.e., $b \geq 36.7647$) confirming that b is strictly positive.

This way, the control law u_{FL} , derived from (3.2), is given by:

$$u = u_{\text{FL}} = \frac{v - f}{b}, \quad (4.9)$$

with scalar v (derived from (3.4)):

$$v = \ddot{\theta}_{3d} - {}^3k_1 \tilde{\theta}_3, \quad (4.10)$$

in which $\tilde{\dot{\theta}}_3 = \dot{\theta}_3 - \dot{\theta}_{3_d}$ is the velocity tracking error, being $\dot{\theta}_{3_d}$ derived from the outer tracking controller and given by (4.7), and the strictly positive constant coefficient 3k_1 is the control gain related to output $\dot{\theta}_3$.

This control gain needs to be chosen so that body C angular velocity dynamics is faster than the outer tracking controller dynamics. Hence, ${}^3k_1 = 40$, resulting in a time constant equal to 0.025 s.

Remark 7. *As T_2 is calculated in the inner velocity controller, which has a faster dynamics than the outer tracking controller, it can be approximated to zero in $\bar{f}_1$ and $\bar{f}_2$ of (4.4).*

All the CMG parameter values used herein are presented at Table 5.

4.3 SM approach

The SM applied to the CMG MIMO is proposed to robustly deal with the tracking problem, as well as to reject disturbances, despite the uncertainty on its viscous friction coefficient values.

Likewise the FL in previous section, the same singularity would appear in the decoupling matrix $\bar{\mathbf{B}}$ if the control input $\mathbf{u} = [T_1 \ T_2]^\top$ was chosen. Therefore, it is again redefined to $\mathbf{u} = [T_1 \ T_{\text{gyro}}]^\top$, forcing the controller to have the same cascade form presented in Figure 8, as well as requiring the extraction of $\dot{\theta}_{3_d}$ as made in (4.7).

Thus, the SM approach is now applied to the outer tracking controller, whereas the inner velocity controller remains the same as developed in Subsection 4.2.2. It was opted since it also complies with Remark 7.

Regarding the CMG case, the relative degree of the outputs $y_i = \theta_i$ ($i = 1, 2$) are $r_1 = r_2 = 2$, as can be seen in (2.26), leading to the following sliding surfaces (3.10):

$$s_i = \dot{\tilde{y}}_i + \lambda_i \tilde{y}_i = \dot{\tilde{\theta}}_i + \lambda_i \tilde{\theta}_i, \quad (i = 1, 2), \quad (4.11)$$

in which $\tilde{\theta}_i = \theta_i - \theta_{i_d}$ is the tracking error of the i^{th} output, and λ_i is limited by the CMG lowest unmodeled structural resonant mode ($v_R = 4.5$ Hz) since it is a mechanical system. Therefore, a reasonable choice is to choose it equal or lower than one third of this frequency in order to avoid structural resonance (SLOTINE; LI, 1991, p. 301-302):

$$\lambda_i \leq \lambda_R \approx \frac{2\pi}{3} v_R = 9.425 \text{ rad/s}, \quad (4.12)$$

and $\lambda_1 = \lambda_2 = 8 \text{ rad/s}$ are chosen for controller design.

By differentiating (4.11) once with respect to time and replacing (2.26) in it, leads to:

$$\dot{s}_i = \underbrace{\ddot{\theta}_i - \ddot{\theta}_{i_d}}_{\ddot{\tilde{\theta}}_i} + \lambda_i \dot{\tilde{\theta}}_i = \underbrace{\frac{\bar{f}_i + g_i(\boldsymbol{\mu}) + b_{i1}T_1 + b_{i2}T_{\text{gyro}}}{\ddot{\theta}_i}}_{\mathfrak{D}} - \underbrace{\ddot{\theta}_{i_d}}_{p_i} + \lambda_i \dot{\tilde{\theta}}_i, \quad (4.13)$$

in which $\bar{f}_i$, $g_i(\boldsymbol{\mu})$, b_{i1} , and b_{i2} are also described in (2.26), being $g_i(\boldsymbol{\mu})$ function of the CMG uncertain parameters, composed of its viscous friction coefficients $\boldsymbol{\mu} = [\mu_1 \ \mu_2 \ \mu_3 \ \mu_4]^\top$.

By rewriting (4.13) in its vector form:

$$\underbrace{\begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \end{bmatrix}}_{\dot{\mathbf{s}}} = \underbrace{\begin{bmatrix} \bar{f}_1/\mathfrak{D} \\ \bar{f}_2/\mathfrak{D} \end{bmatrix}}_{\bar{\mathbf{f}}} + \underbrace{\begin{bmatrix} g_1(\boldsymbol{\mu})/\mathfrak{D} \\ g_2(\boldsymbol{\mu})/\mathfrak{D} \end{bmatrix}}_{\mathbf{g}(\boldsymbol{\mu})} + \underbrace{\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}}_{\mathbf{p}} + \underbrace{\begin{bmatrix} b_{11}/\mathfrak{D} & b_{12}/\mathfrak{D} \\ b_{21}/\mathfrak{D} & b_{22}/\mathfrak{D} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} T_1 \\ T_{\text{gyro}} \end{bmatrix}}_{\mathbf{u}}, \quad (4.14)$$

the control input vector $\mathbf{u}$ appears, which is defined as (3.13):

$$\mathbf{u} = \mathbf{u}_{\text{SM}} = \mathbf{B}^{-1} (\hat{\mathbf{u}}_{\text{eq}} + \mathbf{u}_d), \quad (4.15)$$

in which $\mathbf{B}$ is the same as in (4.4), i.e., it is invertible.

The estimated equivalent control input vector (3.14) is given by:

$$\hat{\mathbf{u}}_{\text{eq}} = -\bar{\mathbf{f}} - \hat{\mathbf{g}} - \mathbf{p}, \quad (4.16)$$

where the estimated vector $\hat{\mathbf{g}}$ is defined as (3.15):

$$\hat{\mathbf{g}} = \mathbf{g}(\boldsymbol{\mu})|_{\mu_k = \frac{\mu_{k_{\max}} + \mu_{k_{\min}}}{2} = \mu_{k_{\text{nom}}}}, \quad (k = 1, 2, 3, 4), \quad (4.17)$$

with $\mu_{k_{\text{nom}}}$ being the nominal value of μ_k , presented in Table 5.

Remark 8. The boundary values of $\mu_{k_{\max}}$ and $\mu_{k_{\min}}$ are defined by considering a maximum estimation error of $\pm 50\%$ of the nominal value, i.e., $\mu_{k_{\max}} = 3\mu_{k_{\text{nom}}}/2$ and $\mu_{k_{\min}} = \mu_{k_{\text{nom}}}/2$.

This yields to the maximum estimation function (3.16):

$$\tilde{\mathbf{g}}_{\max} = \mathbf{g}(\boldsymbol{\mu})|_{\mu_k = \frac{\mu_{k_{\text{nom}}}}{2}}. \quad (4.18)$$

The vector of discontinuous control action (3.20) is given by:

$$\mathbf{u}_d = - \begin{bmatrix} (\tilde{g}_{\max_1} + \eta_1) \text{sat}(s_1/\Phi_1) \\ (\tilde{g}_{\max_2} + \eta_2) \text{sat}(s_2/\Phi_2) \end{bmatrix}. \quad (4.19)$$

The upper bound value of switching height parameter η_i (3.19) is calculated by assuming the maximum possible initial condition values of $\mathbf{y}_i = \begin{bmatrix} \theta_i & \dot{\theta}_i \end{bmatrix}^\top$, i.e., $\tilde{\theta}_{1\max}(0) = \pm 180^\circ$ ($\pm\pi$ rad), $\tilde{\theta}_{2\max}(0) = \pm 80^\circ$ ($\pm 4\pi/9$ rad), and $\dot{\tilde{\theta}}_{i\max}(0) = 0$ ⁴, and by choosing the reaching time as $t_{\text{reach}_1} = t_{\text{reach}_2} = 3$ s, yielding to:

$$\eta_i \leq \frac{|s_i(0)|}{t_{\text{reach}_i}} = \frac{|\tilde{\theta}_{i\max}(0) + \lambda_i \tilde{\theta}_{i\max}(0)|}{t_{\text{reach}_i}} = \frac{8|\tilde{\theta}_{i\max}(0)|}{3} \Rightarrow \begin{cases} \eta_1 \leq 8\pi/3 \approx 8.3776, \\ \eta_2 \leq 32\pi/27 \approx 3.7234, \end{cases} \quad (4.20)$$

where the values $\eta_1 = 8.37$ and $\eta_2 = 3.72$ are chosen.

Regarding the boundary layer (3.21), the maximum acceptable errors for the outputs are defined as $\tilde{\theta}_{1\max} = 12^\circ$ ($\pi/15$ rad) and $\tilde{\theta}_{2\max} = 1^\circ$ ($\pi/180$ rad), leading to:

$$\Phi_i = \tilde{y}_{i\max} \lambda_i^{r_i-1} \Rightarrow \begin{cases} \Phi_1 = \tilde{\theta}_{1\max} \lambda_1 = 1.6755, \\ \Phi_2 = \tilde{\theta}_{2\max} \lambda_2 = 0.1396. \end{cases} \quad (4.21)$$

4.4 PB approach

Regarding the CMG MIMO tracking problem, its general form (2.22) is not fully-actuated, where the outputs to be tracked compose the unactuated subsystem, precluding the PB implementation described in Section 3.3.

To overcome this, it is proposed to employ the CMG reaction torques, which is the main contribution of this thesis as explained in Section 4.1. For this, firstly the coupling terms of (2.22) is separated by rewriting it as:

$$\underbrace{\begin{bmatrix} \mathbf{M}_u & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_a \end{bmatrix}}_{\bar{\mathbf{M}}(\mathbf{q})} \underbrace{\begin{bmatrix} \ddot{\mathbf{q}}_u \\ \ddot{\mathbf{q}}_a \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} \mathbf{V}_u & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_a \end{bmatrix}}_{\bar{\mathbf{V}}(\mathbf{q}, \dot{\mathbf{q}})} \underbrace{\begin{bmatrix} \dot{\mathbf{q}}_u \\ \dot{\mathbf{q}}_a \end{bmatrix}}_{\dot{\mathbf{q}}} = \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{T} \end{bmatrix}}_{\mathbf{u}} + \underbrace{\begin{bmatrix} \boldsymbol{\tau}_{fu} \\ \boldsymbol{\tau}_{fa} \end{bmatrix}}_{\boldsymbol{\tau}_f} + \underbrace{\begin{bmatrix} \mathbf{C}_u \\ \mathbf{C}_a \end{bmatrix}}_{\mathbf{C}}, \quad (4.22)$$

with $\mathbf{C}_u = -\mathbf{M}_c \ddot{\mathbf{q}}_a - \mathbf{V}_{cr} \dot{\mathbf{q}}_a$ and $\mathbf{C}_a = -\mathbf{M}_c^\top \ddot{\mathbf{q}}_u - \mathbf{V}_{cl} \dot{\mathbf{q}}_u$, from which the CMG reaction torques (4.1) and (4.2) will be extracted onwards.

Remark 9. *The coupling terms $\mathbf{C}_u$, $\mathbf{C}_a \in \mathbb{R}^2$ were separated so as not to be function of the desired state dynamics in the controller design, as will be presented ahead.*

It is worth to stress that (4.22) complies with both Properties 1 and 2.

