ARTHUR CASTELLO BRANCO DE OLIVEIRA

Dynamic Bipedal Locomotion Based on Hybrid Zero
Dynamics Control

São Paulo
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Concentration area:

3139 - System engineering

Advisor:

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São Paulo
2019

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São Paulo, _____ de _____ de _____

Assinatura do autor: _____

Assinatura do orientador: _____

Catálogo-na-publicação

Oliveira, Arthur

Dynamic Bipedal Locomotion Based on Hybrid Zero Dynamics Control /
A. Oliveira -- versão corr. -- São Paulo, 2019.

108 p.

Dissertação (Mestrado) - Escola Politécnica da Universidade de São Paulo. Departamento de Engenharia de Telecomunicações e Controle.

1.Robótica 2.Sistemas Dinâmicos 3.Controle (Teoria de Sistemas de Controle) I.Universidade de São Paulo. Escola Politécnica. Departamento de Engenharia de Telecomunicações e Controle II.t.

ACKNOWLEDGMENTS

Escrever este agradecimento me dá uma sensação muito próxima da de dividir meu bolo de aniversário: fico extremamente envergonhado de ter essa responsabilidade; corto os pedaços cada um de um tamanho; deformedo completamente o que um dia foi um belo bolo confeitado; quero muito que todos gostem, mas perco a conta de quem já recebeu seu pedaço no meio do processo. A diferença é que no caso deste agradecimento não vai haver minha mãe para ficar com pena de mim e tomar a espátula para arrumar a bagunça que eu fiz.

E como todo bolo de aniversário, o primeiro pedaço vai para mim! Mesmo que pareça um pouco narcisístico, eu não posso deixar de enfatizar o quão estressante, frustrante, emocionante, divertido e realizador foi realizar este projeto. Mesmo que eu não tenha reconhecido em nenhum momento a vontade de desistir, a disciplina para continuar seguindo, apesar das adversidades, e mesmo que apenas com melhorias incrementais, foi algo que precisei aperfeiçoar dentro de mim. O caminho foi árduo, longo, mas recompensador, como toda boa aventura, e acabo ela mais satisfeito com quem sou do que comecei.

Seguindo o protocolo, devo três grandes pedaços à minha família: minha mãe, Solmara; meu pai, Renato; e meu irmão, Murilo. Eles são o motivo de eu escolher escrever estes agradecimentos em português, mas a inconsistência linguística com o resto do texto é completamente irrelevante comparada à função fundamental que eles tiveram em permitir que o texto exista em primeiro lugar. Para além do sustento financeiro que eles forneceram (que não foi, por medida alguma, trivial), fui abençoado com uma família que é meu seguro de vida. Eu sei que, independentemente dos resultados das minhas escolhas, eu tenho um píer sólido onde posso atracar meu barquinho e suportar as tempestades, e não posso enfatizar o suficiente como isso foi essencial para eu estar onde estou hoje. Por eu sempre ter alguém pra me receber, independente do quão errado tudo desse, eu pude tomar as escolhas que quis tomar, sem sequer olhar o risco associado.

É fortuito que não haja limite de páginas nestes agradecimentos, pois se este fosse um bolo convencional acabaria com ele apenas na gratidão devida nos últimos e próximos três pedaços. Se uma família não fosse fortuna suficiente, eu tenho duas! Minha tia, Solimar, meu tio, Luis Antônio, e meu primo, Luís Henrique, foram o lar longe do lar para este menino do interior que resolveu, depois de dezoito anos morando em uma cidade pequena no interior do Paraná, enfrentar o caos da maior cidade da América do Sul. A mudança de ritmo é esmagadora e por mais que eu tenha aprendido a amar o caos desta cidade, não consigo imaginar a crise de ansiedade que seria desbrava-la sozinho na decadência da puberdade. Se meus pais são um seguro emocional que requerem um sinistro de doze horas de, meus tios são como alguns meses de salários mantidos no tesouro direto, com uma liquidez de uma hora de metrô até a casa deles.

Tradicionalmente seguiriam em ordem família, amigos e trabalho, mas as coisas ficam confusas quando seus amigos vêm do trabalho e são como família para você, então seguindo em desordem, eu devo um pedaço de bolo com morango em cima (dado que ele não gosta de coisa muito doce) ao meu orientador, Bruno. Antes de tomar a decisão de seguir carreira acadêmica ouvi todo o tipo de conselho, mas a ideia de que professores são um

balão de ego e que viam em seus orientandos apenas publicações em potencial era, em certo grau, onipresente. Está certo que o ser humano parece pré-disposto a guardar e reverberar apenas os piores aspectos das coisas, mas não só nenhum professor ao meu redor atendeu a essa expectativa, o Bruno talvez seja a completa antítese disso. O compromisso moral que ele tem com seus alunos é evidente nas suas ações, não só ele se importa, profundamente, com os trabalhos realizados (ao ponto de ser fisicamente perceptível um incomodo quando ele acha que não está colaborando o suficiente) como ele não hesita em se comprometer com seu projeto, dedicando tempo e dinheiro pessoais para fazer tudo a seu alcance para garantir o êxito do projeto. Eu realmente fui muito afortunado em tê-lo como orientador para este projeto.

E talvez uma das principais vantagens de ser orientado do Bruno seja o LCA. Outro preconceito com o qual me deparei sobre a vida acadêmica é a solidão, você não vai trabalhar com pessoas e não vai desenvolver sua capacidade para relacionamento interpessoal. Felizmente isso também se provou falso para mim. Gabriel, Fábio, Mateus (sem h), Fernando, Rafael, Genário, Giovanni, Samuel, Leonardo, André... São muitas as pessoas com quem tive a oportunidade de conviver nestes dois anos e mesmo que algumas prevaleçam sobre outras, todas me ajudaram a crescer como indivíduo e merecem meu agradecimento, em especial os quatro primeiros que me adotaram quando me descobri um completo iletrado em teoria de controle e me guiaram pela mão até eu ser capaz de caminhar sozinho.

E se estamos falando de quem me ensinou controle, acho que é indiscutível dedicar um parágrafo apenas para o Guilherme (a.k.a. $\frac{d}{dt}cabin$). Eu brinco dizendo que esta é a pior pessoa que você pode conhecer na sua vida, ele é mais inteligente que você, mais bonito e, para piorar, mais simpático. Ele me ensinou praticamente todo o ferramental que eu precisei para entender o método desse livro e foi verdadeiramente essencial para este trabalho acontecer.

E se estamos falando de amizades, preciso agradecer à minha terceira trindade, Isabela, Vitor e Ivan (com menção honrosa à Katarina). Parte de mim quer escrever este parágrafo apenas com piadas internas e onomatopeias específicas que só nós entenderíamos, mas em respeito a todos os outros (e à norma culta da língua portuguesa) achei melhor não. Mesmo assim, vocês foram pra mim a valorização do presente enquanto eu trabalhava para o futuro. Perdi a conta de quantas semanas eu aguentei apenas com a expectativa de sairmos para não fazer nada específico no final de semana, ou quantas tardes densas e cansativas foram salvas por um de vocês aparecerem no laboratório.

Finalmente, não sei se é indecoroso agradecer à banca, mas o decoro se foi neste agradecimento a alguns parágrafos. Os professores Paulo, Arturo e Carlos foram mais que meus avaliadores, cada um deles participou da minha jornada no mestrado, seja diretamente no projeto ou me guiando pelos costumes da vida acadêmica.

Este seria o momento em que todos estariam me olhando querendo logo seu pedaço do bolo. Foge à decência gastar o espaço para agradecer justamente a todos que merecem como merecem, por isso me ative àqueles que não podia deixar de mencionar. Para finalizar gostaria de mencionar alguns nomes: Nayara, Bruno, Nayana, Hélder, Neuza, Jéssica, Gustavo, Gabriel, Nayara, Bruno, Fernanda e Raiane. Obrigado a todos que me acompanharam até este momento.

*"The purpose of life is finding the
largest burden that you can bear and
bearing it."*

-Jordan B. Peterson

RESUMO

Esta tese apresenta um método alternativo de síntese de marcha bípede 3D usando controladores independentes projetados para os planos de movimento frontal e sagital. O uso de restrições virtuais no projeto de uma marcha estável para o plano frontal é completamente desenvolvido e estudado neste trabalho. A marcha resultante é simulada e os resultados, apesar de ainda não definitivos, são promissores.

Palavras-Chave – Robô Bípede, Dinâmica Zero, Sistema Híbrido, Ciclo Limite, Restrições Virtuais.

ABSTRACT

This work presents an alternative method for 3D bipedal gait design using independent controllers for the plane of motion frontal and sagittal. The use of virtual constraints to design a stable gait for the frontal system is fully developed and studied in this work and the resulting gait simulated. The results, although not definitive, are promising.

Keywords – Bipedal Robot, Zero Dynamics, Hybrid System, Limit Cycle, Virtual Constraints.

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LIST OF ABBREVIATIONS

ZMP	Zero moment point
FRI	Foot rotation indicator
CoM	Center of mass
CoP	Center of pressure
3D	Three dimensional
HR	Hypothesis about the 3D robot
HR.S	Hypothesis about the sagittal planar robot
HR.F	Hypothesis about the frontal planar robot
HG.S	Hypothesis about the sagittal gait
HG.F	Hypothesis about the frontal gait
HI	Hypothesis about the impact
HSB	Hybrid System's Hypothesis
DOF	Degrees of freedom
LAC	Laboratório de Automação e Controle
LCA	Laboratório de Controle Aplicado
USP	Universidade de São Paulo

LIST OF SYMBOLS

f_i	Nonlinear state space function of continuous phase i of a hybrid system
g_i	Nonlinear state space input function of continuous phase i of a hybrid system
f_{cli}	Closed-loop nonlinear state space function of continuous phase i of a hybrid system
x_i	State vector of phase i of a hybrid system
u_i	Input vector of phase i of a hybrid system
q_i	Vector of relative configuration variables of phase i of a hybrid system (when applicable)
$\mathcal{X}_i$	State manifold of phase i of a hybrid system
n_i	Dimension of the state manifold of phase i of a hybrid system
$\mathcal{U}_i$	Input manifold of phase i of a hybrid system
m_i	Dimension of the input manifold of phase i of a hybrid system
$\mathcal{Q}_i$	Configuration set of phase i of a hybrid system (when applicable)
$\mathcal{S}_i^j$	Transition set from phase i to phase j
Δ_i^j	Transition function from phase i to phase j
H_i^j	Transition equality condition from phase i to phase j
G_i^j	Transition inequality condition from phase i to phase j
$\mathcal{F}_i$	Flow of phase i
Σ_i	Quadruple representing the phase i of a nonlinear hybrid system
D_i	Inertia matrix of phase i of a hybrid system
V_i	Coriolis and centripetal forces vector of phase i of a hybrid system
G_i	Gravity vector of phase i of a hybrid system
B_i	Input to generalized torques matrix of phase i of a hybrid system
V	Potential energy
K	Kinetic energy

$\mathcal{L}$	Lagrangian
x_1	Horizontal position of the stance leg ankle
y_1	Vertical position of the stance leg ankle
θ_1	Absolute orientation of the stance foot
F_1^h	Horizontal component of the stance foot contact forces
F_1^v	Vertical component of the stance foot contact forces
τ_1	Stance foot contact torque
E_2	Unpinned swing foot position and orientation Jacobian
F_1	External forces and torque acting at the stance foot
F_2	External forces and torque acting at the swing foot
$\bar{\Delta}_I$	Impact function before switching leg roles
Δ_I	Impact function after switching leg roles
a_i	i -th link length
α_i	i -th link twist
d_i	i -th link offset
θ_i	i -th joint angle
q_0	Controlled relative configuration variables of a underactuated system
α	Uncontrolled absolute configuration variable of the sagittal system
β	Uncontrolled absolute configuration variable of the frontal system
h_d	Virtual constraint function
h	Output function
$\mathcal{N}$	Phases set of a hybrid system
$\mathcal{T}$	Switching times set of a hybrid system's solution
φ_i	Solution of the i -th continuous phase of a hybrid system
φ	Solution of a hybrid system
$\mathcal{Z}$	Hybrid zero dynamics manifold of a hybrid system
a	Sagittal Beziér polynomial parameters matrix

b	Frontal Beziér polynomial parameters matrix
ξ	Zero dynamics' generalized variable
σ_i	Robot's angular momentum about the point p_i
κ	Zero dynamics' state space functions

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1 INTRODUCTION

The interest in artificial humanoids arose naturally along human history, even before the very concept of robot, as was the case with the homunculus, the golem, the gargoyle, etc. This idea accompanied men until modern Hollywood movies as villains, friends or useful tools.

In a more pragmatic view, a robust bipedal walker could access environments originally designed for human work or hazardous areas where wheeled robots have limited maneuverability. Further on, a humanoid robot could even automatize some services in a more organic way for costumers, which are used to have a human serving them.

It is reasonable to say, then, that the subject of humanoid robots is both interesting and useful, and among different research groups it was divided into simpler problems to be solved. Such are “walking pattern generation”, “path planning”, “environment interaction”, “human interaction”, and others directly related to the problem of creating a successful humanoid robot.

On the subject of walking pattern generation — which is the focus of this work — the problem consists in realizing a stable gait for the system subject to unilateral constraints inherent to the bipedal walking problem in a way so that it is possible to integrate the gait algorithm to the rest of the robot.

This work adopts a very useful (and known in the literature) framework, the use of virtual constraints and hybrid zero dynamics, to propose a design method generating 3D, stable and energy efficient bipedal gaits.

This chapter presents an overview of this subject and how this work fits in this broad research area, serving as both resume and introduction for the rest of the text.

1.1 Literature Review

Bipedal locomotion is one of the big open problems in control theory and, because of that, there are many researches being conducted in this topic.

One of the first solutions proposed to the problem of bipedal locomotion is the use of the Zero Moment Point (ZMP) as a stability criterion. Presented at (VUKOBRATOVIĆ; STEPANENKO, 1972), it is defined so that if the zero moment point (or the center of pressure) is within the convex hull of the contact points with the ground, then the robot can be treated as a traditional pinned manipulator. This method became very popular in bipedal locomotion literature because of its simplicity and good results.

Despite the use of only the ZMP criterion being proved to not be enough to assure the existence of a stable walking gait in latter works, it is still used as a criterion to avoid foot rotation in many projects, two worth noting being the Honda's robot Asimo (presented in Figure 1a), as shown in (TAKENAKA, 2006), and Lola from the Technische Universität München (presented in Figure 1b), as in (BUSCHMANN, 2010).

Figure 1: Robots with ZMP criteria

Source: (a) <http://asimo.honda.com/asimo-specs/> (visited in 18/02/2018)
 (b) <https://www.amm.mw.tum.de/en/research/current-projects/humanoid-robots/>
 (visited in 18/02/2018)

In (WESTERVELT et al., 2007) it is proposed a more mathematical rigorous method using Poincaré functions and virtual constraints to analyze the system restricted to its

hybrid zero dynamics (HZD). The initial results from this method were restricted to planar robots, Mabel (Figure 2a) and Rabbit (Figure 2b) the two most notorious planar robots to use this method.

Figure 2: Robots with Limit Cycle criteria

Source: (a) <http://www.dbusiness.com/November-December-2013/Walk-of-Life/>
 (visited in 18/02/2018)
 (b) (WESTERVELT et al., 2007)

Since it was first proposed, the HZD was used in many works involving bipedal locomotion. In (AMES et al., 2014), for example, the authors extend the class of controllers that can ensure sufficiently rapid attractiveness of the zero dynamics using the concept of control Lyapunov functions (CLF). This paper compares the practical results for a planar robot and the use of CLF resulted in a less oscillatory control signal when compared to the traditional PD controller used in the literature. Furthermore, in (GALLOWAY et al., 2015) this method was even extended to comply with the actuators saturation.

On the subject of uneven terrain, the HZD method was applied in (APOSTOLOPOULOS; BUSS et al., 2016) to obtain a stable gait. In this work the authors proposed an algorithm that changes the constraints parameters in order to allow the robot to overcome irregularities in the walking surfaces.

The HZD method was also used to propose 3D walking gaits design. In (CHEVALLEREAU; GRIZZLE; SHIH, 2009) stable locomotion was obtained for an 8-DOF, 5-link 3D robot with point feet. One conclusion of this work is that, differently from the 2D case, the choice of the controlled variables changed the stability of the gait.

To also obtain 3D stable walking gait, in (HEREID et al., 2016) the authors used a partial hybrid zero dynamics and expressed the gait synthesis problem as a nonlinear programming problem. Using direct collocation, the paper presents a more scalable method

for more complex robots.

A third criterion to walking gait generation was proposed in (PRATT et al., 2006) and in (KOOLEN et al., 2012) using the viability theory applied to bipedal systems. This method presented very good practical results for both walking and fall recovery, and has Atlas, from Boston dynamics (Figure 3), as its most famous application. However, as the ZMP method, it needed a simpler model in order to do online computations.

Figure 3: Atlas

Source: <https://www.bostondynamics.com/atlas> (visited in 18/02/2018)

Even so, in (PRATT et al., 2012) this method was used to obtain stable walking gait with a real robot. In (WIEBER, 2008) and (WIEBER, 2014) the authors proposed the use of model predictive control (MPC) to obtain viability as a consequence, by minimizing the center of mass speed of the model.

Other techniques exist in bipedal locomotion literature. In (NAVEAU et al., 2017) and (BRASSEUR et al., 2015) the MPC was also used to obtain stable locomotion for bipedal systems. In (MOMBAUR et al., 2001) stable gait was obtained for an open loop system. In (SONG; ZEFRAN, 2006) the authors propose a method to generate a stable walking gait for a 3D robot with two degrees of underactuation using the concept of dynamics decomposition extended to hybrid systems with impulse effects.

1.2 Text Structure

The Bipedal Gait design technique presented in this work is very extensive and was divided in nine chapters for clarity, each one about a specific part of the method. The chapters are, in general, organized so that the theory is firstly presented for a generic system, and, then, specified for the sagittal and frontal cases.

This first chapter presents a brief introduction to the bipedal locomotion literature and an initial discussion about the gait design problem that is extended throughout the next chapters. In this chapter it is also defined several of the terms used in this work as well as its main goals.

Chapter 2 has two goals: firstly, to present the systems that will be the subject of this work, the sagittal and frontal planar models, and secondly, to formally define what is a hybrid system for the scope of this work. Finally, the generic definition presented is particularized for the two specific models that are used in this work.

After presenting the system, Chapter 3 presents the first problem of bipedal gait design, the limb coordination. The solution chosen for this problem is the use of virtual constraints, which are also presented in this chapter. The focus is to show when these constraints allow the restriction of the high DOF system to a lower DOF one.

If Chapter 3 shows when to use virtual constraints, Chapter 4 shows how. Here it is shown how the system subjected to those constraints behaves and how it facilitates the computation of a solution of the system. This chapter also presents various useful results from the restriction of the full system.

Chapter 5 presents the second problem of bipedal gait design, the stability criteria. In this chapter the formal definition of a stable gait is presented together with the chosen stability criteria, the existence of a limit cycle. The results of the previous two chapters are, then, applied in order to obtain conditions of existence and stability of a stable gait for a system restricted by virtual constraints.

With the existence conditions for a stable gait, Chapter 6 presents the problem of optimizing the actuator output for the robot's joints while respecting a series of nonlinear constraints imposed by the aforementioned conditions, as well as by the system hypothesis presented in Chapter 2 that were not assured to be respected during design. At the end of this chapter the optimization results are presented for both the sagittal and frontal planar systems.

Consider Figure 4 to better understand how the gait design works so far. The modeling of the planar system returns a nonlinear hybrid system for both the stability criteria and the limb coordination control. The latter, then, returns the adequate virtual constraints also for the stability criteria that, in its turn, computes the existence conditions for a stable gait. The cost function is also computed and the optimization algorithm runs the next round.