With this and keeping in mind Remark 3 for the energy-shaping stage, consider the

⁴it is considered that the CMG system always initiate its operation in static mode.

desired storage function (3.26):

$$\mathcal{V}_e = \frac{1}{2} \dot{\mathbf{e}}^\top \bar{\mathbf{M}}(\mathbf{q}) \dot{\mathbf{e}} + \frac{1}{2} \mathbf{e}^\top \mathbf{K}_p \mathbf{e}, \quad (4.23)$$

with the tracking error vector $\mathbf{e} = \mathbf{q} - \mathbf{q}_d = [\mathbf{e}_u^\top \ \mathbf{e}_a^\top]^\top \in \mathbb{R}^4$ (being $\mathbf{q}_d = [\mathbf{q}_{u_d}^\top \ \mathbf{q}_{a_d}^\top]^\top \in \mathbb{R}^4$ the desired angular position vector, composed of the unactuated $\mathbf{q}_{u_d} = [\theta_{1_d}(t) \ \theta_{2_d}(t)]^\top \in \mathbb{R}^2$ and actuated $\mathbf{q}_{a_d} = [\theta_{3_d}(t) \ \theta_{4_d}(t)]^\top \in \mathbb{R}^2$ components), and the proportional gain matrix $\mathbf{K}_p = \text{diag}(\mathbf{K}_{p_u}, \mathbf{0}) \in \mathbb{R}^{4 \times 4}$ defined as block diagonal, composed of a symmetric positive definite matrix $\mathbf{K}_{p_u} \in \mathbb{R}_+^{2 \times 2}$ and the zero matrix $\mathbf{0} \in \mathbb{R}^{2 \times 2}$.

Note that (4.23) is positive definite since its first quadratic form term is positive definite by Property 2, as well as the second quadratic form term since $\mathbf{K}_{p_u}$ is also positive definite.

Next, by differentiating (4.23) once with respect to time, and using Property 1 and (4.22), leads to:

$$\begin{aligned} \dot{\mathcal{V}}_e &= \dot{\mathbf{e}}^\top \bar{\mathbf{M}}(\mathbf{q}) \ddot{\mathbf{e}} + \overbrace{\frac{1}{2} \dot{\mathbf{e}}^\top \dot{\bar{\mathbf{M}}}(\mathbf{q}) \dot{\mathbf{e}}}^{=\dot{\mathbf{e}}^\top \bar{\mathbf{V}}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{e}}} + \dot{\mathbf{e}}^\top \mathbf{K}_p \dot{\mathbf{e}} = \dot{\mathbf{e}}^\top (\bar{\mathbf{M}}(\mathbf{q}) \ddot{\mathbf{e}} + \bar{\mathbf{V}}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{e}} + \mathbf{K}_p \dot{\mathbf{e}}) \\ &= \dot{\mathbf{e}}^\top \left(\overbrace{\bar{\mathbf{M}}(\mathbf{q}) \ddot{\mathbf{q}} + \bar{\mathbf{V}}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}}^{=\mathbf{u} + \boldsymbol{\tau}_f + \mathbf{C}} - \bar{\mathbf{M}}(\mathbf{q}) \ddot{\mathbf{q}}_d - \bar{\mathbf{V}}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_d + \mathbf{K}_p \dot{\mathbf{e}} \right) \\ &= \dot{\mathbf{e}}^\top (\mathbf{u} + \boldsymbol{\tau}_f + \mathbf{C} - \bar{\mathbf{M}}(\mathbf{q}) \ddot{\mathbf{q}}_d - \bar{\mathbf{V}}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_d + \mathbf{K}_p \dot{\mathbf{e}}), \end{aligned} \quad (4.24)$$

where the control input vector $\mathbf{u}$ appears, which is defined as (3.28):

$$\mathbf{u} = \mathbf{u}_{PB} = -\boldsymbol{\tau}_f - \mathbf{C} + \bar{\mathbf{M}}(\mathbf{q}) \ddot{\mathbf{q}}_d + \bar{\mathbf{V}}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}_d - \mathbf{K}_p \dot{\mathbf{e}} + \mathbf{u}_{\text{damp}}. \quad (4.25)$$

Then, proceeding to the damping injection stage, (4.24) is rendered negative semi-definite by choosing the damping term (3.30):

$$\mathbf{u}_{\text{damp}} = -\mathbf{K}_d \dot{\mathbf{e}}, \quad (4.26)$$

in which the derivative gain matrix $\mathbf{K}_d = \text{diag}(\mathbf{K}_{d_u}, \mathbf{K}_{d_a}) \in \mathbb{R}^{4 \times 4}$ is defined as block diagonal, composed of symmetric positive definite matrices $\mathbf{K}_{d_u}, \mathbf{K}_{d_a} \in \mathbb{R}_+^{2 \times 2}$.

Note that (4.25) can be divided into two subsystems:

- the unactuated one, composed of the first $\mathbb{R}^2$ subsystem:

$$\mathbf{0} = -\boldsymbol{\tau}_{f_u} + \underbrace{\mathbf{M}_c \ddot{\mathbf{q}}_a + \mathbf{V}_{c_r} \dot{\mathbf{q}}_a}_{=-\mathbf{C}_u} + \mathbf{M}_u \ddot{\mathbf{q}}_{u_d} + \mathbf{V}_u \dot{\mathbf{q}}_{u_d} - \mathbf{K}_{p_u} \dot{\mathbf{e}}_u - \mathbf{K}_{d_u} \dot{\mathbf{e}}_u, \quad (4.27)$$

- and the actuated one, comprised of the second $\mathbb{R}^2$ subsystem:

$$\mathbf{T} = -\boldsymbol{\tau}_{\text{fa}} + \underbrace{\mathbf{M}_{\text{c}}^{\top} \ddot{\mathbf{q}}_{\text{u}} + \mathbf{V}_{\text{cl}} \dot{\mathbf{q}}_{\text{u}}}_{= -\mathbf{C}_{\text{a}}} + \mathbf{M}_{\text{a}} \ddot{\mathbf{q}}_{\text{ad}} + \mathbf{V}_{\text{a}} \dot{\mathbf{q}}_{\text{ad}} - \mathbf{K}_{\text{da}} \dot{\mathbf{e}}_{\text{a}}, \quad (4.28)$$

which depends on the $\dot{\mathbf{q}}_{\text{ad}}$ and $\ddot{\mathbf{q}}_{\text{ad}}$ vectors.

It can be seen that, as aforementioned in Remark 9, the coupling terms $\mathbf{C}_{\text{u}}$ and $\mathbf{C}_{\text{a}}$ in (4.27) and (4.28) are function of the states dynamics $\{\ddot{\mathbf{q}}_{\text{a}}, \dot{\mathbf{q}}_{\text{a}}\}$ and $\{\ddot{\mathbf{q}}_{\text{u}}, \dot{\mathbf{q}}_{\text{u}}\}$, respectively, instead of its desired dynamics $\{\ddot{\mathbf{q}}_{\text{ad}}, \dot{\mathbf{q}}_{\text{ad}}\}$ and $\{\ddot{\mathbf{q}}_{\text{ud}}, \dot{\mathbf{q}}_{\text{ud}}\}$, which would occur if $\mathbf{M}(\theta_2, \theta_3)$ of (2.22) were considered in (4.23) rather than $\bar{\mathbf{M}}(\mathbf{q})$ of (4.22), and would thus preclude our following proposal.

To deal with the underactuation, we propose herein to derive the components of $\dot{\mathbf{q}}_{\text{ad}}$ and $\ddot{\mathbf{q}}_{\text{ad}}$ from the unactuated subsystem (4.27) by extracting them from the intrinsic reaction torques stated in Section 4.1, i.e., the gyroscopic torque (4.1) will provide $\dot{\theta}_{3\text{d}}$, and the rotor reaction torque (4.2) will provide $\ddot{\theta}_{4\text{d}}$. Afterwards, both $\ddot{\theta}_{3\text{d}}$ and $\dot{\theta}_{4\text{d}}$ can be obtained, for instance, with backward difference approximation.

It is a reasonable idea since the desired actuated dynamics will this way be function of the desired unactuated ones (derived from the tracking task), and then be used to calculate the control vector $\mathbf{T}$. In a physical interpretation, the motor torques are determined to generate the necessary reaction torques to succeed the tracking task of the unactuated states.

By separating the reaction torque terms in (4.27), leads to:

$$\underbrace{\begin{bmatrix} -c_{\theta_2} c_{\theta_3} & -c_{\theta_2} s_{\theta_3} \\ s_{\theta_3} & -c_{\theta_3} \end{bmatrix}}_{\mathbf{L}_1} \underbrace{\begin{bmatrix} T_{\text{gyro}} \\ T_{\text{wheel}} \end{bmatrix}}_{\boldsymbol{\tau}_{\text{react}}} = \mathbf{r}, \text{ with:}$$

$$\mathbf{r} = -\boldsymbol{\tau}_{\text{fu}} - \bar{\mathbf{C}}_{\text{u}} + \mathbf{M}_{\text{u}} \ddot{\mathbf{q}}_{\text{ud}} + \mathbf{V}_{\text{u}} \dot{\mathbf{q}}_{\text{ud}} - \mathbf{K}_{\text{pu}} \mathbf{e}_{\text{u}} - \mathbf{K}_{\text{du}} \dot{\mathbf{e}}_{\text{u}},$$

in which $\bar{\mathbf{C}}_{\text{u}} = \mathbf{C}_{\text{u}} - \mathbf{L}_1 \boldsymbol{\tau}_{\text{react}}$.

Note that $\mathbf{L}_1$ is invertible since its determinant ($\det(\mathbf{L}_1) = c_{\theta_2}$) is always nonzero during the CMG operation, i.e., Assumption 1 is not violated.

This way, both $\ddot{\theta}_{4\text{d}}$ and $\dot{\theta}_{3\text{d}}$ can be determined by:

$$\underbrace{\begin{bmatrix} J_D \dot{\theta}_4 & 0 \\ 0 & J_D \end{bmatrix}}_{\mathbf{L}_2} \begin{bmatrix} \dot{\theta}_{3\text{d}} \\ \ddot{\theta}_{4\text{d}} \end{bmatrix} = \mathbf{L}_1^{-1} \mathbf{r} \implies \begin{bmatrix} \dot{\theta}_{3\text{d}} \\ \ddot{\theta}_{4\text{d}} \end{bmatrix} = \mathbf{L}_2^{-1} \mathbf{L}_1^{-1} \mathbf{r}, \quad (4.29)$$

where $\mathbf{L}_2$ is also invertible as its determinant $\left(\det(\mathbf{L}_2) = J_D^2 \dot{\theta}_4\right)$ is always nonzero during the CMG operation, i.e., Assumption 3 is not violated.

Finally, both $\ddot{\theta}_{3_d}$ and $\dot{\theta}_{4_d}$ are obtained here considering the backward difference approximation (ATKINSON; HAN; STEWART, 2011):

$$\begin{cases} \dot{\theta}_{4_d}(t) = \ddot{\theta}_{4_d}(t)\Delta t + \dot{\theta}_{4_d}(t - \Delta t) = \ddot{\theta}_{4_d}\Delta t + \dot{\theta}_{4_d}^{\text{prv}}, \\ \ddot{\theta}_{3_d}(t) = \frac{\dot{\theta}_{3_d}(t) - \dot{\theta}_{3_d}(t - \Delta t)}{\Delta t} = \frac{\dot{\theta}_{3_d} - \dot{\theta}_{3_d}^{\text{prv}}}{\Delta t}, \end{cases} \quad (4.30)$$

in which Δt is the difference interval and the superscript ^{prv} stands for the previous value of the variable, i.e., its value in instant $t - \Delta t$.