Figure 4: Flow of the gait design process

At this point, Chapter 7 presents the simulation results for the both planar systems. Here, the frontal results are analyzed and compared with the more consolidated sagittal results. After that, both controllers are applied to the 10 DOF 3D bipedal robot and, as presented, do not generate a stable walking gait.

With the failure to simply apply the sagittal and frontal controllers and obtain a stable walking gait, means of correcting this are considered in Chapter 8. A simple synchronizing controller is proposed and applied to the 3D robot generating stable gait.

Finally, Chapter 9 reviews the proposed methodology to design 3D stable gait, analyses the results obtained and proposes future works that could further improve them.

A final remark, every Figure in this work has its source in its caption except when the source is the author himself.

1.3 Definitions

In this section some terminology commonly used in the bipedal literature is presented in order to make it easier for the reader to understand exactly what is stated. Up until now some terms were used without any definition and should be understood as defined here. These definitions were adapted from (WESTERVELT et al., 2007), Chapter 1, and are used along this work.

Firstly, a *bipedal robot* consists in an open kinematic chain with two subchains called *legs*, usually a subchain called *torso* and optionally other subchains for the arms and neck. The legs and the torso are connected by the *hip*.

The plane that divides the robot into its left and right sides is called the *sagittal plane*, while the one that divides the system into front and back is called *frontal plane*. Lastly, the one that divides into upper and lower parts is the *transversal plane*. In Figure 5 it is possible to see the representation of each plane. A sagittal biped is a robot whose movements are restricted to the sagittal plane, that is, it can not fall sideways along the frontal plane nor turn around along the transversal plane, while a frontal biped is a robot whose movement is restricted to the frontal plane, that is, it can not fall forward or backward nor turn along the transversal plane.

During a step either one or both legs are in contact with the ground. When only one leg is on the ground, the robot is in its *swing phase*, the leg on the ground is called *stance leg* and the one in the air is the *swing leg*. For a frontal robot the swing phase is divided in *rising* and *falling* phases that are defined by the signal of the ankle rotational speed. When both legs are on the ground the robot is in its *stance phase* and the legs are named after their formal role (former stance leg and former swing leg).

Walking can then be defined as the periodic alternation between stance and swing

Figure 5: Human body planes of division

Source: <https://biologydictionary.net/sagittal-plane/> (visited in 18/02/2018)

phases with the restriction that the horizontal (for sagittal robots) or frontal (for three dimensional robots) component of the system's center of mass (CoM) is strictly increasing or stationary (for frontal robots).

Being gait the act of walking, it can be classified according to its stability. Firstly, a *stable gait* is a gait in which the period of the alternation of the swing and stance phases converges to a constant, nonzero value, that is, the steps converge to a nominal step.

In a *statically stable gait* the robot's center of mass projection to the ground plane does not leave the *support polygon* (i.e. the convex hull formed by all the contact points to the ground), while in a *quasi-statically stable gait* it is the robot's center of pressure (CoP) that remains strictly in the support polygon. Finally, a *dynamically stable gait* is a periodic gait in which the robot's CoP is in the boundary of the support polygon for at least part of the gait and yet the gait is stable.

1.4 Problem Formulation and Methodology

This work proposes an extension of the virtual constraints framework to design an energy efficient, dynamically stable, 3D walking gait for bipedal robots.

The method proposed here differs from the one presented in (CHEVALLEREAU; GRIZZLE; SHIH, 2009) in the sense that it uses two independently designed planar controllers for the sagittal and frontal dynamics in order to obtain the 3D gait.

The design process is composed, then, of two independent planar gait designs, one very well documented for the sagittal dynamics, and one much scarcer in the literature for the frontal dynamics.

The use of virtual constraints to generate a stable gait for the frontal system has some theoretical problems and fully developing this is one of the contributions of this work.

Finally the method ends by composing both controllers in order to generate a stable 3D gait. How this composition is done is also subject of analysis in this work.

1.5 Goals

The main goal of this work is to generate a stable 3D walking gait using independently designed planar gaits. In order to do that, the secondary goal of extending the hybrid zero dynamics framework for the frontal system must be achieved first.

The secondary goal can be considered achieved if a method completely analogous to the one proposed in (WESTERVELT et al., 2007) is obtained for the frontal system gait, despite the theoretical complications presented by it.

The main goal of this work will be considered achieved only if a stable walking gait is obtained for a real 10 DOF bipedal robot.

2 MODELS

In this chapter the hypotheses about the system are made explicit, the control design and simulation models are presented, and the system's hybrid nature is discussed. The main goal of this chapter is to present the bipedal system to the reader, and by the end of it the author hopes that the particularities of a bipedal system are made clearer.

2.1 System Hypotheses

The robot hypotheses define the kinematic chain that composes it, how each link interacts with each other and how the robot is constrained in relation to the environment. By hypothesis the robot:

- HR1) is composed by 7 rigid links — both feet, both legs, both thighs and the torso — each with nonzero distributed mass, connected by 6 ideal revolute joints — both ankles (2 DOFs), both knees (1 DOF) and both hips (2 DOFs) — forming a single open kinematic chain. An image of the system is presented in Figure 6;
- HR2) is actuated in each degree of freedom, including the ankles. Furthermore, joints with 2 DOFs have two orthogonal actuators, and every actuator is supposed independent;
- HR3) is 3D, with movement constrained about the world's vertical axis, that is, the robot does not make curves as it walks;
- HR4) is bipedal with two symmetric legs connected at the opposite ends of the torso joint;
- HR5) does not rotate its stance foot during any phase of the step.

These hypotheses relate to the 3D system, however in this work the controllers for the sagittal and frontal dynamics are designed independently and, therefore, models of the robot with movements restricted to these planes of motion are needed. For these models the different hypotheses are presented below.

Figure 6: 10 DOFs Bipedal Robot Kinematics

Source: Adapted from <http://www.takanishi.mech.waseda.ac.jp/top/research/wabian/wabian2_2LL/wabian2_2LL.htm>

The robot restricted to the sagittal plane:

HR1.S) is composed by 7 rigid links — both feet, both legs, both thighs and the torso — each with nonzero distributed mass, connected by 6 ideal revolute joints — both ankles (1 DOF), both knees (1 DOF) and both hips (1 DOF) — forming a single open kinematic chain. An image of the system is presented in Figure 7a;

HR3.S) is planar and restricted to the sagittal plane;

HR4.S) is bipedal with two symmetric legs connect at the same point of the torso joint;

The robot restricted to the frontal plane:

HR1.F) is composed by 5 rigid links — both feet, both legs and the torso — each with nonzero distributed mass, connected by 4 ideal revolute joints — both ankles (1 DOF) and both hips (1 DOF) — forming a single open kinematic chain. An image of the system is presented in Figure 7b;

HR3.F) is planar and restricted to the frontal plane;

With these hypotheses the system's kinematics can be defined. For convenience each link of the 3D system is supposed to be a bar with homogeneous mass distribution, that

Figure 7: Restricted Planar Kinematics

is, its center of mass is located exactly on its geometric center, with the exception of the torso link, that is the composition of two bars.

2.2 Gait Hypotheses

The gait hypotheses are useful to better define what a stable gait for the considered hybrid system is. These hypotheses are guidelines for the controllers and, therefore, are primarily used during the decoupled controller design. For this reason, only the sagittal and frontal gaits hypotheses are presented below.

For the sagittal dynamics:

- HG1.S) The gait is composed of alternating stance (impact) and swing phases;
- HG2.S) During the single support phase (swing phase) the stance leg neither slips nor loses contact with the ground, *i.e.* the vertical component of the forces acting in the stance leg is strictly positive and the ratio of the horizontal and vertical forces does not exceed the static friction coefficient;
- HG3.S) The double support phase (stance phase) is instantaneous and the resulting impact can be modeled as a rigid contact;
- HG4.S) At impact, the former swing leg neither slips nor rebounds and the former stance leg leaves the ground naturally – that is, its post impact vertical speed is positive;

HG5.S) In steady state the gait is symmetric concerning each leg;

HG6.S) The movement is from left to right and the swing leg starts strictly behind the stance leg and ends strictly in front of it;

By these hypotheses, the sagittal dynamics can be modeled as a hybrid system with impulse effects, that is, a single continuous dynamics with a discrete conditional reset rule (the impact). Intuitively, a gait such that the initial conditions before and after a finite fixed number of impacts are the same is a stable one. The definition of hybrid systems will be more formally introduced later in this chapter, and the stability criteria for the gait is better presented in chapter 5.

Hypotheses HG2.S and HG4.S allow the use of a pinned model for the control design of the swing phase, that is, one in which the vertical and horizontal movement of the stance foot are constrained during the modeling process. However, due to the unilateral nature of the foot contact with the ground, both these hypotheses must be verified for a given gait.

For the frontal dynamics:

HG1.F) The gait is composed of four phases: one during which the ankle rotational speed is strictly positive (the rising phase), one during which it is strictly negative (the falling phase), the double support phase (impact) and the reversal phase, when the ankle rotational speed is instantaneously zero;

HG2.F) During the rising, falling and reversal phases, the stance foot neither slips nor loses contact with the ground;

HG3.F) During the gait, the angular moment about the stance ankle is instantaneously zero only during the reversal phase;

HG4.F) At impact, the former swing leg neither slips nor rebound and the former stance leg leaves the ground naturally – that is, its vertical speed is positive;

HG5.F) The reversion occurs once for each step, it is instantaneous and, in steady state, always happens for the same value of ankle angle;

HG6.F) The double support phase is instantaneous and the associated impact can be modeled as a rigid contact;

HG7.F) In steady state, the beginning and end position of the swing leg are strictly the same, resulting in no net movement in the frontal plane;

HG8.F) In steady state the movement is symmetric in relation of both legs;

These hypotheses allow the conclusion that the frontal system can be modeled as a hybrid system with at least two distinct (even if very much alike) continuous phases and two transitions between them. However, if distinct models are considered for each foot on the ground, four continuous phases are needed.

Figure 8: Left and right foot pinned

Note, however, in Figure 8 that although the dynamics differ, from the perspective of the chosen generalized variables, the joint torques are the same except for a minus signal. For this reason, hereinafter in this work it will be considered that the stance leg is the left one, without loss of generality¹.

Like in the sagittal model, hypotheses HG2.F and HG4.F allow the use of the simpler pinned model, but both hypotheses must be verified *a posteriori*. On a final note, hypothesis HG7.F is not a hard constraint in the system's movement, since even if it is not strictly respected, in steady state the net movement of two steps is still zero even if a single step is not.

¹If the right foot is the stance foot, it is only necessary to invert the angles signals when computing the control torques and then invert the required torques signals before applying it to the system

2.3 Impact Hypotheses

The impact hypotheses serve to better define what is expected of the double support phase and, consequently, guide the modeling of this transition. This impact model was taken from (WESTERVELT et al., 2007), chapter 3, and is the same for both the frontal and the sagittal dynamics.

- HI1) An impact results from the simultaneous contact of the swing foot heel and toe with the ground;
- HI2) The impact is instantaneous;
- HI3) The impact results in no rebound, no slipping and no overturn of the swing foot;
- HI4) At the moment of impact, the stance leg lifts from the ground without interaction;
- HI5) The impact forces of the swing foot with the ground can be represented as impulses;
- HI6) The actuators cannot generate impulses and hence can be ignored during impact;
- HI7) The impulsive forces may result in an instantaneous change in the robot's velocities, but there is no instantaneous change in the configuration.

How these assumptions are translated in a mathematical model will be discussed in section 2.6. However, it is worth noting that this set of hypotheses is not exactly true for human walking.

In a typical human gait, the double support phase, although very brief, is not instantaneous and is an important process where the swing heel touches the ground right after the stance one leaves it, then the swing toe hits the ground just before the rising of the stance toe from it.

In this work, however, the double support phase is simplified by this set of hypotheses, and might render a less efficient or robust gait in exchange for a simpler description of the system.

2.4 Hybrid Systems

In this work, a hybrid system is defined as a dynamic system whose continuous evolution suffers a discrete transition at a given condition. This transition can be on the system's time evolution law, on the system's states or in the feedback control law.

More formally, consider a group of n nonlinear affine control system, as presented in Equation (2.1), where $u_i = u_i(x_i)$ is the input signal of the system.

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1) + g_1(x_1)u_1, & x_1 &\in \mathcal{X}_1 \subset \mathbb{R}^{n_1}, & u_1 &\in \mathcal{U}_1 \subset \mathbb{R}^{m_1} \\
 \dot{x}_2 &= f_2(x_2) + g_2(x_2)u_2, & x_2 &\in \mathcal{X}_2 \subset \mathbb{R}^{n_2}, & u_2 &\in \mathcal{U}_2 \subset \mathbb{R}^{m_2} \\
 &\dots \\
 \dot{x}_n &= f_n(x_n) + g_n(x_n)u_n, & x_n &\in \mathcal{X}_n \subset \mathbb{R}^{n_n}, & u_n &\in \mathcal{U}_n \subset \mathbb{R}^{m_n}
 \end{aligned} \tag{2.1}$$

Consider also a group of transition sets $\mathcal{S}_i^j \in \mathcal{X}_i$ with dimension one less than $\mathcal{X}_i$, and a group of transition functions $\Delta_i^j : \mathcal{S}_i^j \rightarrow \mathcal{X}_j$, for $i, j \in \{1, 2 \dots n\}$, constrained in such a way that $\mathcal{S}_i^j \cap \mathcal{S}_i^k = \emptyset$, for $j \neq k$. The transition sets can also be defined by condition functions $H_i^j : \mathcal{X}_i \rightarrow \mathbb{R}$, $G_i^j : \mathcal{X}_i \rightarrow \mathbb{R}^k$, $k \leq n_i$, as $\mathcal{S}_i^j = \{x_i \in \mathcal{X}_i \mid H_i^j(x_i) = 0, G_i^j(x_i) > 0\}$.

Consider also the set of hypotheses ($\forall i, j, k \in \{1, 2, \dots, n\}$, when applicable) presented in accordance to the ones in (WESTERVELT et al., 2007), chapter 4:

HS1) $\mathcal{X}_i$ is a smooth embedded submanifold of $\mathbb{R}^{n_i}$;

HS2) $\mathcal{F}_i : \mathcal{X}_i \rightarrow T\mathcal{X}_i$ is called the flow on $\mathcal{X}_i$, is continuously differentiable, and a solution of $\dot{x}_i = f_i(x_i) + g_i(x_i)u_i(x_i)$ from a given initial condition is unique and depends continuously on the initial condition;

HS3) $\forall \mathcal{S}_i^j$ nonempty, $\forall s_{i,j} \in \mathcal{S}_i^j$, $\frac{\partial H_i^j}{\partial x_i}(s_{i,j}) \neq 0$;

HS4) $\Delta_i^j : \mathcal{S}_i^j \rightarrow \mathcal{X}_j$ is continuous differentiable, where $\mathcal{S}_i^j$ is given the subset topology from $\mathcal{X}_i$;

HS5) $\Delta_i^j(\mathcal{S}_i^j) \cap \mathcal{S}_j^k = \emptyset$.

A nonlinear hybrid system is composed of all the Σ_i with the form presented in Equation (2.2), where $\mathcal{F}_i$ is called the flow on the state manifold $\mathcal{X}_i$, and $\mathcal{T}_i^j$ are the transition from manifold $\mathcal{X}_i$ to any manifold $\mathcal{X}_j$, with $i, j \leq n$.

$$\Sigma_i = \begin{cases} \mathcal{X}_i \subset \mathbb{R}^{n_i} \\ \mathcal{F}_i : \dot{x}_i = f_i(x_i) + g_i(x_i)u_i(x_i), & x_i \notin \mathcal{S}_i^j, j = 1, 2, \dots, n \\ \mathcal{S}_i^j = \{x_i \in \mathcal{X}_i \mid H_i^j(x_i) = 0, G_i^j(x_i) > 0\}, & j = 1, 2, \dots, n \\ \mathcal{T}_i^j : x_j = \Delta_i^j(x_i) & x_i \in \mathcal{S}_i^j, j = 1, 2, \dots, n \end{cases} \tag{2.2}$$

Note that a manifold can transition into itself, in fact a hybrid system with only one manifold that transitions into itself is usually called a system with impulse effects. The hybrid model of walking in the sagittal plane is, in fact, a system with impulse effects, while the one in the frontal plane is a hybrid system with two manifolds that only transition between, never into, themselves.

2.4.1 Sagittal Hybrid Description

The sagittal model of walking is a system with impulse effects, that means that it is composed of a single manifold $\mathcal{X}_s \in \mathbb{R}^{n_s}$, a single flow $\mathcal{F}_s : \dot{x}_s = f_s(x_s) + g_s(x_s)u_s(x_s)$, a single transition set $\mathcal{S}_s = \{x_s \in \mathcal{X}_s \mid H_s(x_s) = 0, G_s(x_s) > 0\}$, and a single transition $\mathcal{T}_s : x_s^+ = \Delta_s(x_s^-)$. As stated before, a system with impulse effects transitions into itself, that is, $\mathcal{T}_s : \mathcal{S}_s \rightarrow \mathcal{X}_s$. In Figure 9 the diagram of a hybrid system with impulse effects is presented.

Figure 9: Diagram of a hybrid system with impulse effects

The physical meaning of each of these components is straightforward: the flow $\mathcal{F}_s$ is the nonlinear swing phase model of the robot and is presented in section 2.5; the manifold $\mathcal{X}_s$ is, therefore, the state manifold of the swing phase and is such that, if $q_s \in \mathcal{Q}_s$ are the joint position vector and manifold, respectively, then $\mathcal{X}_s = T\mathcal{Q}_s$ (where T is the *tangent bundle* operator); the transition $\mathcal{T}_s$ is the impact dynamics and is presented in section 2.6; and the transition set $\mathcal{S}_s$ is the set of states that would result in an impact of the swing foot with the ground, that means that $H_s(x_s)$ is the height of the swing ankle and $G_s(x_s)$ can be chosen as the horizontal position of the swing foot, in order to assure a forward moving gait.

Note that, while technically having a control input signal u_s , once the controller is designed as a function of the system's states $u_s = u_s(x_s)$ (*i.e.* once the system is in closed loop), it can be treated as an autonomous system with impulse effects. This is particularly

useful when studying the existence and stability of a limit cycle in the hybrid system and will be recovered later in this work.

2.4.2 Frontal Hybrid Description

The frontal model of walking is a hybrid system with two continuous phases that can only transition into one another, never into themselves. That means that the gait will have two distinct manifolds, the rising, and the falling manifolds, both presented in Equation (2.3). Like for the sagittal system, a diagram is presented in Figure 10.

$$\begin{aligned} \Sigma_r = & \begin{cases} \mathcal{X}_r = T\mathcal{Q}_r \\ \mathcal{F}_r : \dot{x}_r = f_r(x_r) + g_r(x_r)u_r(x_r), & x_r \notin \mathcal{S}_r^f \\ \mathcal{S}_r^f = \{x_r \in \mathcal{X}_r \mid H_r^f(x_r) = 0, G_r^f(x_r) > 0\} \\ \mathcal{T}_r^f : x_r^+ = x_r^- & x_r^- \in \mathcal{S}_r^f \end{cases} \\ \Sigma_f = & \begin{cases} \mathcal{X}_f = T\mathcal{Q}_f \\ \mathcal{F}_f : \dot{x}_f = f_f(x_f) + g_f(x_f)u_f(x_f), & x_f \notin \mathcal{S}_f^r \\ \mathcal{S}_f^r = \{x_f \in \mathcal{X}_f \mid H_f^r(x_f) = 0\} \\ \mathcal{T}_f^r : x_r^+ = \Delta_f^r x_f^- & x_f^- \in \mathcal{S}_f^r \end{cases} \end{aligned} \quad (2.3)$$

As in the case of the sagittal dynamics: $\mathcal{X}_r$ and $\mathcal{X}_f$ are the rising and falling state manifolds; $\mathcal{F}_r$ and $\mathcal{F}_f$ are the rising and falling continuous dynamics; $\mathcal{S}_r^f$ is the set of states in which a reversion occurs, that is, $H_r^f(x_r)$ is the system's momentum about the stance ankle and $G_r^f(x_r)$ is the horizontal distance between the system's CoM and CoP (the need for this inequality condition will become more evident in section 4.4.2); $\mathcal{S}_f^r$ is the set of states that generate an impact of the swing leg with the ground, similarly with the sagittal dynamics; $\mathcal{T}_r^f$ is a continuous transition from the rising states to the falling states; and $\mathcal{T}_f^r$ is the impact dynamics.