Remark 10. *It is worth mentioning that the difference interval Δt of the backward difference approximation introduced in (4.30) will be limited by the sampling period defined for the simulations and experiments, which is herein chosen equal to the hardware maximum operating sampling time, as will be stated in Section 5.1.*

This way, vectors $\dot{\mathbf{q}}_{a_d}$ and $\ddot{\mathbf{q}}_{a_d}$ are completely determined and then can be used to calculate the control action $\mathbf{T}$ in the actuated subsystem (4.28).

Finally, an analysis of the closed-loop system stability can be done likewise made in Theorem 1. For this, note that Assumption 4 holds for the CMG tracking problem since the desired trajectory $\mathbf{q}_{u_d}(t)$ must be within its operating range, i.e., $|\theta_{1_d}(t)| \leq 180^\circ$ (π rad) and $\theta_{2_d}(t)$ complying with Assumption 1, as well as $\dot{\mathbf{q}}_d(t)$ being bounded.

Theorem 2. *The control law $\mathbf{T}(t)$ in (4.28) with the desired actuated dynamics $\{\ddot{\mathbf{q}}_{a_d}, \dot{\mathbf{q}}_{a_d}\}$ given by (4.29) and (4.30) ensures the asymptotic convergence of $\mathbf{e}_u(t)$ and $\dot{\mathbf{e}}(t)$ to zero, i.e.*

$$\begin{aligned} \lim_{t \rightarrow \infty} \begin{bmatrix} \mathbf{q}_u(t) & \dot{\mathbf{q}}(t) \end{bmatrix}^\top &= \begin{bmatrix} \mathbf{q}_{u_d}(t) & \dot{\mathbf{q}}_d(t) \end{bmatrix}^\top \implies \\ \lim_{t \rightarrow \infty} \begin{bmatrix} \theta_1(t) & \theta_2(t) & \dot{\theta}_i(t) \end{bmatrix}^\top &= \begin{bmatrix} \theta_{1_d}(t) & \theta_{2_d}(t) & \dot{\theta}_{i_d}(t) \end{bmatrix}^\top, \quad (i = 1, 2, 3, 4). \end{aligned} \quad (4.31)$$

Proof. Similarly to the proof of Theorem 1, consider the CMG non-autonomous tracking error dynamics, derived from (4.22), (4.25), and (4.26):

$$\bar{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{e}} = -(\bar{\mathbf{V}}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_d)\dot{\mathbf{e}} - \mathbf{K}_p\mathbf{e}, \quad (4.32)$$

and $\dot{\mathbf{x}} = f(t, \mathbf{x})$ defined implicitly by (4.32), with the state vector $\mathbf{x} = [\mathbf{e}^\top \ \dot{\mathbf{e}}^\top]^\top \in \mathbb{R}^8$.

Consider yet the storage function (4.23) as a Lyapunov function candidate, rewritten as:

$$\mathcal{V}_e(t, \mathbf{x}) = \frac{1}{2} (\dot{\mathbf{e}}_u^\top \mathbf{M}_u \dot{\mathbf{e}}_u + \dot{\mathbf{e}}_a^\top \mathbf{M}_a \dot{\mathbf{e}}_a + \mathbf{e}_u^\top \mathbf{K}_{pu} \mathbf{e}_u), \quad (4.33)$$

where it is defined $\mathbf{K}_{pu} = \text{diag}(k_{p_1}, k_{p_2})$ with strictly positive control gains $k_{p_1}, k_{p_2} \in \mathbb{R}_+^*$.

It is time-varying, positive definite, and satisfies:

$$a_{\min} \|\mathbf{x}\|^2 \leq \mathcal{V}_e(t, \mathbf{x}) \leq a_{\max} \|\mathbf{x}\|^2, \quad \forall t \geq 0, \quad (4.34)$$

with $a_{\min} = \min\{\mathcal{Z}, \mathbf{M}_{u_{\min}}, k_{p_1}, k_{p_2}\}$, $a_{\max} = \max\{1, \mathbf{M}_{u_{\max}}, k_{p_1}, k_{p_2}\}$, where $\mathcal{Z} = I_C + I_D = 0.0272$ (see Table 5), and $\mathbf{M}_{u_{\min}}$ and $\mathbf{M}_{u_{\max}}$ denote, respectively, the smallest and largest eigenvalues of matrix $\mathbf{M}_u$, which is function of $\theta_2(t)$ and $\theta_3(t)$ (see (2.22)).

Then, substituting (4.25), as well (4.26), into its derivative (4.24), leads to:

$$\begin{aligned} \dot{\mathcal{V}}_e(t, \mathbf{x}) &= -\dot{\mathbf{e}}^\top \mathbf{K}_d \dot{\mathbf{e}} = -\left(\sum_{i=1}^4 k_{d_i} \dot{\theta}_i^2 \right) \\ &\leq \mathcal{V}_e^*(\mathbf{x}) \leq 0, \quad \forall t \geq 0, \end{aligned} \quad (4.35)$$

in which it is defined $\mathbf{K}_d = \text{diag}(k_{d_1}, k_{d_2}, k_{d_3}, k_{d_4})$ with strictly positive control gains $k_{d_i} \in \mathbb{R}_+^*$ ($i = 1, 2, 3, 4$), and $\dot{\theta}_i = \dot{\theta}_i - \dot{\theta}_{i_d}$ are the CMG bodies angular velocity errors.

This way, the conditions (1) and (2) of Theorem 3 are satisfied.

Remark 11. *It follows from these two conditions that $\mathbf{e}_u(t)$ and $\dot{\mathbf{e}}(t)$ are bounded. Additionally, $\mathbf{q}_u(t)$ and $\dot{\mathbf{q}}(t)$ are also bounded, since $\mathbf{q}_{ud}(t)$ and $\dot{\mathbf{q}}_d(t)$ were previously assumed to be bounded. Consequently, the right-hand side (RHS) of (4.32) is bounded as well. At last, by considering Property 2, which states that the minimum singular value of $\bar{\mathbf{M}}(\mathbf{q})$ is bounded away from zero, it follows that $\ddot{\mathbf{e}}(t)$ is bounded.*

As $\dot{\mathcal{V}}_e$ is only negative semi-definite, let's look at its higher order derivative when it goes to zero (similarly to what was done in Paden and Panja (1988, p. 1703, 1710)). For this, define function $\mathcal{W}(t, \mathbf{x}) \triangleq \ddot{\mathcal{V}}_e(t, \mathbf{x})$, obtained by differentiating (4.35):

$$\mathcal{W}(t, \mathbf{x}) = -2\dot{\mathbf{e}}^\top \mathbf{K}_d \ddot{\mathbf{e}}, \quad \forall t \geq 0. \quad (4.36)$$

Then, it follows from the bounds on Remark 11 that $|\mathcal{W}(t, \mathbf{x})|$ is also bounded, which satisfies condition (3) of Theorem 3.

Next, let's differentiate (4.36):

$$\dot{\mathcal{W}}(t, \mathbf{x}) = -2\ddot{\mathbf{e}}^\top \mathbf{K}_d \ddot{\mathbf{e}} - 2\dot{\mathbf{e}}^\top \mathbf{K}_d \mathbf{e}^{(3)} \quad (4.37)$$

and substitute the expressions of $\ddot{\mathbf{e}}$ and $\mathbf{e}^{(3)}$ (calculated from (4.32)) into it. By using Lemma 1 in Appendix D, it can be found that $\dot{\mathcal{W}}(t, \mathbf{x})$ is continuous in both arguments and depends on time continuously through a bounded function, and thus satisfying condition (4) of Theorem 3.

Then, by defining $\mathfrak{M} \triangleq \{(\mathbf{e}_u, \dot{\mathbf{e}}) \mid \mathcal{V}_e^*(\mathbf{x}) = 0\}$, where $\mathcal{V}_e^*(\mathbf{x}) = -\dot{\mathbf{e}}^\top \mathbf{K}_d \dot{\mathbf{e}}$, we have that $\dot{\mathbf{e}} = \mathbf{0}$, for all $(\mathbf{e}_u, \dot{\mathbf{e}}) \in \mathfrak{M}$. Taking this into account, by replacing the expression of $\ddot{\mathbf{e}}$, calculated from (4.32), into (4.37) results in:

$$\begin{aligned} \dot{\mathcal{W}}(t, \mathbf{x}) &= -2 (\bar{\mathbf{M}}(\mathbf{q})^{-1} \mathbf{K}_p \mathbf{e})^\top \mathbf{K}_d (\bar{\mathbf{M}}(\mathbf{q})^{-1} \mathbf{K}_p \mathbf{e}) \\ &= -2 (\mathbf{M}_u^{-1} \mathbf{K}_{p_u} \mathbf{e}_u)^\top \mathbf{K}_{d_u} (\mathbf{M}_u^{-1} \mathbf{K}_{p_u} \mathbf{e}_u) \\ &= -2 \mathbf{y}^\top \left(\underbrace{\mathbf{M}_u^{-1} \mathbf{K}_{d_u} \mathbf{M}_u^{-1}}_{\Xi} \right) \mathbf{y}, \quad \forall (\mathbf{e}_u, \dot{\mathbf{e}}) \in \mathfrak{M}, \end{aligned} \quad (4.38)$$

where $\mathbf{y} \triangleq \mathbf{K}_{p_u} \mathbf{e}_u = \begin{bmatrix} k_{p_1} \tilde{\theta}_1 & k_{p_2} \tilde{\theta}_2 \end{bmatrix}^\top$, with the tracking errors $\tilde{\theta}_j = \theta_j - \theta_{j_d}$ ($j = 1, 2$), and Ξ is a symmetric positive definite matrix, recalling that $\mathbf{K}_{p_u} = \text{diag}(k_{p_1}, k_{p_2})$ and $\mathbf{K}_{d_u} = \text{diag}(k_{d_1}, k_{d_2})$, with strictly positive control gains k_{p_1} , k_{p_2} , k_{d_1} and k_{d_2} , and $\mathbf{M}_u$ is a nonsingular matrix, since its determinant ⁵:

$$\det(\mathbf{M}_u) = (9.703 - 1.764s_{\theta_2}^2 - 1.757s_{\theta_3}^2 - 0.585s_{\theta_2}^2 s_{\theta_3}^2) 10^{-3} > 0, \quad \forall (\theta_2, \theta_3), \quad (4.39)$$

is nonzero for all $\theta_2(t)$ and $\theta_3(t)$ during the CMG operation.

Therefore, (4.38) is a negative definite function of $\mathbf{e}_u$, i.e., it is equal to zero if and only if $\mathbf{y} = \mathbf{0} \iff \tilde{\theta}_1 = \tilde{\theta}_2 = 0$, thus accomplishing the tracking task of the unactuated states.

At last, from the closed loop (4.32) and the bounds on Remark 11, the condition (5) of Theorem 3 is verified, thus fulfilling the five conditions of Matrosov's theorem. Besides, the condition (1) of Theorem 3 can be used to find an α and Ω such that, for arbitrary initial conditions, they will be contained in $V_{t_0, \alpha}^{-1}$. Therefore, the origin is a globally asymptotically stable equilibrium point of (4.32).

□

⁵calculated by considering the CMG parameter values in Table 5.

To summarize, Figure 9 illustrates the closed-loop system, where the control vector $\mathbf{T}(t) = [T_2(t) \ T_1(t)]^\top$ given by (4.28), together with the desired actuated dynamics $\{\ddot{\mathbf{q}}_{\text{ad}}, \dot{\mathbf{q}}_{\text{ad}}\}$ given by (4.29) and (4.30), accomplishes the CMG MIMO tracking task by asymptotically converging $\theta_1(t)$, $\theta_2(t)$, and $\dot{\theta}_i(t)$ ($i = 1, 2, 3, 4$) to its desired values.