It might not be clear at this point, why it was chosen to represent the frontal hybrid model as two distinct continuous phases, since both flows are the same for the same input (the rising and falling models are the same) and the transition from rising to falling is continuous. Indeed, it would be possible, at this point, to represent the frontal dynamics as a hybrid system with impulse effects.

This, however, becomes a problem during the controller design and will become clearer

Figure 10: Diagram of the frontal hybrid system

after defining the chosen control law. For now, suffices to say that, since there is at least one reversal at the stance ankle speed, it will be necessary to use two distinct control laws, one before and one after the ankle speed reversal. This results in distinct closed loop flows for the rising and falling phases and, thus, justifies the use of two distinct continuous phases to describe this dynamic.

2.5 Continuous Phases Models

Both the sagittal swing phase model and the frontal rising and falling models are pinned open kinematic chains and can be modeled as a N -link, N -DOF planar manipulator. The kinematic description method for planar robots, used to describe these models as well as the Lagrangian method are better presented in appendix A.

Let $\mathcal{Q}_s \subset \mathbb{R}^6$ and $\mathcal{Q}_f \subset \mathbb{R}^4$ be the configuration space of the sagittal and frontal systems, respectively, and $\mathcal{U}_s \subset \mathbb{R}^6$ and $\mathcal{U}_f \subset \mathbb{R}^4$ be the input space of each system. Furthermore, consider that the subscript s refers exclusively to the sagittal pinned system (with $N_s = 6$ DOFs) and the subscript f for the frontal pinned one (with $N_f = 4$ DOFs).

The method of Lagrange easily yields the models presented in Equation (2.4), where $q_{s,f} \in \mathcal{Q}_{s,f}$ are the configuration variables of the system, $u_{s,f} \in \mathcal{U}_{s,f}$ are the system's input vector and $D_{s,f}$, $C_{s,f}$, $G_{s,f}$ and $B_{s,f}$ are, respectively, the inertia matrix, the coriolis and

centripetal forces vector, the gravitational forces and the input conversion matrix.

$$D_{s,f}(q_{s,f})\ddot{q}_{s,f} + C_{s,f}(q_{s,f}, \dot{q}_{s,f}) + G_{s,f}(q_{s,f}) = B_{s,f}u_{s,f} \quad (2.4)$$

It is assumed that both chosen set of coordinates q_s and q_f are a set of relative coordinates and a single absolute one. Moreover if it is defined that $q_s = [q_1 \ q_2 \ \cdots \ q_6]^\top$ and $q_f = [q_7 \ q_8 \ q_9 \ q_{10}]^\top$, then the two absolute variables are q_6 and q_7 .

Finally, the models can be written in the state space form presented in Equation (2.5), where $x_{s,f} = [q_{s,f} \ \dot{q}_{s,f}]^\top \in \mathcal{X}_{s,f} = T\mathcal{Q}_{s,f}$.

$$\dot{x} = \underbrace{\begin{bmatrix} \dot{q}_{s,f} \\ D_{s,f}^{-1}(-C_{s,f} - G_{s,f}) \end{bmatrix}}_{f_{s,f}(x)} + \underbrace{\begin{bmatrix} 0_{N_{s,f} \times N_{s,f}} \\ D_{s,f}^{-1}B_{s,f} \end{bmatrix}}_{g_{s,f}(x)} u_{s,f} \quad (2.5)$$

2.6 Impact Models

In this section the impact model for both the frontal and sagittal dynamics are computed. This method requires the unpinned models of the systems, that is, the system with the possibility to translate in both the x and y directions, and to rotate the stance foot.

However, the method of computing the impact dynamics for a given unpinned model is always the same, therefore, in this section, the distinction between the sagittal and frontal unpinned models is omitted.

The computation of the unpinned planar model can be easily accomplished by adding three extra DOFs at the stance foot and, consequently, extending the generalized variables vector by adding the horizontal and vertical position of the stance ankle (x_1 and y_1) and the stance foot absolute orientation (θ_1).

Let Equation (2.6) be the resulting unpinned model from the Lagrangian method, where $q_u = [q_{s,f}^\top \ x_1 \ y_1 \ \theta_1]^\top \in \mathcal{Q}_u \subset \mathbb{R}^{N_{f,s}+3}$, $u_u = [u_{s,f}^\top \ 0_{1 \times 3}]^\top$ and δF_1 is a vector of external impulsive forces and torques actuating at the stance ankle, whose integral results in $F_1 = [0_{1 \times N_{s,f}} \ F_1^h \ F_1^v \ \tau_1]^\top$ (respectively, horizontal and vertical forces and torque at the stance ankle), in accord to hypothesis HI5.

$$D_u(q_u)\ddot{q}_u + C(q_u, \dot{q}_u) + G(q_u) = B_u u_u + \delta F_1 \quad (2.6)$$

Equation (2.6) is, then, integrated for the duration of the impact. Since, by hypothesis HI2, the impact is instantaneous (let t_e be the instant at which it occurs), hypotheses HI6 and HI7 allow the conclusion that the integration of the angular positions, velocities and of the input torques during the impact are null, that is, $\int_{t_e^-}^{t_e^+} C_u(q_u, \dot{q}_u) = 0$, $\int_{t_e^-}^{t_e^+} G_u(q_u) = 0$ and $\int_{t_e^-}^{t_e^+} B_u u_u = 0$. The result of the integration of Equation (2.6) is presented in Equation (2.7), where $q_u^- = q_u(t_e^-)$ and $q_u^+ = q_u(t_e^+)$.

$$D_u(q_u^+) \dot{q}_u^+ - D_u(q_u^-) \dot{q}_u^- = F_1 \quad (2.7)$$

Since, by hypothesis HI7, $q_u^- = q_u^+$, Equation (2.7) has $n_{s,f} + 2$ equations and $n_{s,f} + 5$ unknowns ($n_{s,f} + 2$ configuration variables, 2 external forces and 1 external torque). The three extra conditions are computed from hypothesis HI3, that can be mathematically written as $\dot{p}_2(t_e^+) = 0_{3 \times 1}$, where $p_2(t) = [p_2^h(t) \ p_2^v(t) \ \theta_2(t)]^\top$ are the Cartesian position of the swing ankle and the absolute orientation of the swing foot.

Being $E_2(q_u) = \frac{\partial p_2}{\partial q_u}$ hypothesis HI3 can be expressed as in Equation (2.8).

$$E_2(q_u^-) \dot{q}_u^+ = 0_{3 \times 1} \quad (2.8)$$

Moreover, the external forces in Equation (2.7) are applied at the stance ankle, but to verify if the unilateral constraints of the contact of the swing foot (that after the impact becomes the new stance foot) with the ground are satisfied, it is necessary to compute the external forces at the swing ankle F_2 . This is easily accomplished by using the swing foot Jacobian, E_2 , as shown in Equation (2.9).

$$F_1 = E_2^\top(q_u^-) F_2 \quad (2.9)$$

Composing Equations (2.7), (2.8) and (2.9) it is possible to obtain Equation (2.10).

$$\begin{bmatrix} D_u(q_u^-) & -E_2^\top(q_u^-) \\ E_2(q_u^-) & 0_{3 \times 3} \end{bmatrix} \begin{bmatrix} \dot{q}_u^+ \\ F_2 \end{bmatrix} = \begin{bmatrix} D_u(q_u^-) \dot{q}_u^- \\ 0_{3 \times 1} \end{bmatrix} \quad (2.10)$$

Since D_u is positive definite and E_2 always has full rank², then the matrix on the left side of Equation (2.10) is always invertible, meaning that this system always has a solution.

²this proof is omitted, but easily obtained if you consider that only p_2^h depends on p_1^h , only p_2^v depends on p_1^v and only θ_2 depends on θ_1

To simplify the expression of impact dynamics, firstly it is considered that right before the impact the stance leg is on the ground, by hypothesis, that is $q_u^- = [q_{s,f}^-; 0_{3 \times 1}]$. By substituting this Equation in (2.10) Equation (2.11) is obtained.

$$\begin{aligned} \begin{bmatrix} \dot{q}_u^+ \\ F_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} D_u(q_{s,f}^-) & -E_2^\top(q_{s,f}^-) \\ E_2(q_{s,f}^-) & 0_{3 \times 3} \end{bmatrix}^{-1} \begin{bmatrix} D_u(q_{s,f}^-) \begin{bmatrix} I_{N_{s,f} \times N_{s,f}} \\ 0_{3 \times N_{s,f}} \end{bmatrix} \\ 0_{3 \times 1} \end{bmatrix}}_{\bar{\Delta}_I(q_{s,f}^-)} \dot{q}_{s,f}^- \\ \dot{q}_u^+ &= \bar{\Delta}_{\dot{q}_u}(q_{s,f}^-) \dot{q}_{s,f}^- \\ F_2 &= \bar{\Delta}_{F_2}(q_{s,f}^-) \dot{q}_{s,f}^- \end{aligned} \quad (2.11)$$

Finally, consider the need to relabel the legs after impact, that is, leg two is now the stance leg and leg one is now the swing one. This is done using a relabeling matrix that exchange the roles from the leg 1 to the leg 2 and vice-versa. For example, for a 3-link sagittal model where q_1 is the relative angle between the stance leg and the torso, q_2 is between the swing leg and the torso, and q_3 is the absolute torso angle the relabeling matrix would be:

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This concept is easily expanded for more complex systems, only exchanging swing ankle with stance ankle, swing knee with stance knee, and so on. With the relabeling matrix it is possible to define the impact for the pinned system as in Equation (2.12).

$$\begin{bmatrix} q_{s,f}^+ \\ \dot{q}_{s,f}^+ \end{bmatrix} = \Delta_I(q_{s,f}^-, \dot{q}_{s,f}^-) = \begin{bmatrix} \Delta_{q_{s,f}} q_{s,f}^- \\ \Delta_{\dot{q}_{s,f}}(q_{s,f}^-) \dot{q}_{s,f}^- \end{bmatrix} \quad (2.12)$$

where,

$$\begin{aligned} \Delta_{q_{s,f}} &:= R \\ \Delta_{\dot{q}_{s,f}} &:= \begin{bmatrix} R & 0_{N_{s,f} \times 3} \end{bmatrix} \bar{\Delta}_{\dot{q}_u}(q_{s,f}^-) \end{aligned}$$

2.7 3D Robot Model

Although not necessary for the controller design, the 3D 10-DOFs model of the bipedal robot is necessary for the initial validation of the proposed technique. Albeit still an open kinematic chain, the 3D robot is much more complex than the sagittal and frontal planar models, which requires more sophisticated methods of modeling and implementation.

Besides that, to compute the impact model of a 3D system, the unpinned model would require the addition of six degrees of freedom, resulting in a 16-DOF system. This would be impracticable and because of that the impact dynamics of the 3D system was significantly simplified.

2.7.1 Peter Corke Toolbox and Denavit-Hartenberg Parameters

The Peter Corke Robotics Toolbox (CORKE, 2017) is a free toolbox for Matlab® available online that greatly facilitates the process of modeling complex 3D manipulators. It has many features and to cover it goes beyond the scope of this work, suffice to say that the *SerialLink* object of the library was generated using a modified Denavit-Hartenberg description.

Figure 11: Denavit-Hartenberg parameters

Source: (CRAIG, 2005)

Figure 11 illustrates the meaning of each parameter for a link. The exact definition of this kinematic description method is presented in (CRAIG, 2005) and (CORKE, 2017). Using the technique in the bibliography, the resulting D-H parameters for the 10-DOF bipedal system is presented in Table 1.

Table 1: 10-DOFs robot D-H parameters

	α_{i-1} [rad]	a_{i-1} [m]	offset [rad]	d_i [m]
1	$-\frac{\pi}{2}$	0	0	0
2	0	0.4	0	0
3	0	0.4	0	0
4	$\frac{\pi}{2}$	0	0	0
5	0	0.2	$\frac{\pi}{2}$	0
6	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	0
7	0	0.4	0	0
8	0	0.4	0	0
9	$-\frac{\pi}{2}$	0	0	0
10	0	0	0	0

With this parameters it is possible to easily generate the robot's forward kinematics and, with the robot's parameters from Table 2, the toolbox automatically calculates the system dynamics.

Table 2: 10-DOFs robot physical parameters

	CoM (x, y, z) [m]	Inertia (x, y, z) [kg · m ²]	Mass [kg]	Link's Length [m]	Viscous Friction [$\frac{N \cdot m \cdot s}{rad}$]
1 st Link	(0.0, 0.0, 0.0)	(0.000, 0.000, 0.000)	0.00	0.0	0.00
2 nd Link	(0.2, 0.0, 0.0)	(0.015, 0.150, 0.150)	3.00	0.4	0.00
3 rd Link	(0.2, 0.0, 0.0)	(0.033, 0.330, 0.330)	9.15	0.4	5.48
4 th Link	(0.0, 0.0, 0.0)	(0.000, 0.000, 0.000)	0.00	0.0	16.5
5 th Link	(0.1, 0.2, 0.0)	(5.530, 0.553, 0.553)	36.0	0.2	16.5
6 th Link	(0.0, 0.0, 0.0)	(0.000, 0.000, 0.000)	0.00	0.0	16.5
7 th Link	(0.2, 0.0, 0.0)	(0.033, 0.330, 0.330)	9.15	0.4	16.5
8 th Link	(0.2, 0.0, 0.0)	(0.015, 0.150, 0.150)	3.00	0.4	5.48
9 th Link	(0.0, 0.0, 0.0)	(0.000, 0.000, 0.000)	0.00	0.0	5.48
10 th Link	(0.0, 0.0, 0.0)	(0.002, 0.020, 0.020)	0.20	0.0	5.48

Since the 3D model was not used in the controller design, further information about it is not presented in this work. With the parameters of Tables 1 and 2 the model can be directly obtained and simulated with the toolbox.

2.7.2 Impact Simplification

The impact for the 3D 10-DOFs model would require a 16-DOFs unpinned model in order to be computed. This made impracticable a simulation with the correct impact for the 3D model without using better simulators (for example Gazebo, Simbody, etc).

For an initial test of the control design, however, it was important to firstly test it in Matlab®, since it would be better to compare the simulation model and the ones used

by the controllers, and, therefore, identify any source of instability. While better for validating the controller, dedicated simulators like Gazebo make its debug harder.

For these reasons, despite not being able to correctly simulate the impact, it was chosen to compromise the quality of the simulation and apply the decoupled impacts calculated for the sagittal and frontal model during the full robot simulation, that is, the full robot continuous model is used during the swing phase, however at impact, the state variables of each plane of motion are separated, the impact is separately applied for the sagittal and frontal variables and, then, the post impact states are used as initial conditions for the next step.

After the controllers presented good results for this simplified simulation, the controller would be applied at a better simulator, however, due to lack of time for this project, this was not yet done.

3 LIMB COORDINATION - VIRTUAL CONSTRAINTS

The problem of dynamically stable bipedal locomotion can be theoretically solved by a simple system. Indeed, in (MCGEER et al., 1990) it was shown how it can be solved by a 2-link robot for the sagittal dynamics. However, for human-like systems, even if restricted to the sagittal plane, the simplest model has 5 degrees of freedom, thus requiring a way of reducing the system's complexity to solve the bipedal gait problem.

In this work, as in (WESTERVELT et al., 2007), virtual constraints were used to solve this problem and simplify the control model.

3.1 Virtual Constrains

The concept of mechanical constraints is common in multibody systems, where physical links are used to constraint the system's degrees of freedom. One example of usage of physical constraints in the build of walking mechanisms is presented in (KOMODA; WAGATSUMA, 2012), reproduced here in Figure 12.

Figure 12: Complex walker built with physical constraints

Source: (KOMODA; WAGATSUMA, 2012)

This kind of constraint is, however, very difficult to change, *i.e.*, if it is desirable to adjust the system's gait due to irregularities on the walking surface, for example, a new model need to be built.

Using this idea, virtual constraints impose to a system a kinematic constraint analogue to the one imposed by a physical constraint, but much easier to adjust. This is done using a feedback linearization controller to enforce a desired kinematic behavior of each joint, represented by an output function chosen as function of only the joints positions.

More formally, consider a continuous dynamic system as the one in Equation (2.4) (with no subscript to generalize this definition to both frontal and sagittal planes), where $q \in \mathcal{Q} \subset \mathbb{R}^N$ is a set of relative coordinates with only one absolute.

An additional hypothesis made is that the absolute coordinate is not actuated, that is, there is only $(N-1)$ actuators in the robot and none of them are between the robot and the world. If this coordinated is actuated, as in the case with robots with actuated nontrivial feet, just consider that the required torque for it is always zero. This is convenient because it prevents foot rotations while still allowing the design of the limb coordination control. Moreover, this extra actuator can still be considered in the control law for the system after the constraints are applied.

Consider also $H_0 \in \mathbb{R}^{(N-1) \times N}$, such that $q_0 = H_0 q$ are all relative coordinates of the system, and $c \in \mathbb{R}^{1 \times N}$, such that $\alpha = c q$ is a strictly increasing or decreasing absolute coordinate of the system (in this chapter α will be the generic absolute coordinate for the control law, when distinction between frontal and sagittal controllers is needed, α will be the sagittal absolute coordinate and β the frontal one). Consider also that $H = [H_0 ; c]$ has full rank. For this system it is chosen $(N-1)$ outputs as in Equation (3.1):

$$y = h(q) = q_0 - h_d(\alpha) \quad (3.1)$$

where h_d is the desired kinematic for each joint, represented by a $(N-1) \times 1$ vector of polynomials. Note that the hypothesis that α is strictly increasing or decreasing is necessary to assure that $h_d(\alpha)$ is a function.

For the sagittal system the aforementioned variables are $c_s = [-1 \ 0 \ -1/2 \ 0 \ 0 \ -1]$ and $H_{0s} = [I_{5 \times 5} \ 0_{5 \times 1}]$. For the frontal system, $c_f = [1 \ 0 \ 0 \ 0]$ and $H_{0f} = [0_{3 \times 1} \ I_{3 \times 3}]$.

Since $q_0 = q_0(q)$ and $\alpha = \alpha(q)$, the output $y = h(q)$ depends only on the configurations variables q . If this is true, defining the state space as in (2.5), it is possible to obtain the relative degree of the output as in (3.2).

$$\begin{aligned}
\frac{dy}{dt} &= \frac{\partial h}{\partial x} \dot{x} \\
&= \begin{bmatrix} \frac{\partial h}{\partial q} & \frac{\partial h}{\partial \dot{q}} \end{bmatrix} \left[\begin{bmatrix} \dot{q} \\ D^{-1}(-C - G) \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix} u \right] \\
&= \underbrace{\begin{bmatrix} \frac{\partial h}{\partial q} & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ D^{-1}(-C - G) \end{bmatrix}}_{L_f h = \frac{\partial h}{\partial q} \dot{q}} + \underbrace{\begin{bmatrix} \frac{\partial h}{\partial q} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix} u}_{L_g h = 0}
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\frac{d^2 y}{dt^2} &= \frac{d}{dt} \frac{dy}{dt} = \frac{\partial}{\partial x} L_f h \dot{x} \\
&= \begin{bmatrix} \frac{\partial}{\partial q} \left(\frac{\partial h}{\partial q} \dot{q} \right) & \frac{\partial h}{\partial \dot{q}} \end{bmatrix} \left[\begin{bmatrix} \dot{q} \\ D^{-1}[-C - G] \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix} u \right] \\
&= \begin{bmatrix} \frac{\partial}{\partial q} \left(\frac{\partial h}{\partial q} \dot{q} \right) & \frac{\partial h}{\partial \dot{q}} \end{bmatrix} \begin{bmatrix} \dot{q} \\ D^{-1}[-C - G] \end{bmatrix} + \underbrace{\frac{\partial h}{\partial q} D^{-1}B u}_{L_g L_f h \neq 0}
\end{aligned}$$

Therefore, all $(N - 1)$ outputs has relative degree of two resulting in a zero dynamics of second order. $L_g L_f h$ is called the decoupling matrix and, as consequence of results in (ISIDORI, 2013), its invertibility in a point assures the existence and uniqueness of the zero dynamics in a neighborhood of that point.

With a few extra hypotheses proposed below as in (WESTERVELT et al., 2007), chapter 5, it is possible to extend these results to an open subset of the state-space.

HH1) h is function only of the configuration variables;

HH2) Being $q \in \mathcal{Q}$ exists $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ such that for each point $q \in \tilde{\mathcal{Q}}$ the decoupling matrix is invertible;

HH3) There exists a smooth function $\alpha : \tilde{\mathcal{Q}} \rightarrow \mathbb{R}$ such that

$$[h(q); \alpha(q)] : \tilde{\mathcal{Q}} \rightarrow \mathbb{R}^N$$

is a diffeomorphism onto its image;

HH4) There exists at least one point in $\tilde{\mathcal{Q}}$ where h vanishes;

HH5) There exists an unique point $q_0^- \in \tilde{\mathcal{Q}}$, such that $[h(q_0^-); p_2^v(q_0^-); \theta_2(q_0^-)] = 0$, $p_2^h > 0$, and $rank([h; p_2^v; \theta_2]) = N$, where p_2^v , p_2^h and θ_2 are the vertical and horizontal positions of the swing ankle and the absolute angle of the swing foot, respectively.

And so, for this subset, it is possible to ensure traceability of the outputs with a control signal with the form of Equation (3.3).

$$u = L_g L_f h^{-1}(v - L_f^2 h), \quad (3.3)$$

where v is a control signal that stabilizes the system $\ddot{y} = u$, with $y, u \in \mathbb{R}^{N-1}$. With this result, it is possible to constraint the system's degrees of freedom virtually, given a set of functions parametrized by a system variable α .

Mathematically, this restriction represents a submanifold of the state manifold called the zero dynamics (that is, the states that the system can assume while still maintaining $y = 0$). Intuitively, it is reasonable to require that this manifold is invariant (*i.e.* that once into the zero dynamics the closed loop autonomous system never leaves it), this would allow that conclusions taken from the restricted (and, therefore, simpler) system could be applied to the complete system, as long as it is subject to a feedback that would render an invariant zero dynamics.

In chapter 5 the invariance necessity is more formally presented, together with the stability criteria. The invariance conditions, however, depends on the controller design and kinematic constraint specification and therefore will be discussed in this chapter.

3.2 Hybrid Invariance

As hinted before, the existence of a zero dynamics is not enough to successfully solve the limb coordination problem, the resulting zero dynamics must still be invariant and, preferably, finite time attractive.

Typically an invariant manifold is a manifold invariant under the action of the system dynamics. This is particularly tricky for hybrid systems since it must be invariant under the action of all possible dynamics, continuous or discrete. This justifies the study of hybrid invariance as a two part analysis: forward invariance (or invariance of the continuous dynamics); and transition invariance (or invariance of the discrete dynamics).

3.2.1 The Solution of a Hybrid System

In order to better define forward invariance it is first needed to formally define what a solution for a hybrid system is. A solution is defined here in a very similar way as it is

defined in (WESTERVELT et al., 2007), chapter 4.

Consider a hybrid system written as the various Σ_i , as presented in Equation (2.2) for $i \in \mathcal{N} = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$. Consider also that each flow $\mathcal{F}_i$ is in its closed-loop form, that is, $u_i = u_i(x_i)$. This means that $\mathcal{F}_i : \dot{x}_i = f_i(x_i) + g_i(x_i)u_i(x_i) = f_{cli}(x_i)$.

Denote $\mathcal{X} = \bigcup_{i \in \mathcal{N}} \mathcal{X}_i$ the union of all the state manifolds. A function $\varphi : [t_i, t_f) \rightarrow \mathcal{X}$, $t_f \in \mathbb{R} \cup \{\infty\}$, $t_f > t_i$, is a *solution* of the considered hybrid system if, for a closed discrete subset $\mathcal{T} \subset [t_i, t_f)$, $\mathcal{T} = \{t_0 < t_1 < \dots < t_j < \dots\}$, called the switching times set, and for a function $k : \mathcal{T} \rightarrow \mathcal{N}$, which specifies the model's phase after a given transition, if the following conditions hold:

HSol1) $\forall i \geq 0$, φ restricted to $[t_i, t_{i+1})$ takes values in $\mathcal{X}_{k(t_i)}$;

HSol2) $\forall i \geq 0$, φ restricted to $[t_i, t_{i+1})$, is right continuous;

HSol3) $\forall i \geq 0$, φ restricted to (t_i, t_{i+1}) satisfies the differential equation $\dot{\varphi} = f_{clk(t_i)}(\varphi)$;

HSol4) $\forall i \geq 0$ and $\forall t \in (t_i, t_{i+1})$ the limit from the left, $\varphi^-(t) = \lim_{\tau \rightarrow t^-} \varphi(\tau)$, exists and is finite;

HSol5) $\forall i \geq 0$ and $\forall t \in (t_i, t_{i+1})$, $\varphi(t) \notin \mathcal{S}_{k(t_i)}^j$, $\forall j \in \mathcal{N}$;

HSol6) $\forall i \geq 0$, and $t_i < \infty$, $\varphi^-(t_i) \in \mathcal{S}_{k(t_i)}^{k(t_{i+1})}$ and $\varphi^+(t_i) = \Delta_{k(t_i)}^{k(t_{i+1})}(\varphi^-(t_i))$.

Note that from HSH2 the solutions are unique for a given initial condition. For $x_0 \in \mathcal{X}_{k(t_0)} \subset \mathcal{X}$ the solution of the hybrid system corresponding to the initial condition x_0 at the time t_0 (i.e. $x_0 = \varphi(t_0)$) is denoted $\varphi(t, x_0)$.

3.2.2 Forward Invariance and Finite-Time Attractiveness of a Hybrid System

A set $\mathcal{Z} \subset \mathcal{X}$ is *forward invariant* if for each $x_0 \in \mathcal{Z} \cap \bigcap_{j \in \mathcal{N}} \overline{\mathcal{S}}_{k(t_0)}^j$ (that is, for each initial condition that does not generate a transition) exists $t_1 > t_0$ such that $\varphi(t, x_0) \in \mathcal{Z}$, $\forall t \in [t_0, t_1)$. One conclusion that can be drawn from this is that, since this is true for every $x_0 \in \mathcal{Z} \cap \bigcap_{j \in \mathcal{N}} \overline{\mathcal{S}}_{k(t_0)}^j$, then $\varphi^-(t_1, x_0) \in \bigcup_{j \in \mathcal{N}} \mathcal{S}_{k(t_0)}^j$, in other words, $\mathcal{Z}$ is forward invariant if a solution initialized in $\mathcal{Z}$ continues in it during the continuous phase. Moreover, t_1 presented as in this definition is also a switching time, and therefore, $t_1 \in \mathcal{T}$.

Forward invariance can be easily obtained by choosing a control law as presented in Equation (3.3), with the only restriction that $\forall z \in \mathcal{Z}$, $v(z) = 0$. However, as said before,

it is interesting to define the zero dynamics as not only forward invariant, but also locally continually finite time attractive as well.

A submanifold is locally continually finite time attractive if it is forward invariant and there exists an open set $\mathcal{V} \supset \mathcal{Z}$, such that for every $x_0 \in \mathcal{V} \cap \bigcap_{j \in \mathcal{N}} \overline{\mathcal{S}}_{k(t_0)}^j$, $\exists \bar{t} \in (t_0, t_1)$ such that $\varphi(\bar{t}, x_0) \in \mathcal{Z}$. In other words, there exists a open region along the borders of $\mathcal{Z}$ sufficiently small so that every point in this region converges into the zero dynamics in finite time. Moreover, this finite time of convergence is small enough so that for every point in this region, the system converges into the zero dynamics before any transition.

Finite time attractiveness of the continuous phase can be achieved with the choice of v from Equation (3.3). The choice of this work, and the one suggested in (WESTERVELT et al., 2007), chapter 5, uses the result of (BHAT; BERNSTEIN, 2000), where v is given by Equation ((3.4)).

$$v = \Psi(y, \dot{y}) := \frac{1}{\epsilon^2} \begin{bmatrix} \psi_1(y_1, \epsilon \dot{y}_1) \\ \dots \\ \psi_{N-1}(y_{N-1}, \epsilon \dot{y}_{N-1}) \end{bmatrix} \quad (3.4)$$

Where

$$\begin{aligned} \psi_i(y_i, \epsilon \dot{y}_i) &:= -\text{sign}(\epsilon \dot{y}_i) |\epsilon \dot{y}_i|^\alpha - \text{sign}(\phi_i(y_i, \epsilon \dot{y}_i)) |\phi_i(y_i, \epsilon \dot{y}_i)|^{\frac{\alpha}{2-\alpha}} \\ \phi_i(y_i, \epsilon \dot{y}_i) &:= y_i + \frac{1}{2-\alpha} \text{sign}(\epsilon \dot{y}_i) |\epsilon \dot{y}_i|^{2-\alpha} \end{aligned}$$

The rate of convergence, that is, the time $\bar{t}$ at which the solution converges to the zero dynamics, is parametrized by the value of ϵ in the computation of v .

3.2.3 Transition Invariance of a Hybrid System

A subset $\mathcal{Z} \in \mathcal{X}$ is *transition invariant* if for every solution $\varphi(t, x_0)$ with switching set $\mathcal{T}$, $\forall t_i \in \mathcal{T}$, $\lim_{t \rightarrow t_i^-} \varphi(t, x_0) \in \mathcal{Z} \cap \mathcal{S}_{k(t_{i-1})}^{k(t_i)} \implies \varphi(t_i, x_0) \in \mathcal{Z}$. In other words, every transition of this solution that occurs in the zero dynamics results in a new state also in the zero dynamics.

To better understand this, consider a solution $\varphi(t, x_0)$ and an instant t_1 such that $x^-(t_1) = \lim_{t \rightarrow t_1^-} \varphi(t, x_0) \in \mathcal{Z} \cap \mathcal{S}_{k(t_0)}^{k(t_1)}$. Suppose that between t_0 and t_1 there is no transition, and therefore $x^-(t_1) \in \mathcal{X}_{k(t_0)}$, and has the form $x_{k(t_0)} = [q_{k(t_0)}; \dot{q}_{k(t_0)}]$.

Since right before the impact the solution is in the zero dynamics, that means the chosen output function for this continuous phase is null right before the transition ($h_{k(t_0)}(q_{k(t_0)}^-) = 0$). If it is supposed forward invariance during the continuous phase before transition, it means that the solution would keep itself in the zero dynamics, if it was not for the transition, that is $\left. \frac{\partial h}{\partial q} \right|_{q_{k(t_0)}^-} \dot{q}_{k(t_0)}^- = 0$.

This means that during the transitions all the controlled states ($q_0, \dot{q}_0$) can be written as functions of the absolute states ($\alpha, \dot{\alpha}$). This will be better addressed in Chapter 4, where the zero dynamics is formally presented.

For now, consider that during the impact, the impact condition $H_{k(t_0)}^{k(t_1)}(x_{k(t_0)}^-) = 0$ can be written as a function of only the absolute variable and its derivate.

In the cases when the transition function originally depends only on the configuration variables of the robot (as in the sagittal and frontal impact conditions with the swing foot height), then $H_{k(t_0)}^{k(t_1)}(\alpha_{k(t_0)}^-) = 0$ is an equation with a single variable, which can be easily solved in order to find the value of $\alpha_{k(t_0)}^-$ at which the impact occurs.

However, when the transition depends on both the joints positions and velocities, there are more variables than equations. This occurs with the frontal gait design during the reversal phase condition. This problem was solved with a successive approximations algorithm and will be better explained in chapter 4.

Once the value of $x_{k(t_0)}^-$ is determined, it is trivial to calculate $x_{k(t_1)}^+ = \Delta_{k(t_0)}^{k(t_1)}(x_{k(t_0)}^-)$ and verify if $h_{k(t_1)}(x_{k(t_1)}^+) = 0$. This could, on the other hand, be assured during the choices of output functions for each continuous phase. This process is greatly facilitated by choosing convenient polynomials to describe the virtual constraints and will be better addressed in section 3.3.

3.3 Beziér Polynomials

The virtual constraints were expressed as output functions in Equation (3.1), where $h_d(\alpha)$ represents the desired kinematics of the controlled variables. The chosen polynomial to parametrize the desired kinematics of the constraints is the Beziér polynomial and has the form presented in Equation (3.5).

$$h_{di}(s(\alpha)) = \sum_{k=0}^M a_k^i \frac{M!}{k!(M-k)!} s(\alpha)^k (1-s(\alpha))^{M-k}, \quad (3.5)$$

where

$$s(\alpha) = \frac{\alpha - \alpha^+}{\alpha^- - \alpha^+},$$

In Equation (3.5), α is the absolute uncontrolled variable of a given continuous phase of a hybrid system, α^+ and α^- are the values of α just after the transition to the continuous phase and just before the transition to the next phase, respectively. The index i indicates which controlled joint this specific equation constraints; a_k^i are the parameters of this polynomial and M its degree.

Note that, for a Beziér polynomial, $h_{di}(0) = a_0^i$ and $h_{di}(1) = a_N^i$; therefore, if the system is in the zero dynamics, $q_0^+ = [a_0^1; a_0^2; \dots; a_0^N] = a_0$ and $q_0^- = [a_N^1; a_N^2; \dots; a_N^N] = a_N$. This is the first useful property of Beziér polynomials, since for a given set of parameters a , it is straightforward to compute the initial and final values of each joint in a step.

Note also in Figure 13 that the derivate of a Beziér polynomial at the beginning (respectively end) of the graphic is equivalent to the line that crosses the first (respectively last) two points of the curve. This allows one to easily compute the joint speeds at the beginning and end of the step.

Figure 13: Graphic of a Beziér Polynomial with $M=4$

3.3.1 Impact Invariance with Beziér Polynomials

With the results obtained from using Beziér polynomials, it is possible to obtain an impact invariance condition for the constraints parameters. For this, consider an impact of a hybrid system from phase i to phase j , described as an impact as in section 2.6, and both phases with continuous phases constrained by Bézier polynomials parametrized by a^i and a^j , respectively (since it will not be necessary to address a single element of the parameter matrix, the use of the top right corner to indicate phase will not present a problem).

Supposing that the hybrid system is hybrid invariant, it is possible to write the restrictions between $a_{M_i}^i$ and a_0^j for this hypothesis to hold. First, consider the impact written in Equation (3.6), where it is supposed that $x_{i,j} = [q_{i,j}; \dot{q}_{i,j}]$.

$$\begin{aligned} q_j^+ &= \Delta_{i,q}^j q_i^- \\ \dot{q}_j^+ &= \Delta_{i,\dot{q}}^j \dot{q}_i^- \end{aligned} \tag{3.6}$$

with $\Delta_{i,q}^j$ and $\Delta_{i,\dot{q}}^j$ calculated as in Equation (2.12). Hence, it is possible to calculate q_j^+ , q_i^- , $\dot{q}_j^+$ and $\dot{q}_i^-$ as in Equation (3.7), since the system is supposed hybrid invariant.

$$\begin{aligned} q_j^+ &= H_j^{-1} \begin{bmatrix} a_0^j \\ \alpha_j^+ \end{bmatrix} \\ q_i^- &= H_i^{-1} \begin{bmatrix} a_{M_i}^i \\ \alpha_i^- \end{bmatrix} \\ \dot{q}_j^+ &= H_j^{-1} \begin{bmatrix} \frac{\partial h_{d,j}}{\partial \alpha_j} \Big|_{\alpha_j=\alpha_j^+} \\ 1 \end{bmatrix} \dot{\alpha}_j^+ \\ \dot{q}_i^- &= H_i^{-1} \begin{bmatrix} \frac{\partial h_{d,i}}{\partial \alpha_i} \Big|_{\alpha_i=\alpha_i^-} \\ 1 \end{bmatrix} \dot{\alpha}_i^- \end{aligned} \tag{3.7}$$

where $\frac{\partial h_d}{\partial \alpha} \Big|_{\alpha=\alpha^-}$ and $\frac{\partial h_d}{\partial \alpha} \Big|_{\alpha=\alpha^+}$ can be calculated as in Equations (3.8) and (3.9) (it is the same for both phases, therefore, no phase index is shown in these equations).

$$\left. \frac{\partial h_d}{\partial \alpha} \right|_{\alpha=\alpha^-} = \frac{M}{\alpha^- - \alpha^+} (a_N - a_{N-1}) \quad (3.8)$$

$$\left. \frac{\partial h_d}{\partial \alpha} \right|_{\alpha=\alpha^+} = \frac{M}{\alpha^- - \alpha^+} (a_1 - a_0) \quad (3.9)$$

Then, from restriction (3.6) it is calculated that:

$$\begin{bmatrix} a_0^j \\ \alpha_j^+ \end{bmatrix} = H_j \Delta_{i,q}^j H_i^{-1} \begin{bmatrix} a_{M_i}^i \\ \alpha_i^- \end{bmatrix} \quad (3.10)$$

and that:

$$\begin{bmatrix} \frac{M_j}{\alpha_j^- - \alpha_j^+} (a_1^j - a_0^j) \\ 1 \end{bmatrix} \dot{\alpha}_j^+ = H_j \Delta_{i,q}^j \underbrace{H_i^{-1} \begin{bmatrix} \frac{M_i}{\alpha_i^- - \alpha_i^+} (a_{M_i}^i - a_{M_i-1}^i) \\ 1 \end{bmatrix}}_{\omega^-} \dot{\alpha}_i^- \quad (3.11)$$

Simplifying (3.11) it is possible to obtain the Equation (3.12), assuming $c_j \Delta_{i,q}^j \omega^-$ invertible.

$$a_1^j = H_{j,0} \Delta_{i,q}^j \omega^- \frac{\alpha_j^- - \alpha_j^+}{M_j} (c_j \Delta_{i,q}^j \omega^-)^{-1} + a_0^j \quad (3.12)$$

So, for each polynomial, the first and last two parameters are constrained by two equations. Then, by selecting the parameters in a way so that these constraints are respected, it is possible to assure impact invariance. This result also requires $M_{i,j} \geq 4$ in order to be achievable.

Note that, while this supposed that the transition is an impact, if the transition is continuous, as is the case with the reversion, the results of this section are still valid by choosing $\Delta_{i,q}^j = R_i^j$, the relabeling matrix of this transition.

4 SAGITTAL AND FRONTAL HYBRID ZERO DYNAMICS

In chapter 3 it was presented how it was possible to simplify a generic hybrid system into a less complex one using virtual constraints. In this chapter the hybrid system's equations are actually restricted to better understand its particularities and to lay the groundwork for chapter 5, where the stability condition is presented based on such restricted system.