Figure 9: Block diagram of the PB controller for the CMG MIMO tracking task

Source: Author.

The control gains, i.e., the strictly positive components of matrices $\mathbf{K}_{\text{p}_u}$, $\mathbf{K}_{\text{d}_u}$, and $\mathbf{K}_{\text{d}_a}$, were empirically chosen as $k_{\text{p}_1} = 5$, $k_{\text{p}_2} = 4$, $k_{\text{d}_1} = 1.4$, $k_{\text{d}_2} = 1.2$, $k_{\text{d}_3} = 1.6$, and $k_{\text{d}_4} = 0.5$.

5 RESULTS AND DISCUSSION

This chapter presents the validation of the previous designed nonlinear controllers by performing numerical simulations (with the nonlinear model presented in Appendix C and real-time practical experiments (with the real plant). Next, the experimental setup is described, followed by the test procedures considered to validate the controllers. Results are analyzed and compared, allowing an overall discussion.

5.1 Experimental setup

The ECP model 750 CMG system, depicted in Figure 10, is composed of the CMG electromechanical plant and the complementary hardware with the real-time controller (a digital signal processing (DSP) based controller together with a data acquisition board) and the input/output (I/O) electronics (ECP, 1999), connected to a PC, in which the MATLAB[®]/Simulink software (version 2014b) is used to perform both simulations and practical experiments, whose Simulink models are presented in Appendix F. For the real-time control implementation of the latter case, the Real-Time Windows Target, running under windows 7 operating system, is considered.

Figure 10: ECP model 750 CMG system

Source: Author.

In the next procedures, the sampling period is defined as $T = 1$ ms, which is the equivalent hardware maximum operating sampling frequency and is considered small enough for continuous-time implementation.

Remark 12. *The CMG bodies angular positions $\theta_i(t)$ ($i = 1, 2, 3, 4$) are directly obtained (from the nonlinear dynamics model in simulation case and from incremental encoders measurements in experiment case), while its time derivatives $\dot{\theta}_i(t)$ and $\ddot{\theta}_i(t)$ are indirectly obtained (from filtered derivatives in both simulation and experiment, used to mitigate high frequency noise amplification in the latter case), since CMG does not have angular velocity sensors.*

Figure 11 presents the system block diagram used for the performed tests.

Figure 11: System block diagram for the tests

Source: Author.

It illustrates the CMG closed-loop system with a generic nonlinear controller block (to represent each previous designed controller), in which the CMG block represents the **Function Block Parameters** of the Simulink files, being a **S-function** (that contains the CMG nonlinear equations) for the simulation cases and the **Subsystem (mask)** (that communicates with the real plant) for the experiment cases. It also depicts the system signals:

- a) the *tracking signals*, represented by the “desired trajectories” block, in which the desired unactuated state vector $\mathbf{q}_{\text{ud}}(t) = [\theta_{1\text{d}}(t) \ \theta_{2\text{d}}(t)]^\top$ (composed of bodies *A* and *B* desired angular positions $\theta_{1\text{d}}$ and $\theta_{2\text{d}}$) are derived once and twice with respect to time to obtain $\dot{\mathbf{q}}_{\text{ud}}(t)$ and $\ddot{\mathbf{q}}_{\text{ud}}(t)$, respectively;

- b) the *control signals*, which pass through saturation blocks (represented by “sat” blocks) before entering the CMG block as motors torques $T_1(t)$ and $T_2(t)$. This is done to limit the control action with the motor saturation values;
- c) the *CMG bodies angular position signals* $\theta_i(t)$ ($i = 1, 2, 3, 4$), which derivatives $\dot{\theta}_i(t)$ and $\ddot{\theta}_i(t)$ are obtained by derivative blocks (represented by the “d/dt” blocks) that are filtered as abovementioned in Remark 12, though it is not explicitly illustrated in Figure 11;
- d) and the *disturbance signals* $d_{\theta_1}(t)$ and $d_{\theta_2}(t)$ that are added to $\theta_1(t)$ and $\theta_2(t)$, respectively.

With this, two tests are considered here to validate the controller: (i) a tracking test and (ii) a disturbance rejection test. The former is executed to evaluate if the controllers are able to accomplish the tracking of the unactuated states in a wide operating range, while the latter evaluates the controller disturbance rejection capabilities with respect to the output signals.

These tests begin with the procedure of taking the system to its operating point. For it, both axes brakes #1 and #2 are activated, and a simple proportional-integral (PI) controller is employed to lead and to keep the rotor velocity at 400 rpm ($40\pi/3$ rad/s). Then, both axes brakes and the PI controller are deactivated in $t \approx 8$ s, when the designed nonlinear controller is activated. Next:

- (i) for tracking test, both desired trajectories $\theta_{1_d}(t)$ and $\theta_{2_d}(t)$ (which are defined as sine wave signals with amplitudes of 180° (π rad) and 60° ($\pi/3$ rad), respectively, and frequency of 0.05 Hz ($\pi/10$ rad/s)) are simultaneously applied from $t = 10$ s to $t = 70$ s, as presented in Fig. 12a.
- (ii) for disturbance rejection test, filtered squared wave signals are applied to outputs θ_1 , at $t = 15$ s, and θ_2 , at $t = 17.5$ s. These disturbance signals $d_{\theta_1}(t)$ and $d_{\theta_2}(t)$ reach amplitudes of $\pm 45^\circ$ ($\pm\pi/4$ rad) and $\pm 20^\circ$ ($\pm\pi/9$ rad)), respectively, and change every 10 s ($d_{\theta_1}(t)$) and 5 s ($d_{\theta_2}(t)$) until $t = 60$ s, as illustrated in Fig. 12b.

Their results are presented and analyzed in the next sections, where all the graphs present both numerical simulation (red curve) and practical experiment (blue curve) results, and some of them do not begin at $t = 0$ s to emphasize the results.

Figure 12: Signals considered in the tests

Source: Author.

5.2 Tracking test

All the initial conditions herein are defined equal to zero ($\theta_i(0) = \dot{\theta}_i(0) = 0$ ($i = 1, 2, 3, 4$) and $\theta_{1d}(0) = \theta_{2d}(0) = 0$) with no applied disturbances ($d_{\theta_1}(t) = d_{\theta_2}(t) = 0$) in order to evaluate the tracking performance.

Figure 13 brings the plots with FL results, followed by Figure 14 with the SM ones, and Figure 15 depicts the PB ones.

One can see that all designed controllers satisfactorily accomplished the tracking task of both output signals in the first plots (a), with tracking errors presented in the second plots (b). It is worth noting that the SM controller respected both defined boundaries $\tilde{\theta}_{1\max}$ and $\tilde{\theta}_{2\max}$ (dashed black curves).

Their control signals are depicted in the third plots (c), in which they do not saturate in steady-state, and the resulting actuated state and rotor velocity are depicted in the fourth plots (d) ¹. It can be seen that the actuated states of each controller also satisfactorily track their desired trajectories (dashed curves).

¹except for the PB case, in which the actuated state $\ddot{\theta}_4(t)$ is presented instead of the rotor velocity $\dot{\theta}_4(t)$, which is actually depicted in Fig 15f.

Figure 13: Tracking test: Results from FL controller

Source: Author.

The other plots that were not aforementioned are particular results for each controller.

Regarding the SM controller, its sliding surfaces are presented in Figure 14e, in which the boundary layers Φ_1 and Φ_2 (dashed black curves) are also respected in steady-state, and Figure 14f presents the varying viscous friction coefficients for the simulation case ², which values change every 0.5 s within their boundary error values, as defined in Remark 8.

Concerning the PB controller, its storage function is illustrated in Figure 15e together with its derivative. Note that not only $\mathcal{V}_e(t)$ is nonnegative and $\dot{\mathcal{V}}_e(t)$ is nonpositive, but also both rapidly converge to zero.

²as they were considered inaccurate for the SM controller design, it was opted to vary its value in the simulation cases. Although this does not happen in the experiment cases, it was done in order to validate the controller robustness in simulation cases.

Figure 14: Tracking test: Results from SM controller

Source: Author.

Figure 15: Tracking test: Results from PB controller

Source: Author.

Finally, the main results are gathered in Figure 16 for comparison, and to aid it for a quantitative analysis, the maximum absolute error (MAE) and the integral absolute error (IAE) indexes are presented in Table 2. To calculate them, the initial data ($0 \leq t < 11$ s) was ignored to disregard not only the initial procedure of taking the system to its operating point but also the transient state.

Figure 16: Tracking test: Comparing the results of the controllers

Source: Author.

One can see in Figures 16a and 16b, together with the Table 2, that the PB results (blue curves) presented lower $\tilde{\theta}_1$ than the SM (dotted green curves) and FL (dashed magenta curves) controllers, which is more pronounced in the simulation case, as emphasized by their MAE and IAE values presented in Table 2. Regarding $\tilde{\theta}_2$, all of them were considerably low and pretty similar to each other in general. It is worth remarking that the SM results, from the simulation case, presented a higher variance due to the varying viscous friction coefficient values presented in Figure 14f, which did not precluded the

Table 2: Maximum absolute error (MAE) and integral absolute error (IAE) indexes from tracking test results

		Simulation (Figure 16a)			Experiment (Figure 16b)		
		PB	SM	FL	PB	SM	FL
$\tilde{\theta}_1$ (deg)	MAE	0.0402	6.1392	5.1677	1.1735	6.8439	5.6818
	IAE (10^3)	1.219	212.837	180.155	18.305	227.212	184.315
$\tilde{\theta}_2$ (deg)	MAE	0.0262	0.0331	0.0098	1.0618	0.8177	0.8125
	IAE (10^3)	0.826	0.651	0.301	22.686	25.056	29.105

Source: Author.

achievement of the tracking task but instead demonstrated the robustness against the inaccuracy.

Their control signals were also very similar to each other, as depicted in Figures 16c and 16d, except for T_1 of the experiment case, in which the PB one was somehow noisy, what did not affect the overall performance neither its practical viability.

5.3 Disturbance rejection test

In this case, the desired trajectories for the output signals are equal to zero ($\theta_{1d}(t) = \theta_{2d}(t) = 0$) and all initial conditions remain equal to zero, except for the output signals ($\theta_1(0) = 45^\circ$ ($\pi/4$ rad) and $\theta_2(0) = 20^\circ$ ($\pi/9$ rad)), in order to also evaluate the controller performance regarding nonzero initial conditions.

Likewise the previous section, the plots from Figures 17, 18, and 19 present the FL, SM, and PB results, respectively.

The first plots (a) show that the controllers not only converged the output signals to its desired trajectories (dashed black curves), i.e. to zero, right after the controller activation at $t = 8$ s, but also satisfactorily rejected the disturbance signals $d_{\theta_1}(t)$ and $d_{\theta_2}(t)$ (dashed green curves). Their tracking errors are presented in the second plots (b), and regarding the SM controller it is worth noting that its tracking errors are within the specified boundaries $\tilde{\theta}_{1_{\max}}$ and $\tilde{\theta}_{2_{\max}}$ during the steady-state.

The third plots (c) illustrate the control signals, in which they rapidly saturate every time the disturbance signals change (what is normal since these changes are abrupt). This saturation limits the controllers performance, precluding the actuated states to reach their desired values, as depicted in the forth plots (d). Nevertheless, it was still possible to quickly reject significant disturbances in both outputs.

Figure 17: Disturbance rejection test: Results from FL controller

Source: Author.

The others particular results are presented in Figures 18e and 19e.