4.1 Continuous Phase Zero Dynamics

As stated in chapter 3, the absolute angle actuator was not considered during the limb coordination, which results in a second order zero dynamics for a fully actuated system. In this section the continuous phases of this zero dynamics will be made explicit and, since the formulation is the same for every continuous phase, the system is supposed to be in a generic phase i .

By hypothesis HH3, it is possible to write any continuous phase of the system in its normal form using the coordinate change $\Phi_i(q) := [h_i(q_i), \alpha_i(q)]$. With this, the system can be written as in Equation (4.1).

$$\begin{aligned}\eta_{i,1} &= h_i(q), \quad \eta_{i,2} = L_f h_i(q_i, \dot{q}_i) \\ \xi_{i,1} &= \alpha_i(q_i), \quad \xi_{i,2} = L_f \alpha_i(q_i, \dot{q}_i)\end{aligned}\tag{4.1}$$

Enforcing the zero dynamics invariant by using the control law (3.3), the zero dynamics can be obtained imposing $y_i \equiv 0$, as in Equation (4.2), where $u_{i,\alpha}$ is the unused absolute variable control input.

$$\begin{aligned}\dot{\eta}_{i,1} &= 0, \quad \dot{\eta}_{i,2} = 0 \\ \dot{\xi}_{i,1} &= \xi_{i,2}, \quad \dot{\xi}_{i,2} = L_f^2 \alpha_i + LgLf\alpha_i u_i + u_{i,\alpha}\end{aligned}\tag{4.2}$$

Therefore, the zero dynamics can be expressed as a second order system. However, these terms are very computational costly to compute. In (WESTERVELT et al., 2007) it is shown that the zero dynamics can be expressed in a simpler set of coordinates, using $\xi_{i,1}$ as an absolute angle and $\xi_{i,2}$ as the system's angular momentum about that angle, as in Equation (4.3).

$$\begin{aligned}\xi_{i,1} &= \alpha_i, \quad \xi_{i,2} = \gamma_i(q_i, \dot{q}_i) \\ \dot{\xi}_{i,1} &= \kappa_{i,1}(\xi_{i,1})\xi_{i,2}, \quad \dot{\xi}_{i,2} = \kappa_{i,2}(\xi_{i,1}) + u_{i,\alpha}\end{aligned}\tag{4.3}$$

where $\gamma_i(q_i, \dot{q}_i) = \gamma_{i,0}(q_i)\dot{q}_i$ is the last entry of $D(q)\dot{q}$, and

$$\begin{aligned}\kappa_{i,1}(\alpha_i) &= \frac{\partial \alpha_i}{\partial q_i} \left[\frac{\partial h_i}{\partial q_i} \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big|_{\mathcal{Z}} \\ \kappa_{i,2}(\alpha_i) &= - \frac{\partial V_i}{\partial q_{i,N}} \Big|_{\mathcal{Z}}\end{aligned}$$

where $V_i(q)$ is the robot's potential energy. So, this zero dynamics can be calculated with terms of the non-linear model.

4.2 Transition in the Zero Dynamics

To fully compute the hybrid zero dynamics, the restricted transition must be considered. Consider the zero dynamics manifold defined as bellow for a given continuous state i :

$$\mathcal{Z}_i := \{x_i \in \mathcal{X}_i \mid h_i(x_i) = 0, \quad L_f h_i(x) = 0\}$$

And consider a solution $\varphi(t, x_0) \subset \mathcal{Z}$, $t \in (t_0, t_f)$ that contains exactly one transition at t_e . Similarly to $\mathcal{X}$, $\mathcal{Z} = \bigcup_{i \in \mathcal{N}} \mathcal{Z}_i$. For (t_0, t_e) and (t_e, t_f) , φ is a solution of the zero

dynamics and $\lim_{t \rightarrow t_e^-} \varphi(t) = \varphi(t_e^-) \in \mathcal{S}_{k(t_0)} \therefore \varphi(t_e^-) \in \mathcal{S}_{k(t_0)} \cap \mathcal{Z}$ and $\varphi(t_e^+) \in \Delta_{k(t_0)}^{k(t_e)}(\mathcal{S}_{k(t_0)} \cap \mathcal{Z})$ — since it is desired to compute the terms restricted to the zero dynamics, it is natural to suppose impact invariance.

And thus, the transition function of the full model, when restricted to the zero dynamics, is the restricted system impact function. Consider, then, δ_i^j as the transition function from phase i to phase j , restricted to the zero dynamics, and expressed in the coordinates of Equation ((4.3)). The resulting restricted hybrid system is presented in Equation ((4.4)) where $z_i = [\Phi_i(q_i); \dot{\Phi}_i(q_i)]$, $g_i^Z = [0_{N-1 \times 1}; 1]$ and $f_i^Z = (f_i + g_i u_i)|_Z$, being u_i as in ((3.3)), with $v = 0$.

$$\Sigma_i^Z : \begin{cases} \dot{z}_i = f_i^Z(z_i) + g_i^Z u_{i,\alpha} & z_i^- \notin (\bigcup_{j \in \mathcal{N}} \mathcal{S}_i^j) \cap \mathcal{Z} \\ z_j^+ = \delta_i^j(z_i^-) & z_i^- \in \mathcal{S}_i^j \cap \mathcal{Z}, j \in \mathcal{N} \end{cases}, z \in \mathcal{Z} \quad (4.4)$$

4.2.1 Impact in the Zero Dynamics

In the specific case of impact transitions described in section 2.6, the state vector can be divided in angles and velocities, that is $x_i = [q_i; \dot{q}_i]$. Moreover, during an impact there is no instantaneous change in the system's position, meaning that during the transition the changes in q_i is due only to relabeling of the legs, and thus immediate if the condition either right after or right before the leg is known.

For the velocities, the impact transition has the form of Equation (4.5), where the impact matrix $\Delta_{\dot{q}}(q_i^-)$ depends only on the configuration variables right before the impact.

$$\dot{q}_i^+ = \Delta_{\dot{q}}(q_i^-) \dot{q}_i^- \quad (4.5)$$

However, when restricted to the zero dynamics, $q_i^- = q_i(\alpha^-)$ and $\dot{q}_i^- = \frac{\partial q_i}{\partial \alpha}(\alpha^-) \dot{\alpha}^-$, which simplifies (4.5) to (4.6)

$$\dot{\alpha}^+ = \underbrace{\left(\frac{\partial q_i}{\partial \alpha}(\alpha^+) \right)^{-1} \Delta_{\dot{q}}(\alpha^-) \frac{\partial q_i}{\partial \alpha}(\alpha^-)}_{\delta_{\dot{\alpha}}(\alpha^-)} \dot{\alpha}^- \quad (4.6)$$

Therefore, in the zero dynamics, during an impact, the absolute velocity impact function can be expressed only in terms of the absolute configuration variable.

4.3 The Foot Rotation Indicator and Ankle Control

From Equation (4.4) it can be noted that there is still an input control signal that can give some design margin. The ankle torque (u_α for the sagittal gait, u_β for the frontal gait, u_a if a generic approach is wanted) can continuously change a nominal gait speed, increase robustness of the gait, or, in general, allow for some extra design freedom. Specifically for a decoupled frontal gait without knee, however, it is not possible to have a stable gait without ankle torque.

Even so, using the ankle torque in the control design risks overturning the stance foot, that is why it is interesting to study the conditions in which the overturning occurs and include them in the controller design.

One of the indicators of foot rotation in the literature is the foot rotation indicator (or simply FRI), firstly presented in (GOSWAMI, 1999), is a concept very similar to the more broadly used zero moment point (or ZMP), presented in (VUKOBRATOVIĆ; BOROVAC, 2004).

In simple terms, the FRI is defined in a way such that if it is strictly in the interior of the convex hull of the contact points of the robot with the ground, then there is no foot overturning.

In this section the calculations necessary to control the FRI position using the ankle torque are presented. These calculations were taken from (WESTERVELT et al., 2007) and are only reproduced here.

The FRI point is defined as "the point on the foot/ground contact surface, within or outside the convex hull of the foot-support area, at which the resultant moment of the force/torque impressed on the foot is normal to the surface".

Consider, in Figure 14, the vertical projection of the stance ankle to the floor as the origin of the coordinate system, p_{FRI} is the foot rotation indicator position and p_a is the ankle position.

Consider now the unpinned model calculated for this system. The Lagrangian about the absolute angle of the unpinned system (θ_1) results in Equation (4.7).

$$\frac{d}{dt} \frac{\partial \mathcal{L}_u}{\partial \dot{\theta}_1} - \frac{\partial \mathcal{L}_u}{\partial \theta_1} = p_{FRI}^h F_{FRI}^N. \quad (4.7)$$

Since, however, θ_1 is a cyclic variable (therefore, $\frac{\partial K_u}{\partial \theta_1} = 0$), and $\frac{\partial \mathcal{L}_u}{\partial \theta_1} = \sigma_0$, where σ_0 is

Figure 14: Points of notice in the stance foot

the angular momentum about the origin, Equation (4.7) can be rewritten to (4.8), using the angular momentum transfer theorem to transform σ_0 from the origin to the ankle (σ_a).

$$\underbrace{\dot{\sigma}_a - m_{tot} p_a^v \ddot{p}_{cm}^v}_{\dot{\sigma}_0} + m_{tot} g_0 p_{cm}^h = p_{FRI}^h F_{FRI}^N \quad (4.8)$$

Since the stance foot is supposed fixed on the ground at all times during the gait, applying the angular momentum balance theorem to the robot about the supporting ankle yields Equation (4.9).

$$\dot{\sigma}_a = -m_{tot} g_0 p_{cm}^h + (p_{FRI} - p_a) \wedge F_{FRI} \quad (4.9)$$

Furthermore, since the stance foot is supposed in rotational equilibrium about the ankle, Equation (4.10) can be written.

$$0 = -u_a + (p_{FRI} - p_a) \wedge F_{FRI} - m_{foot} g_0 p_{foot,cm}^h \quad (4.10)$$

In these equations m_{tot} is the total mass of the robot, m_{foot} is the mass of the stance foot, p_{cm}^h and p_{cm}^v are the horizontal and vertical positions of the robot's center of mass and g_0 is the local gravity acceleration.

Combining Equations (4.9) and (4.10) yields (4.11), combining this result with (4.8) then results in (4.12).

$$\dot{\sigma}_a = -m_{tot} g_0 p_{cm}^h + m_{foot} g_0 p_{foot,cm}^h + u_a \quad (4.11)$$

$$p_{FRI}^h F_{FRI}^N = m_{foot} g_0 p_{foot,cm}^h + u_a - m_{tot} p_a^v \ddot{p}_{cm}^v \quad (4.12)$$

Since F_{FRI}^N are the vertical component of the reaction forces of the ground, then $F_{FRI}^N = m_{tot} g_0 + m_{tot} \ddot{p}_{cm}^v$, which results, from Equation (4.12), into (4.13).

$$p_{FRI}^h = \frac{-m_{tot} p_a^v \ddot{p}_{cm}^h + m_{foot} g_0 p_{foot,cm}^h + u_a}{m_{tot} g_0 + m_{tot} \ddot{p}_{cm}^v} \quad (4.13)$$

Equation (4.13) is valid for the complete system. Restricting the values of p_{cm} to the zero dynamics results in $p_{cm} = p_{cm}(\xi_1)$ and $\dot{p}_{cm} = \frac{\partial p_{cm}}{\partial \xi_1} \kappa_1 \xi_2 = [\lambda_{cm}^h(\xi_1); \lambda_{cm}^v(\xi_1)] \xi_2$. Equation (4.14) presents the expression of $\ddot{p}_{cm}$ restricted to the zero dynamics.

$$\ddot{p}_{cm} = \begin{bmatrix} \lambda_{cm}^h(\xi_1)(\kappa_2(\xi_1) + u_a) + \frac{\partial \lambda_{cm}^h(\xi_1)}{\partial \xi_1} \kappa_1 \xi_2^2 \\ \lambda_{cm}^v(\xi_1)(\kappa_2(\xi_1) + u_a) + \frac{\partial \lambda_{cm}^v(\xi_1)}{\partial \xi_1} \kappa_1 \xi_2^2 \end{bmatrix} \quad (4.14)$$

Substituting Equation (4.14) into (4.13) and rearranging the terms, results in equation (4.15).

$$u_a = \frac{\gamma_0(\xi_1) + \gamma_1(\xi_1) \xi_2^2}{\gamma_2(\xi_1)} \quad (4.15)$$

where

$$\begin{aligned} \gamma_0(\xi_1) &= p_{FRI}^{h,d}(\xi_1) m_{tot} (g_0 + \lambda_{cm}^v(\xi_1) \kappa_2(\xi_1)) + p_a^v m_{tot} \lambda_{cm}^h(\xi_1) \kappa_2(\xi_1) - p_{foot,cm}^h m_{foot} g_0 \\ \gamma_1(\xi_1) &= p_{fri}^{h,d}(\xi_1) m_{tot} \frac{\partial \lambda_{cm}^v(\xi_1)}{\partial \xi_1} \kappa_1(\xi_1) + p_a^v m_{tot} \frac{\partial \lambda_{cm}^h(\xi_1)}{\partial \xi_1} \kappa_1(\xi_1) \\ \gamma_2(\xi_1) &= 1 - p_{FRI}^{h,d}(\xi_1) m_{tot} \lambda_{cm}^v(\xi_1) - p_a^v m_{tot} \lambda_{FRI}^h(\xi_1) \end{aligned}$$

It is clear, then, that as long as $\gamma_2(\xi_1) \neq 0$ it is possible to track a desired FRI position by choosing u_a as in Equation (4.15).

Note that in the zero dynamics, the relation between the ankle momentum and the momentum about the FRI point (obtained from the moment transfer theorem) is the one presented in Equation (4.16).

$$\sigma_{FRI} = \underbrace{(1 - p_{FRI}^{h,d}(\xi_1) m_{tot} \lambda_{cm}^v(\xi_1) - p_a^v m_{tot} \lambda_{FRI}^h(\xi_1))}_{\gamma_2(\xi_1)} \sigma_a \quad (4.16)$$

This implies that if the moment about the FRI point is never zero, then $\gamma_2(\xi_1)$ is never

zero. This is true by hypothesis for the sagittal gait, however in the frontal gait there is a reversion of the angular moment about the ankle, this implies that the momentum about the FRI can be zero in this gait and, therefore, this condition must be verified for each set of parameters.

4.4 Hybrid Zero Dynamics Closed-Form Solution

Since, by hypothesis, the system's zero dynamics intercepts the switching set for any given continuous phase, the restricted system also has a hybrid solution. Hence, any solution of the hybrid zero dynamics has continuous parts and discrete transitions. In this section the specific cases of frontal and sagittal gaits will be focused and the transitions and continuous solutions will be presented.

4.4.1 Swing, Rising and Falling Phases Solutions

All three continuous phases zero dynamics in sagittal and frontal gait have the same form presented in Equation (4.17).

$$\begin{aligned}\dot{\xi}_1 &= \kappa_1 \xi_2 \\ \dot{\xi}_2 &= \kappa_2 + u_a\end{aligned}\tag{4.17}$$

Considering that it is desired to track the FRI position in order to avoid foot over-turning while still using the ankle actuator to facilitate the gait, the feedback presented in Equation (4.15) can be applied, resulting in Equation (4.18).

$$\begin{aligned}\dot{\xi}_1 &= \kappa_1 \xi_2 \\ \dot{\xi}_2 &= \kappa_3 + \kappa_4 \xi_2^2\end{aligned}\tag{4.18}$$

where,

$$\begin{aligned}\kappa_3 &= \kappa_2 + \frac{\gamma_0}{\gamma_2} \\ \kappa_4 &= \frac{\gamma_1}{\gamma_2}\end{aligned}$$

Equation (4.18) is written with ξ_2 being the system's angular momentum about the stance ankle. With a little change in the equations it is possible to write it with ξ_2^{FRI} being the system's angular momentum about the FRI point, as presented in Equation (4.19).

$$\begin{aligned}\dot{\xi}_1 &= \kappa_1^{FRI} \xi_{2FRI} \\ \dot{\xi}_{2FRI} &= \kappa_2^{FRI} + \kappa_3^{FRI} \xi_{2FRI}^2\end{aligned}\tag{4.19}$$

where,

$$\begin{aligned}\kappa_1^{FRI} &= \frac{\kappa_1}{\gamma_2} \\ \kappa_2^{FRI} &= -m_{tot} g_0 (p_{cm}^h - p_{FRI}^{h,d}) \\ \kappa_3^{FRI} &= -m_{tot} \frac{\partial p_{FRI}^{h,d}}{\partial \xi_1} \lambda_{cm}^v \kappa_1 \left(\frac{1}{\gamma_2}\right)^2\end{aligned}$$

Equation (4.20), then, presents the derivative of ξ_{2FRI} in relation to ξ_1 .

$$\frac{d\xi_{2FRI}}{d\xi_1} = \frac{\kappa_2^{FRI}}{\kappa_1^{FRI} \xi_{2FRI}} + \frac{\kappa_3^{FRI}}{\kappa_1^{FRI}} \xi_{2FRI}\tag{4.20}$$

In this system it is possible to apply a coordinate change in the form of $\frac{(\xi_{2FRI})^2}{2} = \zeta_{FRI}$, being $\xi_{2FRI} d\xi_{2FRI} = d\zeta_{FRI}$. After the coordinate change, Equation (4.20) simplifies to (4.21) (where the dependencies of each term is made explicit).

$$\frac{d\zeta_{FRI}}{d\xi_1} = \frac{\kappa_2^{FRI}(\xi_1)}{\kappa_1^{FRI}(\xi_1)} + 2 \frac{\kappa_3^{FRI}(\xi_1)}{\kappa_1^{FRI}(\xi_1)} \zeta_{FRI}\tag{4.21}$$

Notice that Equation (4.21) is linear in ζ_{FRI} with ξ_1 -varying parameters, and has an explicit solution of the form presented in Equation (4.22).

$$\zeta_{FRI}(\xi_1) = (\delta_{FRI}(\xi_1))^2 \zeta_{FRI}^+ - V_Z^{FRI}(\xi_1)\tag{4.22}$$

where ζ_{FRI}^+ is the initial value of ζ_{FRI} and,

$$\begin{aligned}\delta_{FRI}(\xi_1) &= \exp\left(\int_{\xi_1^+}^{\xi_1} \frac{\kappa_3^{FRI}(\tau_1)}{\kappa_1^{FRI}(\tau_1)} d\tau_1\right) \\ V_{\mathcal{Z}}^{FRI}(\xi_1) &= -\int_{\xi_1^+}^{\xi_1} \exp\left(\int_{\tau_2}^{\xi_1} \frac{\kappa_3^{FRI}(\tau_1)}{\kappa_1^{FRI}(\tau_1)} d\tau_1\right) \frac{\kappa_2^{FRI}(\tau_2)}{\kappa_1^{FRI}(\tau_2)} d\tau_2\end{aligned}$$

Note that Equation (4.22) is written about the FRI point. It is possible to rewrite it about the ankle when considering the relation between ξ_2 and ξ_{2FRI} presented in Equation (4.16). Relabeling $\gamma_2(\xi_1)$ to $\delta_a^{FRI}(\xi_1)$ and defining $\delta_{FRI}^a = (\delta_a^{FRI})^{-1}$ Equation (4.22) can be written to (4.23), where $\zeta_2 = \frac{\xi_2^2}{2}$.

$$\zeta_2(\xi_1) = \underbrace{(\delta_{FRI}^a(\xi_1))^2 (\delta_{FRI}(\xi_1))^2 (\delta_a^{FRI}(\xi_1^+))^2}_{\delta^2(\xi_1)} \zeta_2^+ - (\delta_{FRI}^a(\xi_1))^2 V_{\mathcal{Z}}^{FRI}(\xi_1) \quad (4.23)$$

Considering this solution for the sagittal system, notice that if the desired FRI position is constant, then $\kappa_3^{FRI} \equiv 0$. Since the ankle torque is not necessary for the sagittal gait design, the desired FRI position for this gait is chosen to be zero for every step, this is roughly equivalent to the point feet gait presented in (WESTERVELT et al., 2007) with the obvious difference of the swing foot link.

With this simplification, the sagittal zero dynamics solution have the form presented in Equation (4.24).

$$\zeta_{2s}(\xi_1^s) = \delta_s^2(\xi_1^s) \zeta_{2s}^+(\xi_1^s) - (\delta_{FRI}^a(\xi_1^s))^2 V_{\mathcal{Z}}^{FRIs}(\xi_1^s) \quad (4.24)$$

where,

$$V_{\mathcal{Z}}^{FRIs}(\xi_1^s) = -\int_{\xi_1^{s+}}^{\xi_1^s} \frac{\kappa_2^{FRIs}(\tau)}{\kappa_1^{FRIs}(\tau)} d\tau \quad (4.25)$$

The frontal gait, however, has a more serious problem with its solution. Equation (4.20) is only defined for $\xi_{2FRI} \neq 0$. While in the sagittal gait the system's momentum is never zero, it is zero at the frontal reversal and thus the solution is undefined at that point.

Being ξ_1^{r+} and ξ_1^{r-} the values of ξ_1^r at the beginning and at the end of the rising phase. For the full system, it is known that $\dot{\xi}_1^r = 0$ implies $\xi_{2FRI}^r = 0$, that is equivalent to $\zeta_{FRI}^r = 0$. Indeed, from Equation (4.22), for a given value of $\xi_1^r = \xi_1^{r-}$, it is possible that

$$\zeta_{rFRI}(\xi_1^{r-}) = \delta_{rFRI}(\xi_1^{r-})^2 \zeta_{rFRI}^+ - V_{\mathcal{Z}}^{FRIr}(\xi_1^{r-}) = 0$$

meaning that, despite Equation (4.20) having a singularity for $\xi_{2FRI}^r = 0$, Equation (4.21) allows the computation of its solution for $\zeta_{rFRI}(\xi_1^{r-}) = 0 \iff \xi_{2FRI}^r(\xi_1^{r-}) = 0$. Due to the continuity of $\xi_{2FRI}^r(\xi_1^r)$ and $\zeta_{rFRI}(\xi_1^r)$ with relation to ξ_1^r , since

$$\frac{\xi_{2FRI}^r(\xi_1^r)^2}{2} = \zeta_{rFRI}(\xi_1^r), \quad \xi_1^r \in [\xi_1^{r+}, \xi_1^{r-}[$$

then

$$\lim_{\xi_1^r \rightarrow \xi_1^{r-}} \frac{\xi_{2FRI}^r(\xi_1^r)^2}{2} = \lim_{\xi_1^r \rightarrow \xi_1^{r-}} \zeta_{rFRI}(\xi_1^r) = \zeta_{rFRI}(\xi_1^{r-}) = 0$$

thus,

$$\lim_{\xi_1^r \rightarrow \xi_1^{r-}} \xi_{2FRI}^2(\xi_1^r) = 0 \triangleq \xi_{2FRI}^r(\xi_1^{r-})$$

which is consistent with the expected result. Therefore, the reversal is well defined for the frontal restricted system. A point ξ_1^{r-} is, then, called a reversal point of the hybrid system if at it:

HIP1) the system is halted, that is, $\dot{\xi}_1^r(\xi_1^{r-}) = 0 \iff \xi_{2FRI}^r(\xi_1^{r-}) = 0 \iff \zeta_{rFRI}(\xi_1^{r-}) = 0$;

HIP2) the ankle rotational speed changes direction, that is $\dot{\xi}_{2FRI}^r(\xi_1^{r-}) < 0 \iff \ddot{\xi}_1^r(\xi_1^{r-}) < 0$ (considering the left foot on the ground and using the chosen angle and axis conventions).

However, even if it is well defined, a reversal point might not exist for a given set of parameters.

4.4.2 Reversal Point Existence

Differently from the impact, it is much harder to verify for a given set of parameters the existence of the reversal point. This is due to the fact that the reversal condition function restricted to the zero dynamics has two variables (ankle position and speed), but only one equation.

Once the stability condition is defined for the zero dynamics in chapter 5, an iterative approximations algorithm can be used to find the reversal point, if it exists. However, if

the existence is not assured, the algorithm may never converge or try to verify a candidate for the reversal point in which the solution is not well defined.

Due to these problems, in this section an interval in which the reversal point must be, if it exists, will be defined. Although calculated for the restricted system, the results of this chapter depend on the physical interpretation of ξ_1 as the absolute frontal plane ankle angle and ξ_{2FRI} as the system's angular moment about the FRI point. Therefore, the notation of β for ξ_1 , and σ_{FRI} for ξ_{2FRI} will be used here.

Let β_r^+ be the value of β at the beginning of the rising phase, β_r^- the value at the ending of the rising phase, β_f^+ the value at the beginning of the falling phase and β_f^- the value at the ending of the falling phase, with β_r^+ and β_f^- known *a priori* from the impact condition.

Since, by the measurement conventions, $\beta < 0$, suppose, without loss of generality, that $\beta_r^+ > \beta_f^-$ (if not, the proof is analogous). If exists β_r^- that satisfies HIP1 and HIP2, then $\beta_r^- > \beta_r^+$. This is straightforward and the proof is not shown here, but can be easily concluded if it is considered that $\beta_r^- < \beta_r^+$ results in a fall after impact, contradicting HI4 and HG4.F.

Furthermore, consider $q_{0r}^- = q_{0f}^+$ the controlled configuration variables right at the end of the rising phase and at the beginning of the falling phase, respectively, also consider $p_{rFRI}^{h,d-}$ the desired horizontal FRI position at the end of the rising phase, and assume that $p_{cm}^h([\beta_r^+ ; q_{0r}^-]) > p_{rFRI}^{h,d-}$. Considering β_r^{max} so that,

$$p_{cm}^h([\beta_r^{max} ; q_{0r}^-]) = p_{rFRI}^{h,d-},$$

then, if there exists β_r^- that satisfies HIP1 and HIP2, $\beta_r^- \in]\beta_r^+, \beta_r^{max}[$.

To prove that, suppose that $\exists \beta_r^- > \beta_r^{max}$ that is a reversal point, *i.e.*, that satisfies HIP1 and HIP2. Then, using Equation (4.19), it follows that:

$$\dot{\sigma}_{rFRI}^- = \kappa_2^{FRI}(\beta_r^-) + \kappa_3^{FRI}(\beta_r^-) \underbrace{\sigma_{rFRI}^-}_{=0}.$$

Substituting the expression of κ_2^{FRI} results in

$$\dot{\sigma}_{rFRI}^- = -m_{tot} g_0 (p_{cm}^h([\beta_r^- ; q_{0r}^-]) - p_{rFRI}^{h,d-})$$

Notice, that, since the system's pose (q_{br}^-) is fixed, the horizontal position of the center of mass is proportional to minus the cosine of β — as illustrated by Fig. 15 — and is

Figure 15: Illustration of different poses and center of mass position of the robot for the same q_{0r} but for different values of β

monotonically increasing for β in the last two quadrants (since by convention, $\beta < 0$ in Figure 15). Therefore, supposing that $-\pi < \beta_r^- < 0$, if $p_{cm}^h([\beta_r^+ ; q_{0r}^-]) > p_{rFRI}^{h,d-}$ and $p_{cm}^h([\beta_r^{max} ; q_{0r}^-]) = p_{rFRI}^{h,d-}$, and if $\beta_r^- > \beta_r^{max}$, then $p_{cm}^h([\beta_r^- ; q_{0r}^-]) < p_{rFRI}^{h,d-}$ resulting in a positive value of $\dot{\sigma}_{FRI}^-$ contradicting the hypotheses HIP2. Therefore, if there exists a reversal point it must be in the set $]\beta_r^+, \beta_r^{max}[$.

4.5 Useful Results From the Zero Dynamics

Although the restricted system is already defined, some other results are useful for computing the optimization algorithm. These results are presented here.

4.5.1 Time to Impact

The time to impact function can be calculated from the hybrid zero dynamics equations, as shown in Equation (4.26), with $\xi_2 = \sqrt{2\zeta_2(\xi_1^+, \xi_2^-)}$ from Equation (4.22).

$$\begin{aligned} \frac{d\xi_1}{dt} &= \kappa_1(\xi_1)\xi_2(\xi_1, \xi_2^-) \\ dt &= \frac{d\xi_1}{\kappa_1(\xi_1)\xi_2(\xi_1, \xi_2^-)} \\ T_I(\xi_2^-) &= \int_{\xi_1^+}^{\xi_1^-} \frac{1}{\kappa_1(\xi_1)\xi_2(\xi_1, \xi_2^-)} d\xi_1 \end{aligned} \quad (4.26)$$

With this it is possible to calculate the duration of the impact without needing to

simulate it, but still requiring the computation of the solution $\xi_2(\xi_1, \xi_2^-)$. For the sagittal system the solution computation is greatly simplified by imposing the FRI position to be constant during the continuous phase, but that is not true for the frontal system.

4.5.2 Stance Leg Contact Forces

A hypothesis that needs to be verified for each solution is the contact between the stance leg and the ground. To calculate this, consider the full $N + 3$ DOF model with input u as in Equation (3.3), with $v = 0$. The new inputs are the external forces of the system. Since the calculations for the $N + 3$ DOF model considered the position of the stance leg foot as the two extra variables, the external forces are the ones applied to the stance leg foot. The resulting closed loop system is:

$$f_e(x_u) + g_e(x_u)[F_1^N; F_1^T; \tau_1],$$

where x_u is the unpinned model state space variables. Being the output vector $y_e = h_e(x_u) = [p_1^v; p_1^h; \theta_1]$, it is verifiable that the decoupling matrix $L_{g_e}L_{f_e}h_e$ is invertible, which allows the calculation of the forces as bellow.

$$\begin{bmatrix} F_1^N \\ F_1^T \\ \tau_1 \end{bmatrix} = -(L_{g_e}L_{f_e}h_e)^{-1}L_{f_e}^2h_e$$

Hence, it is possible to calculate this expression restricted to the zero dynamics and calculate the forces on the stance foot for each value of ξ_1 .

4.5.3 Computing the Zero Dynamics Terms

Consider q as the set of relative coordinates used for modeling the system, q_0 as in Equation (3.1) being the controlled quantities and α the quantity that parametrizes the virtual constraints, then there exists $H = [H_0; c]$ invertible, so that:

$$\tilde{q} = \begin{bmatrix} q_0 \\ \alpha \end{bmatrix} = Hq$$

Hence, the inertia matrix on the new set of coordinates is:

$$\tilde{D}(\tilde{q}) = (H^{-1})^\top D(q) H^{-1} \Big|_{q=H^{-1}\tilde{q}} \quad (4.27)$$

Therefore, as presented in (WESTERVELT et al., 2007), the terms of the hybrid zero dynamics can be calculated as in Equation (4.28).

$$\begin{aligned} \kappa_1(\xi_1) &= \frac{1}{I(\xi_1)}, \\ \kappa_2(\xi_1) &= m_{tot} \vec{g} p_{cm}^h(h_d(\xi_1), \xi_1), \\ \delta &= 1 - m_{tot} L_s \lambda_{cm}^v - J_{foot} \omega_0, \end{aligned} \quad (4.28)$$

where the virtual inertia $I(\xi_1)$ is given by:

$$I(\xi_1) = \left(\tilde{D}(N, N) + \begin{bmatrix} \tilde{D}(N, 1) & \tilde{D}(N, 2) & \cdots & \tilde{D}(N, N-1) \end{bmatrix} \frac{\partial h_d(\xi_1)}{\partial \xi_1} \right) \Big|_{q_0=h_d(\xi_1)},$$

5 LIMIT CYCLE STABILITY CRITERIA

This chapter aims to formally define what constitutes a stable gait for the purpose of this work. It is defined what is a periodic orbit in the hybrid dynamics and the framework used to search for limit cycles in a hybrid system.

By the end of this chapter it will be clear what are the sagittal and frontal gaits criteria for existence and stability, allowing the formulation of the optimization problem. In this chapter it will also be discussed how to verify the existence of and to compute the reversal point for the frontal dynamics.

5.1 Limit Cycle in a Hybrid System

Consider a solution $\varphi(t, x_0)$ of a hybrid system as presented in section 3.2.1, where $t \in [t_0, \infty[$, $x_0 = \varphi(t_0)$. The solution is *periodic* if $\exists \bar{t} > t_0$ and $\exists T_s \in \mathbb{R}$, such that $\forall t \in [\bar{t}, \infty[$, $\varphi(t, x_0) = \varphi(t + T_s, x_0)$. A set $\mathcal{O} \subset \mathcal{X}$ is, then, a nontrivial periodic orbit of the system if it has more than one point and exists a periodic solution $\varphi(t, x_0)$ such that $\mathcal{O} = \{\varphi(t, x_0) \mid t > \bar{t}\}$.

A periodic orbit of a hybrid system can be transversal to a switching set $\mathcal{S}_i^j$, $i, j \in \mathcal{N}$, if $\mathcal{O} \cap \mathcal{S}_i^j = \bar{x}_i$ is a single point and $L_f H_i(\bar{x}_i) \neq 0$, where H_i is the transition condition function. For the frontal and sagittal hybrid systems as presented in this work, a nontrivial periodic orbit that is transversal to every transition defined is a periodic locomotion.

Note that a periodic orbit of a hybrid system is not supposed to be a closed set. In fact, if it intercepts any transition set (as is the case of a periodic locomotion) it will not be closed. This can be seen for both periodic orbits of the sagittal and frontal systems in Figures 16a and 16b.

To illustrate these definitions, consider that a human walking in a flat homogeneous surface is a periodic orbit, since the legs repeat the same motion every two steps. In this example the switching set is the states that result in impact, that is, every state where the

Figure 16: Representation of periodic locomotions

(a) Sagittal Periodic Locomotion

(b) Frontal Periodic Locomotion

human has its swing foot on the ground. The person's gait is transversal to the switching set if the vertical velocity of the swing foot, just before the impact, is negative. This means that the person must, in some sense, stomp the swing foot during impact, instead of gently placing it on the ground.

A periodic orbit (and consequently a periodic locomotion) can be locally stable and asymptotically stable in the sense of Lyapunov. Consider an open set $\mathcal{V} \supset \mathcal{O}$ sufficiently small so that, $\exists \epsilon > 0$ and $\exists \varphi(t, x_0)$, the orbit $\mathcal{O}$ is stable in the sense of Lyapunov if $\forall x_0 \in \mathcal{V}$ and $\forall t \in [t_0, \infty[$, $\text{dist}(\varphi(t, x_0), \mathcal{O}) < \epsilon$, where the distance between a point and a set of points is the minimum distance between that point and every point in the set and the distance between points is the usual Euclidean one.

A periodic orbit can be locally attractive if, for the same conditions as before, $\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, x_0), \mathcal{O}) = 0$ holds. Finally, a periodic orbit is locally asymptotically stable in the sense of Lyapunov if it is both locally stable and attractive.

5.2 Poincaré Method for Hybrid Systems

Consider a solution of a hybrid system defined as in section 3.2.1. For any given continuous phase i of a hybrid system, consider $\varphi_i(t, x_{i0})$ the solution of the continuous phase in a way that, $\forall t \in [t_i, t_{i+1}[$, $\varphi_{k(t_i)}(t, x_{k(t_i)0}) = \varphi(t, \varphi(t_i, x_0)) \iff x_{k(t_i)0} = \varphi(t_i, x_0)$.

Similarly to $\mathcal{X}$ and $\mathcal{Z}$, consider $\mathcal{S} = \bigcup_{i,j \in \mathcal{N}} \mathcal{S}_i^j$. To simplify the notation, consider also that for $x \in \mathcal{S}$, Δ is the adequate switching function, such that, $\Delta(\mathcal{S}) = \bigcup_{i,j \in \mathcal{N}} \Delta_i^j(\mathcal{S}_i^j)$. Consider, also, a similar notational simplification for the transition condition function H_i^j .

For any continuous phase i of the hybrid system it is also possible to define a *time to transition* function T_{Ti} as in Equation (5.1).

$$T_{Ti} := \begin{cases} \inf \{ \bar{t} \geq t_i \mid \lim_{t \rightarrow \bar{t}^-} \varphi_i(t, x_{i0}) \in \mathcal{S}_i^j, \forall j \in \mathcal{N} \} & \text{if } \exists \bar{t} \\ \infty & \text{otherwise} \end{cases} \quad (5.1)$$

Notice that $\forall t \in [t_i, t_{i+1}[$, $t + T_{Ti}(\varphi(t, x_{i0})) = t_{i+1}$, and $\forall x_i \in \mathcal{X}_i$, $0 < T_{Ti}(x_i) < \infty \iff \exists j \in \mathcal{N}$ so that $\lim_{t \rightarrow T_{Ti}(x_i)} \varphi(t, x_i) \in \mathcal{S}_i^j$. That is, if for a given point x_i its time to impact is real and strictly positive, then the solution of the continuous phase i , that has x_i as initial condition, results in a transition.

It is possible to define $\tilde{\mathcal{X}}_i := \{x_i \in \mathcal{X}_i \mid 0 < T_{Ti}(x_i) < \infty, \text{ and } L_f H(\varphi(T_{Ti}(x_i), x_i)) \neq 0\}$

as the set of all points whose solutions results in a transition and are transversal to that transition set.

Consider $\tilde{\mathcal{S}}_i = \{x \in \mathcal{S} \mid \Delta(x) \in \tilde{\mathcal{X}}_i\}$, and $\mathcal{S}^i = \{x \in \mathcal{X}_i \cap \mathcal{S} \mid x = \varphi_i(T_{T_i}(\tilde{x}_i), \tilde{x}_i), \tilde{x}_i \in \tilde{\mathcal{X}}_i\}$. It is possible to define the *Poincaré phase i map* $P_i : \tilde{\mathcal{S}}_i \rightarrow \mathcal{S}^i$ as in Equation (5.2).

$$P_i(x) = \varphi_i(T_{T_i}(\Delta(x)), \Delta(x)) \quad (5.2)$$

The Poincaré function for phase i is well defined continuous and continuously differentiable in this manner.

To define a Poincaré function for the hybrid system, however, it is necessary to suppose either that there is only one possible way for a periodic orbit that contains transitions to transit between phases (for example, if from phase 1 it is only possible to transit to phase 3, then to phase 2, then to phase 4 and finally back to phase 1), or that there is a specific order of transition for a desired periodic orbit (in the last example, if it could be possible to transit from phase 2 directly to phase 1, but it is desired for a solution to pass through phase 4 before going back to phase 1).

The second condition will not be treated for two reasons: first, it requires special care with the undesired transition points; second, both hybrid systems of this work fall under the first condition (the sagittal system can only transit into itself and the two frontal system phases can only transition into each other).

Consider, then, that for this hybrid system, any periodic solution $\varphi(t, x_0)$ of period T_s that contains a transition, with a switching time set $\mathcal{T} = \{t_0 < t_1 < \dots < t_m < t_0 + T_s < t_1 + T_s < \dots\}$, such that $k(t_i) = k(t_i + T_s)$, has the same ordered set $\mathcal{K} = \{k(t_0), k(t_1), \dots, k(t_m)\}$. That is, every periodic orbit of this hybrid system must transit between the same phases in the same ordered to complete a cycle.

Then, considering P_i the phase i Poincaré function, then the *Poincaré return map* of this hybrid system $P : \tilde{\mathcal{S}}_{k(t_0)} \rightarrow \mathcal{S}_{k(t_0)}$ is defined as in Equation (5.3).

$$P := P_{k(t_m)} \circ P_{k(t_{m-1})} \circ \dots \circ P_{k(t_0)} \quad (5.3)$$

Note that under the hypothesis that every periodic orbit of this hybrid system must transit between the same phases in the same order, $\mathcal{S}_{k(t_0)} \subset \mathcal{S}_{k(t_m)}^{k(t_0)}$, because otherwise there would be more than one way to return to phase 1, contradicting the hypothesis.