The former regards the SM sliding surfaces, in which the controller provides a finite-time convergence of the sliding surfaces every time it surpasses the boundary layers ϕ_1 and ϕ_2 , respecting the specified reaching time ($t_{\text{reach}} \leq 3$ s).

And the latter concerns the PB storage function and its derivative, in which, similarly to the tracking test results, $\mathcal{V}_e(t)$ is nonnegative and $\dot{\mathcal{V}}_e(t)$ is nonpositive, and they quickly converge to zero every time the disturbance signals change.

Figure 18: Disturbance rejection test: Results from SM controller

Source: Author.

Figure 19: Disturbance rejection test: Results from PB controller

Source: Author.

Likewise the tracking test, the main results are gathered in Figure 20 for comparison, and both MAE and IAE indexes are presented in Table 3. To calculate them, the initial data ($0 \leq t < 14$ s) was ignored to disregard the inaccuracies on the initial condition positioning that occurred in the experiment procedure.

Figure 20: Disturbance rejection test: Comparing the results of the controllers

Source: Author.

Table 3: Maximum absolute error (MAE) and integral absolute error (IAE) indexes from disturbance rejection test results

		Simulation (Figure 20a)			Experiment (Figure 20b)		
		PB	SM	FL	PB	SM	FL
$\tilde{\theta}_1$ (deg)	MAE	41.1353	43.1438	41.4512	42.5740	54.5394	46.5809
	IAE (10^3)	44.033	106.237	46.576	58.719	174.950	88.116
$\tilde{\theta}_2$ (deg)	MAE	17.4643	17.5220	17.4581	18.7492	19.5118	19.1928
	IAE (10^3)	40.375	40.435	24.204	94.582	104.714	63.652

Source: Author.

Figures 20a and 20b presents the tracking errors obtained in the simulations and experiments, respectively, in which their $\tilde{\theta}_1$ were very alike, being the PB and FL results very similar to each other and the SM ones higher than both previous, as emphasized by their MAE and IAE indexes presented in Table 3. On the other hand, their $\tilde{\theta}_2$ were overall higher in the experiments due to longer saturation of control signals (Figure 20d), specially of T_1 , when compared to the simulation ones (Figure 20c).

Regarding the control signals in Figures 20c and 20d, there were some coupling effects when T_1 actuates against the disturbance applied in θ_1 (at instants $t = \{15, 25, 35, 45\}$ s) also affecting $\tilde{\theta}_2$, being more pronounced in the SM case, e.g., at $t = 25$ s of experiment case (Figure 20b), the $\tilde{\theta}_2$ absolute overshoot values are approximately 2.9° (PB), 11.0° (SM), and 2.7° (FL). It is also possible to observe that the T_1 from PB not only was somehow noisy again but also its total saturation time (TST)³ was higher in both simulation and experiment cases, as presented in Table 4, causing higher overshoots of its $\tilde{\theta}_2$. On the other hand, the low TST of SM case yields to a slower error convergence of its both $\tilde{\theta}_1$ and $\tilde{\theta}_2$, as demonstrated by its higher IAE values in Table 3.

Table 4: Total saturation time (TST) of motors #1 and #2 from disturbance rejection test results

	Simulation (Figure 20c)			Experiment (Figure 20d)		
	PB	SM	FL	PB	SM	FL
T_1 (s)	2.800	0.448	1.162	6.438	0.618	3.222
T_2 (s)	0.196	0	0.3850	0.580	0.110	0.647

Source: Author.

5.4 Overall analysis and discussion

Despite the approximations adopted herein to achieve the practical viability, as the use of filtered derivatives to obtain the angular velocities and accelerations of the CMG bodies (since there are only incremental encoders to measure its angular positions), as well as the employment of the backward difference approximation to calculate the desired actuated angular velocities and accelerations for the controllers control law, the results obtained from the three designed nonlinear controllers were overall satisfactory, where not only the tracking task in a wide operating range was accomplished but also a disturbance rejection was achieved within the CMG hardware limitations, i.e., the limited torque

³to calculate it, the initial data ($0 \leq t < 14$ s) was also ignored.

provided by the DC motors.

Concerning the FL and SM controllers, it is worth remarking the presence of steady-state errors in both outputs, yielding not only to a periodic behavior of their $\tilde{\theta}_1$ in the tracking test, in both simulation and experiment cases, but also to a constant error during the steady-state of their $\tilde{\theta}_2$ that is more pronounced in the experiment results, suggesting that an adoption of an integral term in their control designs to improve their tracking performance would be interesting (SLOTINE; LI, 1991; HAMAYUN et al., 2016).

6 CONCLUSIONS

In order to improve the performance of the ECP CMG tracking task to get a wider operating range, it has been designed three nonlinear controllers, each based on feedback linearization, sliding-mode and passivity-based approaches. To deal with this underactuated problem that involves the tracking task of the unactuated states, it was proposed herein to employ the intrinsic reaction torques of the CMG as internal control signals, which is one of the main contributions of this thesis. This enabled the design of the nonlinear controllers for this particular class of systems, which practical feasibility was provided by adopting some approximations stated in Section 5.4.

Regarding the CMG nonlinear model obtained herein, it satisfactorily attended the application, even though some nonlinear dynamics were not considered in the modeling process, such as the current drive amplifier, Coulomb friction and motor dead-zones.

In general, the three designed nonlinear controllers accomplished the tracking task in a wide range of operation with satisfactory results. It is worth emphasizing the enabled stability analysis for the PB closed-loop system, which provided a good tracking performance that converges the tracking errors to zero, even though it has presented significant overshoots in the disturbance rejection test, caused by the longer saturation of its control signals, and one of its control action has been somehow noisy in all experiments.

Regarding the comparison made of the results of the controllers, it is worth emphasizing that the idea was not to acknowledge which one is better, but rather to contrast their differences, once each of them has its particular characteristics, as well as their control gains have been empirically chosen to obtain satisfactory results.

Thus, considering the control gains chosen herein for each controller, as well as their control design development, and their obtained results, the author suggests that the PB have reached better results for the tracking task, limited by the lack of the integral term in the FL and SM controller designs. Regarding the disturbance rejection test, the author suggests that the FL controller have presented better rejection, where its overshoots were overall lower and smoother than the ones from the other two controllers.

These main characteristics of each controller may be considered and explored depending on the CMG application, as the robustness provided by the SM controller, or the tracking error convergence provided by the PB controller, or the overall lower and

smoother tracking errors from disturbance rejection provided by the FL controller.

Obviously these suggestions are made based on the results obtained herein, that may be improved by the following author's suggestions for future research.

Regarding both FL and SM control designs, an integral term may be added in their control designs to improve their tracking performance, as aforementioned in Section 5.4. In a general view, the control saturation or a bounded control action may be considered in the three control design procedures developed herein to reduce the control saturation during the transient-state, as well as the use of optimization techniques to find the control gains to improve the overall control performance. Additionally, the discrete-time version of the designed nonlinear controllers may also improve their performance, once the real-time application is discrete. Specifically regarding the SM, another improvement that may be considered is the employment of the higher-order sliding-mode approaches, as stated in Section 3.2, which (depending on the unmodeled dynamics) may reduce the chattering phenomenon.

Concerning the modeling process, the quaternion algebra may be considered in both kinematics and dynamics stages to avoid coordinate singularities, which occurs whenever a small change in the physical configuration of the represented system cannot be represented by a small change in coordinates. If this model is used as base for the nonlinear controllers, it may improve their performance, once the quaternion algebra not only has lower parameters to represent all orientations without singularity (ANGELES, 2013; STUELPNAGEL, 1964), but also is computationally faster compared to the homogeneous representation (SAHU; BISWAL; SUBUDHI, 2008). Moreover, as the designed controllers are governed by nonlinear equations, there may be some singularities arisen from the aforementioned coordinate singularities that may cause motor saturation, which could preclude a wider operating range and/or better tracking performance.

Another improvement that may be considered for the modeling process is to consider some nonlinear dynamics, e.g., the current drive amplifier dynamics (in which an analog PI controller is implemented for tracking the demanded current (ECP, 1999, p. 55)), the Coulomb friction action, the DC motors transmission dynamics and its dead-zones, to name a few. Although it may yield to a more complex control action as a result of more nonlinear terms being added to it, a performance improvement is expected.

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APPENDIX A – PUBLISHED AND SUBMITTED PAPERS

During this thesis development, a total of seven papers were derived from it:

- four papers were submitted for international conferences and are already published in their conference proceedings;
- three articles were submitted for journals and are already accepted.

These papers are listed below:

- a) **paper:** Modeling of a 4-DoF Control Moment Gyroscope (TORIUMI et al., 2017);
 - **conference:** 24th ABCM International Congress of Mechanical Engineering (COBEM 2017);
 - **date/place:** Dec. 3-8, 2017, Curitiba, Paraná, Brazil;
 - **link:** <http://www.sistema.abcm.org.br/articleFiles/download/11021>;
 - **doi:** 10.26678/ABCM.COBEM2017.COB17-1054;
 - **abstract:** Control moment gyroscopes (CMGs) are used as actuators in some applications like satellites and spacecrafts attitude control, and stabilization. A good system model is fundamental for control applications. This paper presents a detailed modeling process of a four degrees of freedom (DOF) CMG unit, which kinematics stage is based on a defined convention using quaternion algebra, and dynamics stage is based on Gibbs-Appell equations in Kane's proposed dynamical equations form. Two tests that analyze the main characteristics of the plant are considered in order to validate the generated nonlinear model, which numerical simulations results are compared to the practical experiments.
- b) **paper:** Robust Nonlinear Control Applied to a Control Moment Gyroscope with SISO Configuration (TORIUMI; ANGÉLICO, 2018);

- **conference:** 9th IFAC Symposium on Robust Control Design (ROCOND 2018);
 - **date/place:** Sept. 3-5, 2018, Florianópolis, Santa Catarina, Brazil;
 - **link:** <https://www.sciencedirect.com/science/article/pii/S2405896318327587>;
 - **doi:** 10.1016/j.ifacol.2018.11.097;
 - **abstract:** This paper presents a robust nonlinear control design for a control moment gyroscope (CMG) in its single-input/single-output (SISO) configuration. The CMG unit, model 750 from Educational Control Products, is described, as well as its SISO dynamics. The control design is based on a sliding-mode control approach and aims at tracking a desired output trajectory in a wide operational range, despite uncertainties on some model parameters and output disturbances. Results are obtained from both numerical simulations and real-time practical experiments to validate the controller
- c) **paper:** Feedback Linearization Approach Applied to a Control Moment Gyroscope with SISO Configuration (TORIUMI; ANGÉLICO; TANNURI, 2018);
- **conference:** 13th IEEE/IAS International Conference on Industry Applications (INDUSCON 2018);
 - **date/place:** Nov. 11-14, 2018, São Paulo, São Paulo, Brazil;
 - **link:** <https://ieeexplore.ieee.org/document/8627190>;
 - **doi:** 10.1109/INDUSCON.2018.8627190;
 - **abstract:** Control moment gyroscopes are widely used as actuators in some applications like satellites and spacecrafts attitude control, which involves a large operational range problem, thus posing additional challenges to the control design. A modeling process of a control moment gyroscope, model 750 from Educational Control Products, is presented and its multiple-input multiple-output (MIMO) nonlinear model is obtained, as well as one of its single-input single-output (SISO) configuration model. A feedback linearization, which is an approach to nonlinear control design, is proposed to control the single output of CMG SISO configuration. Both numerical simulation and practical experiment results are obtained to evaluate the designed nonlinear controller.
- d) **paper:** Sliding Mode Control Applied to a Multivariate Underactuated Control Moment Gyroscope (TORIUMI; ANGÉLICO, 2019b);
- **conference:** 58th IEEE Conference on Decision and Control (CDC 2019);