Note that the Poincaré function is a discrete function that, given the initial condition of the system in phase $k(t_0)$, returns the state of the system right after it transits again to phase $k(t_0)$. Since for, every initial condition, the solution of the hybrid system is unique by hypothesis, $P(x^*) = x^* \iff \varphi(t, x^*)$ is a periodic solution of the hybrid system. Moreover, if x^* is an asymptotically stable fixed point of P , then $\mathcal{O} = \{\varphi(t, x^*) \mid t > t_0\}$ is an asymptotically stable periodic orbit of the hybrid system.

This is a powerful result, since it allows the problem of finding a limit cycle in a continuous hybrid system to be simplified to finding a fixed point of a discrete time system. However, to compute the phase i Poincaré functions, it is necessary to know the closed form solution of each continuous phase of the hybrid system, which is not practical for higher order systems.

From previous chapters of this work, however, it was possible to calculate the closed form solution of the sagittal and frontal restricted systems, and this motivates the study of the conditions that assure that the Poincaré map of the restricted system is equivalent to the Poincaré map of the full system restricted to the zero dynamics.

5.3 Restricted Poincaré Method

Even if the Poincaré function is very useful, computing the analytical expression of $P(x)$ still requires the solution of the full hybrid system, which is not usually available. That is, possibly, one of the main reasons why the limit cycle robots were firstly designed as simpler systems, with fewer degrees of freedom.

To further simplify the process of finding a limit cycle of the full system, the Poincaré return map is restricted to a lower dimension manifold of $\mathcal{S}_{k(t_0)}$, resulting in a restricted Poincaré function.

Consider a the hypothesis bellow of a submanifold $\mathcal{Z}$:

- HZ1) $\mathcal{Z}$ is an embedded submanifold of $\mathcal{X}$;
- HZ2) $\mathcal{Z} \cap \mathcal{S}_{k(t_0)}$ is an embedded submanifold with dimension one less than $\mathcal{Z}$;
- HZ3) $\mathcal{Z}$ is finite-time attractive;
- HZ4) $\mathcal{Z}$ is hybrid invariant.

For any submanifold of the state space respecting these conditions, the restricted Poincaré map can be defined as in Equation (5.4).

$$\rho : \mathcal{S}_{k(t_0)} \cap \mathcal{Z} \rightarrow \mathcal{S}_{k(t_0)} \cap \mathcal{Z}, \quad \rho(x) := P(x) \Big|_{x \in \mathcal{S}_{k(t_0)} \cap \mathcal{Z}} \quad (5.4)$$

So, ρ can be calculated using the system restricted to a lower dimension subset of the state space, simplifying the problem.

To show that, consider the hybrid system restricted to $\mathcal{Z}$ as in Equation ((5.5)).

$$\Sigma|_{\mathcal{Z}} : \begin{cases} \dot{z}(t) = f|_{\mathcal{Z}}(z(t)) & z^-(t) \notin \mathcal{S} \cap \mathcal{Z} \\ z^+(t) = \Delta|_{\mathcal{S} \cap \mathcal{Z}}(z^-(t)) & z^-(t) \in \mathcal{S} \cap \mathcal{Z} \end{cases} \quad (5.5)$$

It is evident that the restricted hybrid dynamics is itself a system with impulse effects and can have a Poincaré function defined for it. Note also that, since $\mathcal{Z}$ is hybrid invariant, a periodic orbit in the restricted dynamics implies a periodic orbit in the full system. Finally, it is provable that, under this conditions, the Poincaré function for the restricted dynamics is the full model Poincaré function restricted to $\mathcal{S} \cap \mathcal{Z}$.

Therefore, by calculating the Poincaré function for the (simpler) restricted dynamics and finding its equilibrium points, then a limit cycle for the full model is also found.

And so, by defining a subset $\mathcal{Z}$ hybrid invariant of the state space, the problem of verifying the existence and stability of a limit cycle can be greatly simplified. Note that the existence of a limit cycle in the restricted dynamics is enough to exist the limit cycle in the full model, but is not necessary. This, however, did not present itself as a problem during this work. With this result it is possible to describe the restricted Poincaré function for the frontal and sagittal dynamics using the closed form solution of the continuous phases presented in chapter 4.

5.4 Sagittal Restricted Poincaré Function

Consider the restricted sagittal hybrid system with closed form solution as in Equation (4.24). Consider also the impact condition function for the sagittal dynamics $H_s(x_s)$.

Since the impact condition function is the height of the swing foot, it actually depends only on the configuration variables and not on the joints velocities, that is $H_s(x_s) = H_s(q_s)$. Furthermore, since the virtual constraints are parametrized by Beziér polynomials, the configuration of the robot at the end of the step, q_{0s}^- , is known.

Therefore, the impact condition has only one variable, $H_s(q_s^-) = H_s(\alpha^-) = 0$. The

equation can be solved numerically and, despite having more than one result due to the presence of sines and cosines, there are only two between $-\pi$ and π , and one breaks the other constraint that requires the horizontal position of the swing foot to be positive at impact.

Knowing the full configuration of the robot at the end of the step and supposing that the polynomials parameters were selected in a way that assured hybrid invariance (respecting Equations (3.10) and (3.12)), one must only apply the relabeling matrix, R , to find the initial value of the absolute coordinate, α^+ .

With α^- , α^+ , the restricted impact function δ_s , and the solution of the continuous phase as of Equation (4.24), it is possible to define the restricted Poincaré function as in Equation (5.6) (note that since $\delta_s \xi_{2FRI}^{s-} = \xi_{2FRI}^{s+}$, then $\delta_s^2 \zeta_{sFRI}^- = \zeta_{sFRI}^+$).

$$\rho_s(\zeta_{2s}^-) = \delta_s^2(\xi_1^-) \delta_\alpha^2 \zeta_{2s}^- - \delta_{FRI}^a(\xi_1^{s-})^2 V_{\mathcal{Z}}^{FRIs}(\xi_1^{s-}) \quad (5.6)$$

With the Poincaré function defined it is possible to calculate the fixed point ζ_{sFRI}^* as in Equation (5.7) (hereinafter consider the notation simplification $\delta_s(\xi_1^{s-}) = \delta_s$, and $\delta_{FRI}^a(\xi_1^{s-}) = \delta_{FRI}^{a-}$).

$$\zeta_{2s}^* = - \frac{(\delta_{FRI}^{a-})^2 V_{\mathcal{Z}}^{FRIs}(\xi_1^{s-})}{1 - \delta_s^2 \delta_\alpha^2} \quad (5.7)$$

Equation (5.7) is a closed form equation to calculate a fixed point for the restricted Poincaré function for any given set of parameters. Since the variable ζ_2 is defined as $\frac{\xi_2^2}{2}$, the first restriction is that for the nominal step the value of ζ_{2s} is never lesser than or equal to zero. This assures the existence of the limit cycle. The stability condition comes from the discrete system theory, both conditions are written bellow:

C1.S The limit cycle for the restricted system exists for a given set of parameters if and only if $\zeta_{2s}^* > 0$ and $\max_{\xi_1^{s+} \leq \xi_1^s \leq \xi_1^{s-}} V_{\mathcal{Z}}^{FRIs}(\xi_1^s) = V_{\mathcal{Z},max}^{FRIs} < \frac{\delta_s^2 \delta_\alpha^2 \zeta_{sFRI}^*}{(\delta_{FRI}^{a-})^2} = - \frac{\delta_s^2 \delta_\alpha^2}{1 - \delta_s^2 \delta_\alpha^2} \frac{V_{\mathcal{Z}}^{FRIs}(\xi_1^{s-})}{(\delta_{FRI}^{a-})^2}$,

C2.S If the limit cycle for the restricted system exists, it is asymptotically stable if and only if $\delta_s^2 \delta_\alpha^2 < 1$.

It is possible to write these conditions as nonlinear constraints for an optimization algorithm, together with the other hypothesis (as the unilateral constraints of the contact of the stance foot with the ground, and others), and optimize the system's gait in relation to a cost function, usually the square energy cost or the square sum of the joint's torques.

5.5 Frontal Restricted Poincaré Function

With Equation (4.23) it is possible to calculate the values of ζ_{FRI} at the end of each continuous phase, given their value at the beginning and the values of $\xi_1^{r,f}$ at each transition, that is,

$$\zeta_{r,f}^- = \delta_{r,f}^2(\xi_1^{r,f-})\zeta_{r,f}^+ - \delta_{FRI}^a(\xi_1^{r,f-})^2 V_Z^{FRI}(\xi_1^{r,f-}) \quad (5.8)$$

Furthermore, substituting the transition functions from rising to falling (1) and from falling to rising ($\delta_{\dot{\beta}}$), it is possible to calculate ζ_{FRI} at the beginning of each phase, knowing its value at the end of the previous phase, that is,

$$\rho_r(\zeta_f^-) \triangleq \zeta_r^- = \delta_r^2(\xi_1^{r-}) \delta_{\dot{\beta}}^2(\xi_1^{r-}) \zeta_f^- - \delta_{FRI}^a(\xi_1^{r-})^2 V_Z^{FRIr}(\xi_1^{r-}) \quad (5.9)$$

$$\rho_f(\zeta_r^-) \triangleq \zeta_f^- = \delta_f^2(\xi_1^{f-}) \zeta_r^- - \delta_{FRI}^a(\xi_1^{f-})^2 V_Z^{FRI f}(\xi_1^{f-}) \quad (5.10)$$

Finally, composing Equations (5.9) and (5.10) it is possible to obtain the hybrid Poincaré function shown in Equation (5.11) and with domain of definition $\mathcal{D} := \{\zeta_f^- > 0 \mid \zeta_{f,r}(\zeta_f^-) \geq 0, \text{ and } \exists! \xi_1^{r-} \rightarrow \xi_1^{r,f}(\xi_1^{r-}) = 0, \ddot{\xi}_1^{r,f}(\xi_1^{r-}) < 0\}$ (note that hereinafter any explicit dependency on $\xi_1^{r,f-}$ will be omitted when not ambiguous).

$$\begin{aligned} \rho(\zeta_f^-) &= \rho_f \circ \rho_r(\zeta_f^-) = \\ &\delta_f^2 \delta_r^2 \delta_{\dot{\beta}}^2 \zeta_f^- - \delta_f^2 \delta_{FRI}^a(\xi_1^{r-})^2 V_Z^{FRIr}(\xi_1^{r-}) - \\ &\quad - \delta_{FRI}^a(\xi_1^{f-})^2 V_Z^{FRI f}(\xi_1^{f-}) \end{aligned} \quad (5.11)$$

With this Poincaré function it is possible to find the closed form of the fixed point, such that $\rho(\zeta_f^*) = \zeta_f^*$, as presented in Equation (5.12).

$$\zeta_f^* = - \frac{\delta_f^2 \delta_{FRI}^a(\xi_1^{r-})^2 V_Z^{FRIr}(\xi_1^{r-}) + \delta_{FRI}^a(\xi_1^{f-})^2 V_Z^{FRI f}(\xi_1^{f-})}{1 - \delta_f^2 \delta_r^2 \delta_{\dot{\beta}}^2} \quad (5.12)$$

Then, the limit cycle existence and stability constraints can be written as bellow:

C1.F The limit cycle for the restricted system exists for a given set of parameters if and only if $\zeta_{2f}^* > 0$ and $\zeta_f^* \in \mathcal{D}$;

C2.F If the limit cycle for the restricted system exists, it is asymptotically stable if and only if $0 < \delta_f^2 \delta_r^2 \delta_\beta^2 < 1$.

Although there is a closed form for the fixed point, it depends on the value of the reversal point. While the impact can be calculated *a priori*, since it depends only on the robot's configuration — that is known a priori for the restricted system if the constraints are parametrized by a Beziér polynomial — the reversion depends mainly on the post impact velocity of the system and, therefore, can not be calculated with only the configuration at the beginning or end of each phase.

To calculate the reversal point, the marginal condition $\zeta_r^- = 0$ is applied to Equation (5.9) together with the closed form of the fixed point ($\zeta_f^- = \zeta_f^*$), resulting in Equation (5.13).

$$\begin{aligned} Z(\xi_1^{r-}) &\triangleq (1 - \delta_f^2 \delta_r^2 \delta_\beta^2) \rho_r(\xi_1^{r-}) = \\ &= -\delta_r^2 \delta_\beta^2 \delta_{FRI}^a(\xi_1^{f-})^2 V_{\mathcal{Z}}^{FRI f}(\xi_1^{f-}) - \\ &\quad - \delta_{FRI}^a(\xi_1^{r-})^2 V_{\mathcal{Z}}^{FRI r}(\xi_1^{r-}) \end{aligned} \quad (5.13)$$

By definition $Z(\xi_1^{r-}) = 0$ if the system is halted for that value of ξ_1^{r-} . Furthermore, being $q_r^- = [\xi_1^{r-} ; q_{br}^-]$, where q_{br}^- is known from the Beziér parameters, then if $p_{cm}^h(q_r^-) > p_{FRI}^{h,d}(\xi_1^{r-})$, $\xi_1^{r-} > \xi_1^{r+}$ and $Z(\xi_1^{r-}) = 0$, ξ_1^{r-} is a reversal point.

Therefore, to find the reversal point one needs only to numerically find the zero of $Z(\xi_1)$. However, in order for the limit cycle in the zero dynamics to be also a limit cycle in the full model, the parameters need to be hybrid invariant. While the forward invariance can be easily enforced by the choice of feedback law, the transition invariance requires a changing in the first two columns of the Beziér parameters matrix b , in accordance to Equations (3.10) and (3.12) (with little change for the reversal invariance, where there is no impact).

The problem is that, to compute the transition invariance restrictions for the parameters, the values of ξ_1^{r-} and ξ_1^{f+} (that are equal due to the reversal description as continuous) are necessary, and the invariant parameters are needed so that the computation of $Z(\xi_1)$, and consequently the computed transition point, is valid not only for the restricted system but for the complete system as well.

In this work, to solve this problem, an algorithm of successive approximations was used in order to find ξ_1^{r-} , that is, the invariance is imposed on the parameters using an

initial guess of $\bar{\xi}_1^{r-}$, then the zero of $Z(\xi_1)$ was found for this set of parameters and the invariance was recalculated for the new value of ξ_1^{r-} .

Although the convergence conditions of this algorithm were not carefully delimited, the algorithm converged in less than 10 steps for every value of b that had an unique reversal point (the parameters were given by the Matlab *fmincon* function during parameter optimization). However, if the reversal point does not exists or if $\gamma_2 \neq 0$ does not hold for the hole search interval of ξ_1^{r-} , $Z(\xi_1)$ might not be defined for some points.

In this work this problem was solved by using the interval as described in section 4.4.2 and verifying if $\gamma_2(\xi_1) \neq 0$ holds for the hole interval.

6 PARAMETER OPTIMIZATION

With the results presented so far it is possible to write optimization algorithms for both the frontal and sagittal decoupled systems.

In this chapter the optimization problem will be presented together with the chosen cost function, all the relevant nonlinear constraints and references on where in this text each of them was dealt with.

6.1 Optimization Problem Formulation

The optimization problem consists in finding the set of parameters a_0 (for the sagittal gait) and b_0 (for the frontal gait) that minimizes the quadratic sum of joint torques of the restricted system, while respecting a set of nonlinear constraints NLCS01-06 (for the sagittal system) and NLCS01-07 (for the frontal system), at this point no restriction are imposed to the actuators.

Note that a_0 and b_0 are only the last $N - 2$ columns of the total parameters matrix, since the first two are completely defined by the transition invariance constraints. The notation a and b will be used to refer to the parameters matrix with these two first columns already calculated.

Note also that $a \in \mathbb{R}^{5 \times M_s}$ despite having six actuators, due to the fact that the FRI position was specified a priori to be zero along the step. Also, $b \in \mathbb{R}^{8 \times M_f}$ even if having only 4 actuators, because it has two continuous phases and, therefore, two set of parameters to optimize for each actuator.

More simplifications could be specified to decrease the numbers of optimization parameters for the frontal system. For example, it could be specified that the virtual constraints (not the FRI positions) were to be the same for the rising and falling phases, imposing a symmetrical pose for the controlled joints and decreasing the number of lines in the parameter matrix b from eight to five. The optimization was done both with and without

this restriction and it was observed that when restricted in this way the optimization algorithm converged more quickly to good solutions.

The cost function is the one presented in Equation (6.1) with the control vector u calculated as in (3.3) with $v \equiv 0$ and $T_I(\xi_2^-)$ calculated as in Equation (4.26).

$$J = \frac{1}{p_2^h(q^-)} \int_0^{T_I(\xi_2^-)} \|u(t)\|^2 dt \quad (6.1)$$

The nonlinear constraints for the sagittal system are presented below.

- NLC.S01) The conditions for existence and stability of the limit cycle in the hybrid zero dynamics as in **C1.S** and **C2.S**;
- NLC.S02) HH2 and HH4, that is, the invertibility of the decoupling matrix along the step;
- NLC.S03) HG2.S, that is, during the step the stance leg neither slips nor loses contact with the ground;
- NLC.S04) HG4.S, that is, the swing leg neither slips nor rebound during the impact and the vertical component of the post impact speed of the new swing leg is positive;
- NLC.S05) The desired step duration;
- NLC.S06) That both ends of the swing foot touch the ground simultaneously.

Bellow are the constraints for the frontal system.

- NLC.F01) The conditions for existence and stability of the limit cycle in the hybrid zero dynamics as in **C1.F** and **C2.F**;
- NLC.F02) The existence of the reversal point;
- NLC.F03) HH2 and HH4, that is, the invertibility of the decoupling matrix along the step;
- NLC.F04) HG2.F, that is, during the step the stance leg neither slips nor loses contact with the ground;
- NLC.F05) HG4.F, that is, the swing leg neither slips nor rebound during the impact and the vertical component of the post impact speed of the new swing leg is positive;
- NLC.F06) The desired step duration;

NLC.F07) Both ends of the swing foot touch the ground simultaneously.

Note that those are the minimum necessary constraints, but other could be added in order to achieve a desired walking style. For example, it could be specified that the knees should not turn forward in a non-human like way, or that the hip must be maintained above a minimum height at all times.

One of the most important constraints for this work is the desired time constraint. Since it is desired to apply both set of constraints to a 3D robot in order to obtain a 3D gait, it is necessary to assure that both nominal gaits are nominally synchronized.

It could be possible to join both set of constraints and cost functions in a single optimization algorithm and specify that both gaits must take the same amount of time instead of a fixed desired step time. However, individually, each optimization process was already computationally demanding, due to that and lack of time this was not tried in this work.

6.2 Optimization Results

In this section the resulting cost and parameters for each gait is presented. The discussion will be delayed to the next chapter, where the simulations are also presented. The order chosen for the Beziér polynomials were seven for both systems and the desired gait period was 0.7 seconds.

For the sagittal gait the final parameters matrix is the one presented, with 4 decimal places, in Equation (6.2).

$$a = \begin{bmatrix} 3.6782 & 3.7913 & 3.5480 & 3.3542 & 3.4202 & 3.3256 & 3.0809 & 3.0005 \\ 3.0005 & 2.9142 & 3.8201 & 2.8429 & 4.1239 & 3.8666 & 3.6480 & 3.6782 \\ -0.5022 & -0.9079 & -0.4677 & -0.3762 & -0.7069 & -0.5338 & -0.3568 & -0.3769 \\ -0.3769 & -0.4115 & -1.8166 & -0.0124 & -1.5363 & -0.8878 & -0.5829 & -0.5022 \\ -1.0528 & -0.9447 & -0.5396 & -1.3399 & -1.1257 & -1.5783 & -1.4894 & -1.6052 \end{bmatrix} \quad (6.2)$$

These parameters resulted in a cost of 7945.5506 for the restricted system. For a comparison, the initial gait given to the optimization algorithm disrespected the desired time constraint and the friction cone constraint of the stance foot along the step, and had a cost of 26713.5326. This means that the optimization algorithm not only found a

feasible solution from an unfeasible initial condition, but significantly decreased the gaits cost in the process.