- **date/place:** Dec. 11-13, 2019, Nice, France;
 - **link:** <https://ieeexplore.ieee.org/document/9029582>;
 - **doi:** 10.1109/CDC40024.2019.9029582;
 - **abstract:** It is proposed a nonlinear controller based on sliding mode control (SMC) to accomplish the tracking problem of a control moment gyroscope (CMG). This is a challenging task for practical implementation since it is an underactuated mechanical system and also leads to a coupled nonlinear MIMO system that works in a wide range of operation. To avoid a singularity that would arise in the control signal computation, a cascade structure is proposed with two loops: one based on sliding mode control to deal with tracking and uncertainties on some model parameters, and other based on feedback linearization, which is an inner velocity control. Numerical simulations and practical experiments with the real plant are performed to evaluate the proposed controller.
- e) **article:** Nonlinear Controller Design for Tracking Task of a Control Moment Gyroscope Actuator (TORIUMI; ANGÉLICO, 2020a);
- **journal:** IEEE/ASME Transactions on Mechatronics (Qualis A1, ENG. IV);
 - **submission date:** Apr. 9, 2018 (accepted Sept. 30, 2019, published online Oct. 14, 2019, and printed in Volume 25, Issue 1, Feb. 13, 2020);
 - **link:** <https://ieeexplore.ieee.org/document/8867857>;
 - **doi:** 10.1109/TMECH.2019.2946804;
 - **abstract:** Control moment gyroscopes (CMG) are widely used as actuators for attitude control of satellites and spacecraft, which requires the controller to fulfill the tracking task of a CMG in a wide operating range, despite its highly coupled nonlinear behavior. A variety of control techniques were proposed for tracking task of the CMG unit, model 750, from Educational Control Products (ECP), but a few accomplished it in a wide operating range, e.g., the linear parameter-varying approach. Another one that may accomplish it is the nonlinear control approach. Therefore, this article proposes a nonlinear controller design based on feedback linearization for the tracking task of the CMG unit in a wide operating range. To cope with a singularity that would appear in the control signal computation, the proposed controller has a cascade structure composed of an outer tracking controller and an inner velocity controller, both

based on the input–output linearization approach. The designed controller is validated via numerical simulations and real-time practical experiments.

f) **article:** Passivity-Based Nonlinear Control Approach for Tracking Task of an Underactuated CMG (TORIUMI; ANGÉLICO, 2020b);

- **journal:** IEEE/ASME Transactions on Mechatronics (Qualis A1, ENG. IV);
- **submission date:** Apr. 8, 2020 (accepted Nov. 3, 2020, and published online Nov. 9, 2020);
- **link:** <https://ieeexplore.ieee.org/document/9252108>;
- **doi:** 10.1109/TMECH.2020.3036765;
- **abstract:** This paper proposes a passivity-based nonlinear control approach to deal with the tracking task of a Control Moment Gyroscope (CMG), which is an underactuated nonlinear MIMO system with coupled dynamics, where the main goal is the tracking of the unactuated states in a wide range of operation. To overcome this underactuation, the proposed nonlinear controller employs the CMG intrinsic reaction torques. The closed-loop system stability is analyzed by invoking the Lyapunov stability and Matrosov’s theorem. Numerical simulations and real-time practical experiments with the real plant are performed to validate the proposed nonlinear controller, as well as its performance is analyzed by contrasting the experiment results with the ones obtained by authors’ previous established nonlinear controllers.

g) **article:** Lyapunov-Based Nonlinear Control Applied to a Control Moment Gyroscope;

- **journal:** SBA Journal of Control, Automation and Electrical Systems (Qualis B1, ENG. IV);
- **submission date:** May 15, 2020 (accepted Dec. 9, 2020);
- **doi:** 10.1007/s40313-020-00681-x;
- **abstract:** One of the main applications of a mechanical control moment gyroscope (CMG) as actuator is the attitude control of spacecraft and satellites, which requires a wide operating range. Motivated by this, this paper investigates the feasibility of a Lyapunov-based nonlinear control applied to the tracking task of a CMG unit from Educational Control Products, model 750, in its SISO configuration. This problem involves an underactuated system, in

which the output signal is the unactuated state, posing additional challenges to control design. To deal with it, an additional internal control signal is employed and the stability analysis of the closed-loop system is given by invoking both Lyapunov stability and Matrosov's theorem. To validate the proposed controller, numerical simulations and practical experiments were performed and contrasted with the authors' previous designed nonlinear controllers.

Additionally, four more papers were developed in co-authoring:

- two papers were published in conference proceedings;
- and two articles were submitted for journals (one was accepted and already published and the other is currently in peer-review process).

These papers are described below:

- a) **paper:** Sliding Mode Control with Linear Quadratic Regulator Augmented with Integrators Applied to a 2DOF Helicopter (BARBOSA; TORIUMI; ANGÉLICO, 2018);
 - **conference:** XXII Congresso Brasileiro de Automática (CBA 2018);
 - **date/place:** Sept. 9-12, 2018, João Pessoa, Paraíba, Brazil;
 - **link:** <https://plataforma.swge.com.br/PROCEEDINGS/PDF/CBA2018-1049.pdf>;
 - **doi:** 10.20906/CPS/CBA2018-1049;
 - **abstract:** It is proposed the use of Sliding Mode Control (SMC) with Linear Quadratic Regulator (LQR) when augmented with integrators: the former guarantees robustness against model uncertainty and external disturbance while the latter adds tracking capability with null steady-state error for step inputs. Besides, only the first derivative of the reference is required for trajectory tracking. The η -reachability condition is studied for a manifold based on the tracking error. The methodology of inserting integrators at the input of the linear controller is also described, and the proposed control scheme is applied to a 2DOF Helicopter.
- b) **paper:** Optimum Decentralized Control of a CMG with Minimum Variance using Particle Swarm Optimization (TORIUMI; ANGÉLICO, 2019a);

- **conference:** 18th European Control Conference (ECC 2019);
 - **date/place:** June 25-28, 2019, Naples, Italy;
 - **link:** <https://ieeexplore.ieee.org/document/8796001>;
 - **doi:** 10.23919/ECC.2019.8796001;
 - **abstract:** This paper considers the application of Particle Swarm Optimization (PSO) to decentralized control of a Control Moment Gyroscope (CMG). The PSO is used to obtain optimized gains for a decentralized PID controller. A cost function with ITAE index combined with the control signal variance is proposed for the system. A simple velocity feedback is proposed for damping resonant poles prior to the decentralized control. Simulation and practical results are presented in order to validate the methodology.
- c) **article:** On Guaranteeing Convergence of Discrete LQG/LTR When Augmenting it With Forward PI Controllers (ANGÉLICO et al., 2017);
- **journal:** IEEE Access (Qualis A2, ENG. IV);
 - **submission date:** Aug. 21, 2017 (accepted Oct. 17, 2017, published online Oct. 30, 2017, and printed in Volume 5, Dec. 22, 2017);
 - **link:** <https://ieeexplore.ieee.org/document/8089725>;
 - **doi:** 10.1109/ACCESS.2017.2768160;
 - **abstract:** Using the loop transfer recovery (LTR) method to recover the linear quadratic Gaussian (LQG) robustness properties is a well-established procedure, as well as augmenting the system with integrators at the plant input to deal with steady-state error. However, when using the discrete version of the LQG/LTR controller, simply using integrators discretized by the forward Euler method does not guarantee recovery convergence. This paper presents a solution: augmenting the system with a PI controller. A control moment gyroscope is used to apply this technique, and its modeling process is showed, along with its linearization and discretization. Particularly, it presents a resonance due to nutation frequency, which is damped in an inner loop prior to the robust control design by simple velocity feedback. Particle swarm optimization is applied aiming to shape the target open loop and to guarantee set point, disturbance and measurement noise robustness. At last, real experiments are conducted to corroborate the presented method.

- d) **article:** Model-based Fault Detection and Isolation Filter for Markovian Jump Linear Systems applied to a Control Moment Gyroscope;
- **journal:** European Journal of Control (Qualis A2, ENG. IV);
 - **submission date:** Feb. 19, 2020 (currently in peer-review process);
 - **abstract:** We study a fault detection problem in the context of Markovian Jump Linear Systems (MJLS). There are two major contributions in this paper. The first one is the design of a mixed H_∞/H_- Fault Detection Filter (FDF) under the MJLS formulation, which is obtained by solving Linear Matrices Inequalities (LMI) optimization problems. The second contribution is the implementation of the theoretical results in a Control Moment Gyroscope unit, which is extensively used as an actuator for attitude control of spacecraft and satellites. This class of problem is within the framework of MJLS, since the loss of communication among the various components of the control system could be modeled by a Markov chain, so that reliable solutions in the design of FDFs are required. From the practical implementations it is possible to observe that the proposed mixed H_∞/H_- fault detection solution outperformed the approach using only the H_∞ norm, showing that it could be a valuable alternative for the fault detection problem.

APPENDIX B – CMG INERTIA MATRICES

Due to the convention adopted herein to attach the frames to the CMG bodies, they differ from the ECP manual (ECP, 1999, p. 60), as can be seen in Figure 21.

Figure 21: CMG bodies frames adopted herein and in ECP manual

Source: (a) Author; (b) Adapted from ECP (1999, p. 60).

Thus, the inertia matrices also differs from the ECP manual ones (ECP, 1999, p. 61):

$$\begin{aligned} \mathbf{I}^A &= \begin{bmatrix} I_A & 0 & 0 \\ 0 & J_A & 0 \\ 0 & 0 & K_A \end{bmatrix}, \quad \mathbf{I}^B = \begin{bmatrix} I_B & 0 & 0 \\ 0 & J_B & 0 \\ 0 & 0 & K_B \end{bmatrix}, \\ \mathbf{I}^C &= \begin{bmatrix} I_C & 0 & 0 \\ 0 & J_C & 0 \\ 0 & 0 & K_C \end{bmatrix}, \quad \text{and} \quad \mathbf{I}^D = \begin{bmatrix} I_D & 0 & 0 \\ 0 & J_D & 0 \\ 0 & 0 & K_D \end{bmatrix}, \end{aligned} \quad (\text{B.1})$$

where $K_D = I_D$ since body D is axisymmetric.

With the change-of-basis matrices $\mathbf{T}_A$, $\mathbf{T}_B$, $\mathbf{T}_C$, and $\mathbf{T}_D$:

$$\begin{aligned} \begin{bmatrix} \hat{X}_1 \\ \hat{Y}_1 \\ \hat{Z}_1 \end{bmatrix} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}_A} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} \hat{X}_2 \\ \hat{Y}_2 \\ \hat{Z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{T}_B} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \\ \begin{bmatrix} \hat{X}_3 \\ \hat{Y}_3 \\ \hat{Z}_3 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{T}_C} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \hat{X}_4 \\ \hat{Y}_4 \\ \hat{Z}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{T}_D} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \end{aligned} \quad (\text{B.2})$$

the following change of basis (LIPSCHUTZ; LIPSON, 2009, p. 381) can be performed:

$$\mathcal{I}_1 = \mathbf{T}_A \mathbf{I}^A \mathbf{T}_A^\top, \quad \mathcal{I}_2 = \mathbf{T}_B \mathbf{I}^B \mathbf{T}_B^\top, \quad \mathcal{I}_3 = \mathbf{T}_C \mathbf{I}^C \mathbf{T}_C^\top, \quad \text{and} \quad \mathcal{I}_4 = \mathbf{T}_D \mathbf{I}^D \mathbf{T}_D^\top, \quad (\text{B.3})$$

yielding to the inertia matrices (2.14) used herein.

APPENDIX C – CMG NONLINEAR MODEL

The resulting nonlinear model are composed of the following four equations:

$$\ddot{\theta}_1 = \frac{\mathfrak{N}_1}{\mathfrak{D}}, \quad \ddot{\theta}_2 = \frac{\mathfrak{N}_2}{\mathfrak{D}}, \quad \ddot{\theta}_3 = \frac{\mathfrak{N}_3}{\mathcal{Z}\mathfrak{D}}, \quad \text{and} \quad \ddot{\theta}_4 = \frac{\mathfrak{N}_4}{J_D\mathfrak{D}},$$

being:

$$\begin{aligned} \mathfrak{D} &= \mathcal{V} (\mathcal{F} + \mathcal{E}c_{\theta_3}^2) + \mathcal{F}\mathcal{G}c_{\theta_2}^2 + \mathcal{E}\mathcal{H}c_{\theta_2}^2c_{\theta_3}^2, \\ \mathfrak{N}_1 &= -\mathcal{F} \left(T_1 - \mu_4\dot{\theta}_4 \right) c_{\theta_2}s_{\theta_3} + (\mathcal{F} + \mathcal{E}c_{\theta_3}^2) \left(T_2 - \mu_3\dot{\theta}_3 \right) s_{\theta_2} \\ &\quad + \mathcal{E}\mu_2\dot{\theta}_2c_{\theta_2}c_{\theta_3}s_{\theta_3} - (\mathcal{F} + \mathcal{E}c_{\theta_3}^2) \mu_1\dot{\theta}_1 + [\mathcal{E}(\mathcal{Y} + \mathcal{W}) + J_D\mathcal{W}] \dot{\theta}_1^2c_{\theta_2}^2c_{\theta_3}s_{\theta_2}s_{\theta_3} \\ &\quad + [(\mathcal{Y} + \mathcal{G})(\mathcal{F} + \mathcal{E}c_{\theta_3}^2) + J_D\mathcal{W}c_{\theta_3}^2] \dot{\theta}_1\dot{\theta}_2c_{\theta_2}s_{\theta_2} \\ &\quad - [\mathcal{E}(2\mathcal{W} + \mathcal{Y} - \mathcal{G}) + J_D\mathcal{W}] \left(\dot{\theta}_1c_{\theta_2}s_{\theta_3} + \dot{\theta}_2c_{\theta_3} \right) \dot{\theta}_3c_{\theta_2}c_{\theta_3} \\ &\quad - \mathcal{F}(\mathcal{Y} - \mathcal{G}) \dot{\theta}_2\dot{\theta}_3c_{\theta_2} + J_D\mathcal{W} \left(\dot{\theta}_1s_{\theta_2} - \dot{\theta}_3 \right) \dot{\theta}_4c_{\theta_2}c_{\theta_3}, \\ \mathfrak{N}_2 &= -[\mathcal{V} + (\mathcal{Y} + \mathcal{Z})c_{\theta_2}^2] \left(T_1 - \mu_4\dot{\theta}_4 \right) c_{\theta_3} - \mathcal{E} \left(T_2 - \mu_3\dot{\theta}_3 \right) c_{\theta_2}c_{\theta_3}s_{\theta_2}s_{\theta_3} \\ &\quad - [\mathcal{V} + (\mathcal{G} - \mathcal{E}c_{\theta_3}^2)c_{\theta_2}^2] \mu_2\dot{\theta}_2 + \mathcal{E}\mu_1\dot{\theta}_1c_{\theta_2}c_{\theta_3}s_{\theta_3} - \mathcal{V}(\mathcal{Y} - \mathcal{X} + \mathcal{X}c_{\theta_3}^2) \dot{\theta}_1^2c_{\theta_2}s_{\theta_2} \\ &\quad - \mathcal{G}(\mathcal{Y} - \mathcal{X}) \dot{\theta}_1^2c_{\theta_2}^3s_{\theta_2} + [\mathcal{E}(\mathcal{Y} + \mathcal{G}) + J_D\mathcal{G}] \dot{\theta}_1^2c_{\theta_2}^3c_{\theta_3}^2s_{\theta_2} \\ &\quad - [J_D(\mathcal{V} + \mathcal{G}c_{\theta_2}^2) + \mathcal{E}(\mathcal{Y} + \mathcal{G})c_{\theta_2}^2] \dot{\theta}_1\dot{\theta}_2c_{\theta_3}s_{\theta_2}s_{\theta_3} \\ &\quad - \mathcal{V}[\mathcal{X} + \mathcal{Z} + (2\mathcal{E} + J_D)c_{\theta_3}^2] \dot{\theta}_1\dot{\theta}_3c_{\theta_2} - \mathcal{G}(\mathcal{X} + \mathcal{Z}) \dot{\theta}_1\dot{\theta}_3c_{\theta_2}^3 \\ &\quad - [\mathcal{E}(\mathcal{Y} + \mathcal{G}) + J_D\mathcal{G}] \dot{\theta}_1\dot{\theta}_3c_{\theta_2}^3c_{\theta_3}^2 + \mathcal{V}(2\mathcal{E} + J_D) \dot{\theta}_2\dot{\theta}_3c_{\theta_3}s_{\theta_3} \\ &\quad + [\mathcal{E}(\mathcal{Y} + \mathcal{G}) + J_D\mathcal{G}] \dot{\theta}_2\dot{\theta}_3c_{\theta_2}^2c_{\theta_3}s_{\theta_3} - J_D(\mathcal{V} + \mathcal{G}c_{\theta_2}^2) \left(\dot{\theta}_1s_{\theta_2} - \dot{\theta}_3 \right) \dot{\theta}_4s_{\theta_3}, \end{aligned}$$

$$\begin{aligned}
\mathfrak{N}_3 = & -\mathcal{Z}\mathcal{F} \left(T_1 - \mu_4 \dot{\theta}_4 \right) c_{\theta_2} s_{\theta_2} s_{\theta_3} + \mathcal{Z}\mathcal{E} \mu_2 \dot{\theta}_2 c_{\theta_2} c_{\theta_3} s_{\theta_2} s_{\theta_3} - \mathcal{Z} \left(\mathcal{F} + \mathcal{E} c_{\theta_3}^2 \right) \mu_1 \dot{\theta}_1 s_{\theta_2} \\
& + \left[(\mathcal{V} + \mathcal{Z}) \left(\mathcal{F} + \mathcal{E} c_{\theta_3}^2 \right) - \mathcal{F} (\mathcal{Z} - \mathcal{G}) c_{\theta_2}^2 - \mathcal{E} (\mathcal{Z} - \mathcal{H}) c_{\theta_2}^2 c_{\theta_3}^2 \right] \left(T_2 - \mu_3 \dot{\theta}_3 \right) \\
& + \mathcal{E} \left[\mathcal{Z} (\mathcal{Y} + \mathcal{F} + \mathcal{E}) + \mathcal{V} (\mathcal{F} - \mathcal{X} c_{\theta_3}^2) \right] \dot{\theta}_1^2 c_{\theta_2}^2 c_{\theta_3} s_{\theta_3} + J_D (\mathcal{V}\mathcal{F} + \mathcal{Z}\mathcal{W}) \dot{\theta}_1^2 c_{\theta_2}^2 c_{\theta_3} s_{\theta_3} \\
& - \mathcal{E} \left[(\mathcal{F} - \mathcal{Z}) (\mathcal{Z} - \mathcal{G}) + \mathcal{X}\mathcal{H} c_{\theta_3}^2 \right] \dot{\theta}_1^2 c_{\theta_2}^4 c_{\theta_3} s_{\theta_3} \\
& + J_D [\mathcal{G} (J_B + K_C) - I_C \mathcal{W} + I_D \mathcal{H}] \dot{\theta}_1^2 c_{\theta_2}^4 c_{\theta_3} s_{\theta_3} \\
& + \left[\mathcal{Z} (\mathcal{V} + \mathcal{Y} + \mathcal{G}) (\mathcal{F} + \mathcal{E} c_{\theta_3}^2) + \mathcal{V}\mathcal{F} (\mathcal{X} + 2\mathcal{E} c_{\theta_3}^2) + J_D (\mathcal{Z}\mathcal{W} + 2\mathcal{V}\mathcal{F}) c_{\theta_3}^2 \right] \dot{\theta}_1 \dot{\theta}_2 c_{\theta_2} \\
& - \left[\mathcal{F} (\mathcal{Y} - \mathcal{G}) (\mathcal{Z} - \mathcal{G}) + J_D \mathcal{F}\mathcal{G} + \mathcal{E} (\mathcal{Z}\mathcal{Y} + 2\mathcal{X}\mathcal{H} c_{\theta_3}^2) c_{\theta_3}^2 \right] \dot{\theta}_1 \dot{\theta}_2 c_{\theta_2}^3 \\
& - \mathcal{V}\mathcal{X}\mathcal{E} \dot{\theta}_1 \dot{\theta}_2 c_{\theta_2} c_{2\theta_3} c_{\theta_3}^2 + (\mathcal{E} + J_D) [3\mathcal{F}\mathcal{G} - \mathcal{W} (2\mathcal{Z} + \mathcal{Y})] \dot{\theta}_1 \dot{\theta}_2 c_{\theta_2}^3 c_{\theta_3}^2 \\
& - \mathcal{Z} [\mathcal{W} (\mathcal{E} + J_D) + \mathcal{E} (\mathcal{F} - \mathcal{Z})] \left(\dot{\theta}_1 c_{\theta_2} s_{\theta_3} + \dot{\theta}_2 c_{\theta_3} \right) \dot{\theta}_3 c_{\theta_2} c_{\theta_3} s_{\theta_2} \\
& + \mathcal{X} \left[\mathcal{V} (\mathcal{F} + \mathcal{E} c_{\theta_3}^2) + (\mathcal{F}\mathcal{G} + \mathcal{E}\mathcal{H} c_{\theta_3}^2) c_{\theta_2}^2 \right] \dot{\theta}_2^2 c_{\theta_3} s_{\theta_3} - \mathcal{Z}\mathcal{F} (\mathcal{Y} - \mathcal{G}) \dot{\theta}_2 \dot{\theta}_3 c_{\theta_2} s_{\theta_2} \\
& + J_D \left[\mathcal{V} (\mathcal{F} + \mathcal{E} c_{\theta_3}^2) + \mathcal{Z}\mathcal{W} \right] \dot{\theta}_1 \dot{\theta}_4 c_{\theta_2} c_{\theta_3} \\
& + J_D [\mathcal{G} (J_B + K_C) - I_C \mathcal{W} + I_D \mathcal{H} + \mathcal{E}\mathcal{H} c_{\theta_3}^2] \dot{\theta}_1 \dot{\theta}_4 c_{\theta_2}^3 c_{\theta_3} \\
& - J_D \left[\mathcal{V} (\mathcal{W} - \mathcal{E} s_{\theta_3}^2) + (\mathcal{F}\mathcal{G} + \mathcal{E}\mathcal{H} c_{\theta_3}^2) c_{\theta_2}^2 \right] \dot{\theta}_2 \dot{\theta}_4 s_{\theta_3} - J_D \mathcal{Z}\mathcal{W} \dot{\theta}_3 \dot{\theta}_4 c_{\theta_2} c_{\theta_3} s_{\theta_2}, \\
\mathfrak{N}_4 = & \left[\mathcal{V} (\mathcal{F} - \mathcal{X} c_{\theta_3}^2) + \mathcal{F} (\mathcal{G} + J_D) c_{\theta_2}^2 - \mathcal{X}\mathcal{H} c_{\theta_2}^2 c_{\theta_3}^2 \right] \left(T_1 - \mu_4 \dot{\theta}_4 \right) \\
& - J_D \mathcal{F} \left(T_2 - \mu_3 \dot{\theta}_3 \right) c_{\theta_2} s_{\theta_2} s_{\theta_3} + J_D \left[\mathcal{V} + (\mathcal{Y} + \mathcal{Z}) c_{\theta_2}^2 \right] \mu_2 \dot{\theta}_2 c_{\theta_3} + J_D \mathcal{F} \mu_1 \dot{\theta}_1 c_{\theta_2} s_{\theta_3} \\
& + J_D \mathcal{V} (\mathcal{Y} - \mathcal{X} + \mathcal{X} c_{\theta_3}^2) \dot{\theta}_1^2 c_{\theta_2} c_{\theta_3} s_{\theta_2} \\
& + J_D \left[(\mathcal{F} - \mathcal{Z}) (\mathcal{Y} + \mathcal{Z}) + \mathcal{H} (\mathcal{G} + J_D + \mathcal{X} c_{\theta_3}^2) \right] \dot{\theta}_1^2 c_{\theta_2}^3 c_{\theta_3} s_{\theta_2} \\
& + J_D \left[\mathcal{V} (\mathcal{F} - \mathcal{X} c_{\theta_3}^2) - (\mathcal{Y}\mathcal{F} + \mathcal{X}\mathcal{H} c_{\theta_3}^2) c_{\theta_2}^2 \right] \dot{\theta}_1 \dot{\theta}_2 s_{\theta_2} s_{\theta_3} \\
& - J_D \mathcal{V} (\mathcal{F} - \mathcal{X} - \mathcal{Z} + \mathcal{X} c_{\theta_3}^2) \dot{\theta}_1 \dot{\theta}_3 c_{\theta_2} c_{\theta_3} \\
& - J_D \left[(\mathcal{F} - \mathcal{Z}) (\mathcal{Y} + \mathcal{Z}) - \mathcal{X}\mathcal{H} s_{\theta_3}^2 \right] \dot{\theta}_1 \dot{\theta}_3 c_{\theta_2}^3 c_{\theta_3} \\
& + J_D \left[\mathcal{V} (\mathcal{F} + \mathcal{X} c_{\theta_3}^2) + (\mathcal{Y}\mathcal{F} + \mathcal{X}\mathcal{H} c_{\theta_3}^2) c_{\theta_2}^2 \right] \dot{\theta}_2 \dot{\theta}_3 s_{\theta_3} \\
& + J_D^2 (\mathcal{V} + \mathcal{H} c_{\theta_2}^2) \left(\dot{\theta}_1 s_{\theta_2} - \dot{\theta}_3 \right) \dot{\theta}_4 c_{\theta_3} s_{\theta_3},
\end{aligned}$$

with:

$$\mathcal{X} = K_C - J_C - J_D + I_D, \quad \mathcal{Y} = K_B - I_B + K_C - I_C,$$

$$\mathcal{Z} = I_C + I_D, \quad \mathcal{W} = J_B + J_C, \quad \mathcal{V} = K_A + I_B,$$

$$\mathcal{E} = J_C - K_C - I_D, \quad \mathcal{F} = J_B + K_C + I_D, \quad \mathcal{G} = K_B - I_B + J_C, \quad \mathcal{H} = K_B - J_B - I_B.$$

Remark 13. By analyzing the denominator $\mathfrak{D}$ and considering the SMI values in Table 5:

$$\mathfrak{D} = \mathcal{V} (\mathcal{F} + \mathcal{E} c_{\theta_3}^2) + \mathcal{F}\mathcal{G} c_{\theta_2}^2 + \mathcal{E}\mathcal{H} c_{\theta_2}^2 c_{\theta_3}^2 \approx (2343.84 c_{\theta_2}^2 - 473.86 c_{\theta_3}^2 + 4199.38) 10^{-6},$$

it follows that $\mathfrak{D} > 0$, $\forall (\theta_2(t), \theta_3(t))$, i.e., $\mathfrak{D}$ is strictly positive during CMG operation.

APPENDIX D – MATROSOV’S THEOREM

As this thesis deals with the tracking task problem, the desired trajectory may be time-varying. If it evoked a regulation problem, where the desired position(s) is(are) a constant value(s), then it would be possible to invoke the LaSalle’s invariance principle (KHALIL; GRIZZLE, 2002) to analyze the stability of the tracking error dynamics of the closed-loop system. But, in the CMG case, the desired trajectory is actually time-varying, yielding to a non-autonomous tracking error dynamics. Thus, one option to analyze its stability is to invoke the Matrosov’s theorem (MATROSOV, 1962, p. 1339) instead (as done in some applications, like Nicosia and Tomei (1995), Zhang et al. (2008), and Kristiansen and Nicklasson (2009), to name a few).

There are some works in the literature that present it with slightly different notations, as Rouché (1968, p. 305), Fossen (2002, p. 521), Maghenem, Loría and Panteley (2017, p. 3), and Paden and Panja (1988, p. 1708-1709), being the latter adopted herein.

But before presenting it, Paden and Panja (1988) define the following notation:

- a) $\mathbb{R}$: real numbers;
- b) $\mathbb{R}_+$: non-negative real numbers;
- c) $\mathcal{C}^n$: continuously differentiable n times;
- d) $d(x, \Omega)$: distance from the point x to the set Ω ;
- e) B_α : open ball of radius α centered at the origin;
- f) $\bar{S}$: closure of the set S ;
- g) a function of “class $\mathcal{K}$ ”: a continuous function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\alpha(p)$ is strictly increasing and $\alpha(0) = 0$.

Then, they rely on Matrosov’s theorem concerning the differential equation $\dot{x} = f(t, x)$ (defined implicitly by the tracking error dynamics equation) with an equilibrium point at zero at time t_0 .

Thus, the following Matrosov's theorem is stated by Paden and Panja (1988, p. 1708-1709)

Theorem 3. *Let $\Omega \in \mathbb{R}^n$ be an open connected region in $\mathbb{R}^n$ containing the origin. If there exist two $\mathcal{C}^1$ functions $V : [t_0, \infty) \times \Omega \rightarrow \mathbb{R}$, $W : [t_0, \infty) \times \Omega \rightarrow \mathbb{R}$; a $\mathcal{C}^0$ function $V^* : \Omega \rightarrow \mathbb{R}$; three functions a, b, c of class $\mathcal{K}$ such that, for every $(x, t) \in [t_0, \infty) \times \Omega$,*

- (1) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$
- (2) $\dot{V}(t, x) \leq V^*(x) \leq 0$. Define $E \triangleq \{x \in \Omega \mid V^*(x) = 0\}$.
- (3) $|W(t, x)|$ is bounded
- (4) $\max \left(d(x, E), |\dot{W}(t, x)| \right) \geq c(\|x\|)$
- (5) $\|f(t, x)\|$ is bounded.

Choosing $\alpha > 0$ such that $\bar{B}_\alpha \subset \Omega$, define for all $t \in [t_0, \infty)$

$$V_{t,\alpha}^{-1} = \{x \in \Omega : V(t, x) \leq a(\alpha)\}.$$

Then

- (a) For all $x_0 \in V_{t_0,\alpha}^{-1}$, $x(t)$ tends to zero uniformly in t_0, x_0 as t tends to infinity.
- (b) The origin is uniformly asymptotically stable.

Proof. See Rouche, Habets and Laloy (1977). □

[...]

Condition (4) of the theorem is easier for us to verify if we use the following lemma.

Lemma 1. *Condition (4) of the theorem is satisfied if conditions (4)' below are satisfied.*

- 4'(a) $\dot{W}(x, t)$ is continuous in both arguments and depends on time in the following way, $\dot{W}(x, t) = g(x, \beta(t))$ where g is continuous in both of its arguments. $\beta(t)$ is also continuous and its image lies in a bounded set K_1 . (For simplicity, we simply say that $\dot{W}(x, t)$ depends on time continuously through a bounded function.)
- 4'(b) There exists a class $\mathcal{K}$ function, k , such that $|\dot{W}(x, t)| \geq k(\|x\|) \forall x \in E$ and $t \geq t_0$.

Proof. Since $\beta(t) \subset K_1$ and $\dot{W}$ is continuous in β for all $t \geq t_0$ we have that $Y(x) \triangleq \min_{\beta \in K_1} |g(x, \beta)|$ exists and is continuous. Now consider

$$c_1(\alpha) \triangleq \min_{\eta \in \Omega, \|\eta\| \geq \alpha} \max(d(\eta, E), |Y(\eta)|) \quad (\text{D.1})$$

which is defined for all $\alpha = \|x\|$, x in Ω . The set over which the minimization occurs is decreasing with increasing α , so $c_1(\alpha)$ is non-decreasing with respect to α . Observe that $c_1(\alpha) \geq 0$ and that $c_1(\alpha) = 0$ implies that the minimum in (D.1) is attained for $\eta = 0$ by condition 4'(b).

Thus, in addition to being non-decreasing, $c_1(0) = 0$, and $c_1(p) > 0$ for all $p > 0$. To show condition (4) define the *strictly* increasing function $c(\cdot)$ by

$$c(p) \triangleq \int_0^p c_1(\sigma) \exp [-(p - \sigma)] d\sigma$$

It follows from the properties of c_1 that $c(\alpha) \leq c_1(\alpha)$ and $c(p)$ satisfies the conditions of the definition of a function of class $\mathcal{K}$. Thus, condition (4) holds:

$$\max \left(d(x, E), |\dot{W}(t, x)| \right) \geq c(\|x\|)$$

□

APPENDIX E – CMG PARAMETERS

The CMG parameters are the SMI of each body and the VFC of each axis, beyond the encoder and motor gains. Their values are described in Table 5, which some of them (I_B , J_B , K_B , K_C , I_D and J_D) are given in plant manual (ECP, 1999, p. 80), and others (K_A , I_C and J_C) are obtained through procedures described in it (ECP, 1999, p. 71-74), which development is presented in Toriumi (2016, p. 90-98).

Table 5: CMG Parameter Values

Parameter	Symbol	Values	Unit
Body scalar moment of inertia (SMI)	K_A	0.0698	kg·m ²
	I_B	0.0119	
	J_B	0.0178	
	K_B	0.0297	
	I_C	0.0124	
	J_C	0.0278	
	K_C	0.0188	
	I_D	0.0148	
Axis viscous friction coefficient (VFC)	J_D	0.0273	$\frac{\text{N}\cdot\text{m}\cdot\text{s}}{\text{rad}}$
	μ_1	0.0027	
	μ_2	0.0027	
	μ_3	0.0118	
	μ_4	0.000187	
Encoder gains	k_{e_A}	81504	$\frac{\text{counts}}{\text{rad}}$
	k_{e_B}	81504	
	k_{e_C}	124256	
	k_{e_D}	33952	
Motor gains	k_{m_1}	2.1544×10^{-5}	$\frac{\text{N}\cdot\text{m}}{\text{counts}}$
	k_{m_2}	8.9355×10^{-5}	

Source: Author (except for the VFC values that were extracted from Abbas et al. (2014)).

APPENDIX F – SIMULINK MODELS

Figure 22: Simulink model for simulations of the tracking test

Source: Author.

Source: Author.

Figure 24: Simulink model for experiments of the tracking test

Source: Author.

Figure 25: Simulink model for experiments of the disturbance rejection test

Source: Author.

Figure 26: FL nonlinear controller block from simulink models

Source: Author.

Figure 27: PB nonlinear controller block from simulink models

Source: Author.

Figure 28: SM nonlinear controller block from simulink models

Source: Author.