For the frontal gait the resulting parameters matrix is the one presented, with 4 decimal places, in Equation (6.3).

$$b = \begin{bmatrix} 1.6726 & 1.6755 & 1.6962 & 1.6617 & 1.6621 & 1.6374 & 1.6672 & 1.6576 \\ 1.6726 & 1.6703 & 1.6756 & 1.6597 & 1.6272 & 1.5867 & 1.5409 & 1.6678 \\ 1.6726 & 1.6626 & 1.6435 & 1.6788 & 1.6766 & 1.7103 & 1.6411 & 1.6832 \\ 0.0005 & 0.0284 & 0.0917 & 0.0706 & 0.0687 & -0.0206 & 0.0069 & 0.0338 \\ 1.6576 & 1.6481 & 1.6374 & 1.6621 & 1.6617 & 1.6962 & 1.6768 & 1.6726 \\ 1.6678 & 1.7947 & 1.5867 & 1.6272 & 1.6597 & 1.6756 & 1.7042 & 1.6726 \\ 1.6832 & 1.7252 & 1.7103 & 1.6766 & 1.6788 & 1.6435 & 1.6747 & 1.6726 \\ 0.0338 & 0.0120 & 0.1168 & 0.0173 & -0.0881 & -0.0108 & -0.0532 & 0.0005 \end{bmatrix} \quad (6.3)$$

These parameters resulted in a cost of 4748.7133, compared to 16267.7787 of the initial parameters.

7 DECOUPLED GAIT SIMULATION RESULTS

In this chapter the simulation results for the decoupled gaits will be shown and discussed, with focus on the frontal gait and how the reversal point influences in the full model simulation. The first attempts of applying both controllers to a 3D robot will also be presented and discussed.

7.1 Sagittal Gait Results

In Figure 17 it is possible to compare, during the simulated gait, the actual versus the desired controlled joints angles. Notice that even during impact the invariance condition holds for all the angles, the FRI position is instantaneously away from zero during impact but still within the foot (that has limits in -0.06 meters and 0.1 meters for the frontal dynamics) .

The simulated gait resulted in a mean cost of 7807.4471 and a step time of 0.699716 seconds. In Figure 18 it is possible to view that the normal contact force of the contact foot with the ground is positive along the step and at the impact, and Figure 19 shows that the required transversal force was within the specified friction cone also both during the step and at the impacts.

Finally, Figure 20 shows a stick figure of the sagittal gait with a snap period of 0.1314 seconds.

With these results, the optimized sagittal gait is considered successfully generated.

7.2 Frontal Gait Results

Similarly to the sagittal gait, in Figure 21 it is possible to compare, during the simulated gait, the actual versus the desired controlled joints angles and FRI position.

There is a noticeable difference in the frontal plane results when compared to the

Figure 17: Sagittal Simulation Results - Desired and Simulated Joints Angles

Figure 18: Sagittal Simulation Results - Normal Contact Forces of the Stance Leg ($\times 100$)

Figure 19: Sagittal Simulation Results - Stance Leg Friction ($\times 100$)

Figure 20: Sagittal Simulation Results - Stick Figure

sagittal dynamics, as can be seen in the zoomed plots of Figure 22.

Notice from Figure 22 that in the q_{10} time evolution there is an instant in which the system leaves the zero dynamics. This effect occurs due to the approximation in the calculation of the reversal point.

In Figure 23 it is possible to verify that, for this gait, the estimation of the reversal point was actually very good, that is why this effect was slight and only noticeable for one of the joint angles.

In numbers, the successive approximations estimated that the reversion would occur for an ankle angle of -1.581175652049738 radians and the peaks of Figure 23 occurs at a mean of -1.581174025175828 with a standard deviation of $8.667411704406329 \cdot 10^{-6}$,

Figure 21: Frontal Simulation Results - Desired and Simulated Joints Angles

Figure 22: Frontal Simulation Results - Desired and Simulated Joints Angles

which is a very good approximation. However, for other set of parameters simulated there was often bigger errors between the estimated reversal point and the actual simulated one, which resulted in loss of invariance during reversion. The finite-time attractiveness of the system, however, was enough to quickly return it to the zero dynamics, without causing great problems for the gait.

The important observation to make is the necessity to saturate the Beziér polynomials

Figure 23: Frontal Simulation Results - Absolute Ankle Position

because of this difference. Since the polynomials were regressed for the variable s between 0 and 1 (that is, for ξ_1^r between ξ_1^{r+} and ξ_1^{r-}) if the variable exits, this region the polynomial can very rapidly increase its value. The way that this problem was solved in this work was by saturating the value of ξ_1^r before the calculation of the Beziér polynomials.

Furthermore, the simulated gait resulted in a cost of 4262.3768 and a step time of 0.6960. Although these two results are still very close to the predicted by the restricted system calculations, there is more difference for the frontal dynamics than for the sagittal. Again, this is attributed to the imprecision of the reversal point computation.

Note that the actual cost was lesser than the expected optimization cost of 4748.7133. This is due to the fact that, for this case, the actuation was less than expected to lead the system back to the zero dynamics. There is no guarantee that the practical system is more cost efficient for every gait, only that this difference tends to increase the more the system exits the zero dynamics.

In Figure 24 it is possible to view that the normal contact force of the contact foot with the ground is positive along the step and at the impact, and Figure 25 shows that the required transversal force was within the specified friction cone also both during the step and at the impacts.

Finally, Figure 26 shows a stick figure of the frontal gait for the first two steps (one with the left foot on the ground and one with the right) with a snap period of 0.007 seconds.

Figure 24: Frontal Simulation Results - Normal Contact Forces of the Stance Leg ($\times 100$) Figure 25: Frontal Simulation Results - Stance Leg Friction ($\times 100$)

Figure 26: Frontal Simulation Results - Stick Figure

With these results, the optimized frontal gait is considered successfully generated.

7.3 3D Gait First Attempts

With the good results obtained for both the sagittal and frontal gaits, the same controllers were applied to the 3D system (built with the same physical parameters). Figures 27 and 28 show, respectively, the sagittal and frontal absolute angles.

Note that, after a single step, the system overturns during the rising phase (due to how the simulation was implemented, if the rising phase do not ends, the impact condition is never checked and, thus, the system explodes). Despite that, Figures 29 and 30 show that the virtual constraints were still respected (considering the saturation described in section 7.2).

This overturn occurs because the frontal gait is sensible to disturbances, more specifically, the reversal point is sensible to disturbances. From Figure 28 it is evident that the first step for the frontal angles not only did not reach the precomputed reversal point, but also ended the step much earlier than expected by the sagittal gait.

Figure 27: 3D Simulation Results - Absolute Sagittal Angle

Figure 28: 3D Simulation Results - Absolute Frontal Angle

The main hypothesis for this difference in the coupled simulation versus the decoupled frontal simulation is that, since the sagittal gait is flexing the knees, the system's parameters are changing during the gait and affecting when the frontal gait inverts.

Figure 29: 3D Simulation Results - Desired and Simulated Sagittal Joints Angles

Figure 30: 3D Simulation Results - Desired and Simulated Frontal Joints Angles

Despite that, note that the first step had a step time of 0.6, very closely to the nominal step time. It is also highlighted that, despite the coupling effects from the sagittal in the frontal dynamics, the ankle angle reached fairly closely to the expected reversal point. Both observations suggest the need of little actuation to reject the coupling effect and

actually stabilize the gait.

This motivates the search for a way to stabilize the 3D gait, which is explored in the next chapter.

8 3D GAIT SYNCHRONIZATION CONTROL AND RESULTS

As presented in chapter 7, simply composing both sagittal and frontal controllers for the 3D system did not produce a stable 3D gait even if the impact applied was the simplified one presented in chapter 2.

This chapter presents a solution for this problem together with simulation results, and ends discussing other possible solutions that could render even better results but were not verifiable in the available time.

8.1 Synchronism Problem Formulation

The problem observed during the initial simulations of the 3D gait was the inability of the frontal plane control to deal with the disturbances introduced by the coupling of the movement with the frontal dynamics.

This can be interpreted as the influence of the knee in the frontal gait, that is, if the frontal gait ended before the sagittal one, the impact could occur when the lengths of the legs for the frontal plane were not the same. This happens when the sagittal gait is still ongoing, and therefore has the foot above the ground, but the frontal gait has already finished and thus is falling more sideways than what was pretended in the nominal gait.

This results in an impact with more energy than what was expected and in a few steps results in overturn in the frontal dynamics. This happens even for small variations on the step, that is, even if the first step is very close to the nominal step, the gait still becomes unstable after a few steps.

The virtual constraints, however, were still respected during all times. This led to the conclusion that simply composing both gaits without synchronizing them in any way resulted in an unstable gait in the zero dynamics.

To formulate the synchronizing control law, consider the decoupled zero dynamics

shown in Equation (8.1), where the difference between rising and falling parameters is only the set of parameters used.

$$\begin{aligned}
 \dot{\xi}_1^s &= \kappa_1^s(\xi_1^s)\xi_2^s \\
 \dot{\xi}_1^{r,f} &= \kappa_1^{r,f}(\xi_1^{r,f})\xi_2^{r,f} \\
 \dot{\xi}_2^s &= \kappa_2^s(\xi_1^s) \\
 \dot{\xi}_2^{r,f} &= \kappa_2^{r,f}(\xi_1^{r,f}) + u_\beta
 \end{aligned} \tag{8.1}$$

During this chapter, the labels r, f for *rising* and *falling* phases will be substituted by only f , for frontal gait. Consider that at each moment, if the robot is falling or rising in the frontal dynamics, the correct set of parameters will be applied.

Note that the feedback control law that imposed the desired FRI position was not applied for the frontal gait in Equation (8.1). Formulating the optimization while tracking the FRI position resulted in a stable gait for the decoupled frontal system that did not overturn the stance foot. However, the ankle actuator will be needed to synchronize the gaits and thus must be redesigned.

A natural question at this point is why, then, design the frontal gait independently from the sagittal gait in the first place, that is, why not describe the frontal virtual constraints as functions of ξ_1^s (including an extra virtual constraint for the frontal stance ankle) and optimize under the constraint that the frontal FRI position must be within the stance foot?

This was, indeed, the first approach of this work (since it is the most intuitive one), but, for the problem formulated in this way, the optimization algorithm did not obtain any feasible set of parameters. The author supposes that this is not a very good way of formulating the optimization problem for the algorithm.

Using the approach presented in the previous chapters, a set of virtual constraints for the frontal gait that already generates a stable gait while underusing the ankle actuator (since it was optimized for actuator effort) was obtained. Moreover, in the first simulations of the 3D gait, the difference in the nominal decoupled gaits and the simulated first step were not too big, indicating that little change in the frontal ankle torque outline during the gait might render the gait stable.

8.1.1 Synchronizing Control Law

The idea of this controller is to parametrize the nominal trajectory of ξ_1^f by the nominal trajectory of ξ_1^s . The graph obtained plotting those simulated trajectories is presented in Figure 31.

Figure 31: Graph of the nominal trajectory of the frontal absolute angle in function of the sagittal absolute angle

With this it is possible to regress a Beziér polynomial for $h_{\xi_1^f}^d(\xi_1^s)$ and define for the system presented in Equation (8.1) the output as $h_{zd}([\xi_1^s; \xi_1^f]) = \xi_1^f - h_{\xi_1^f}^d(\xi_1^s)$.

Generating the new virtual constraint in a similar way as it was done in Chapter 3, results in the control law presented in Equation (8.2).

$$u_\beta = -\frac{1}{\kappa_1^f(\xi_1^f)}(T_1 + T_2 + T_3 + T_4 + v) \quad (8.2)$$

where,

$$\begin{aligned}
T_1 &= - \left(\frac{\partial^2 h_{\xi_1^f}^d}{(\partial \xi_1^s)^2} + \frac{\partial h_{\xi_1^f}^d}{\partial \xi_1^s} \frac{\partial \kappa_1^s}{\partial \xi_1^s} \right) \kappa_1^s (\xi_2^s)^2 \\
T_2 &= \frac{\partial \kappa_1^f}{\partial \xi_1^f} \kappa_1^f (\xi_2^f)^2 \\
T_3 &= - \frac{\partial h_{\xi_1^f}^d}{\partial \xi_1^s} \kappa_1^s \kappa_2^s \\
T_4 &= \kappa_1^f \kappa_2^f \\
v &= \frac{1}{\epsilon^2} K_p (\xi_1^f - h_{\xi_1^f}^d(\xi_1^s)) + \frac{1}{\epsilon} \left(\dot{\xi}_1^f - \frac{\partial h_{\xi_1^f}^d}{\partial \xi_1^s} \dot{\xi}_1^s \right)
\end{aligned}$$

Finally, this controller, although synchronizes the gait, can cause foot overturn. To solve this, a simple saturation is added to the system. To saturate the ankle torque it is calculated the necessary ankle torque in order for the FRI to be at each edge of the foot at each iteration and limit the torque between those values.

8.2 Final 3D Simulation Results

The proposed control resulted in a intuitively successful gait, although a periodic limit cycle was not characterized at a single step, as per design. In Figures 32 and 33 it is possible to see the desired and simulated joints positions during the steps.

Note that at almost every step the frontal stance ankle angle (q_7 or β) surpasses the calculated reversal point, indicating that even the proposed controller did not successfully synchronized the gaits completely.

Note also that the sagittal angles are relatively well behaved when compared to the frontal angles, this fortifies even further the hypothesis that the sensibility of the frontal dynamics was responsible for the observed behavior of the system.

As with the impact dynamics, the contact forces need the unpinned model to compute and were not verified. The main argument to justify this is that the Matlab simulation was never intended to be the final means of validation, but rather a first validation that was easy to debug.

Finally, although omitted in the text, the system was simulated for 100 seconds and walked successfully during the whole simulation.

Figure 32: 3D Robot Final Results - Desired and Simulated Sagittal Joints Angles

Figure 33: 3D Robot Final Results - Desired and Simulated Frontal Joints Angles

9 CONCLUSIONS AND FUTURE WORKS

The main goal of this project was to propose a method to generate 3D, stable and energy efficient bipedal gaits using the hybrid zero dynamics framework. In this chapter are presented a brief review of the proposed method, an analysis of the results obtained and suggestions of future works that could further improve the method proposed by this work.

9.1 Method Review

The proposed method consists in applying decoupled gait designs for the sagittal and frontal 2D restricted systems simultaneously in a 3D bipedal robot in order to generate a stable 3D walking gait.

The proposed framework is the hybrid zero dynamics that consists in applying virtual constraints to a system via feedback linearization law and, thus, restricting the robot to a lower dimensional manifold, where the closed form solution of the system is known.

The hybrid zero dynamics is vastly used to generate 2D sagittal walking gaits, however the frontal locomotion has the inherent problem of movement reversion during the step, and, to the best of the author's knowledge, there is no similar work in the literature where hybrid zero dynamics was used to generate a stable walking gait for the frontal dynamics.

To deal with the reversion of movement in the frontal dynamics, the modeling of its dynamics as a two continuous phase hybrid system was used. Separating the frontal movement in rising, reversal, falling and impact made possible to parametrize the system's controlled variables by an absolute angle of the system and, thus, restrict it to a hybrid zero dynamics, despite the fact that said absolute angle is not strictly increasing or decreasing along the whole step but only along each continuous phase.

The singularity of the zero dynamics during the reversal was dealt with and the Poincaré functions of both the sagittal and frontal dynamics. They were presented to-

gether with the closed form of the fixed point of the Poincaré functions and with the conditions of existence and stability of the limit cycle in the zero dynamics.

After presenting the optimization problem formulation and the simulation results for the sagittal and frontal systems, the problem of generating stable 3D bipedal locomotion was approached. It was shown that simply composing both controllers in a 3D walker was not enough to generate a stable gait.

A simple solution for this problem, synchronizing both the frontal and sagittal absolute angles via feedback control law in the zero dynamics resulted in a stable gait, although the periodicity of the gait was not a single step.

9.2 Results Analysis

This work had two major goals: first the proposal of a method to apply the hybrid zero dynamics framework (common in the literature for sagittal gaits) to design a stable walking gait for the frontal dynamics; and second, the design of a 3D stable gait using gaits designed using the decoupled sagittal and frontal models.

The first goal presented very good results despite the problems introduced by the fact that the frontal gait has a reversion in its spatial evolution, the constraints were respected and the specifications for the restricted system did present themselves as good predictors for the same specifications for the full system.

It was, however, detected some sensibility with regard to the reversal point. In this stage the main problem was that the numeric imprecisions resulted in a brief loss of invariance after the reversion, but even so, the results are considered more than satisfactory.

The second goal did not presented as good results as the first. The first approach of simply applying both decoupled planar control laws into the full system did not generated a stable gait due to the system passing the frontal potential energy barrier.

This, actually, highlights an important difference between the sagittal and frontal gaits: while the sagittal gait must be sure to always have energy after the step to surpass the potential energy barrier, the frontal gait must assure the exact opposite.

The sensitivity of the frontal gait to parameter variation was verified in the last month of this work and was not completely addressed. Even so, the proposed controller did solve the overturn of the 3D system, while still assuring that the FRI point remains strictly inside the support foot. However, it is not possible to certify that the resulting gait was

stable as defined in this work, since the limit cycle was not well defined.

Moreover, the system was not completely validated due to simplifications in the impact dynamics and to the fact that the contact forces were not verified for the 3D system. Due to these observations, the second goal was not considered satisfactory.

Even so, the presented results are promising and some other strategy might render good enough results that justify the use of better simulation software.

9.3 Future Works

As said in the previous section, most of suggested future work of this project relates to the fact that the composition of both planar controllers results in the overturn of the robot about the frontal stance ankle.

The first proposed approach to solve this problem resides in the sensitivity of the frontal gait's reversal point to disturbances. The seemingly most promising way to deal with this problem is to insert the knees to the frontal model as linear actuators. This would allow the design of a stable gait without relying to the ankle torque, which would be available to enforce the nominal reversal point against mild disturbances.

The addition of a knee as a linear actuator, however, would generate a coupling between the frontal and sagittal gaits design, this would require the simultaneous optimization of both gaits and there would be no guarantees *a priori* that there is a constraints to the knees that would generate stable walking gait for both the frontal and sagittal gaits simultaneously without the use of the stance ankle.

The fact, however, that initial simulations of the 3D system without synchronizing control resulted in overturn about the frontal stance ankle indicates that the knee constraints that generate stable walking gait for the sagittal system also injects energy to the frontal one, rendering the use of the ankle to do that redundant.

It would also be worth exploring the possibility to use a 3D model feedback linearization control law to enforce the virtual constraints obtained for the planar systems. In this work the virtual constraints were enforced considering the decoupled planar systems, but the use of the coupled model to do so might reduce the coupling effect on each plane and, thus, allow for a stable gait to arise.

Other works could also explore the theoretical deficiencies in the frontal gait design, more specifically searching for conditions for the uniqueness of the reversal point and for

$\gamma_2 \neq 0$. Both these conditions were numerically verified for the interval of existence of the reversal point but this is computationally consuming and the formulation of clear conditions would quicken the optimization process.

Furthermore, this work did not suppose that the frontal FRI position was constant during each continuous phase, but this supposition could greatly simplify the calculations of the frontal zero dynamics restrictions while still generating a stable gait if the position of the FRI during the rising and falling phases differ.

Finally, this work still lacks better validation due to the poor results obtained for the 3D system. If any of the suggestions mentioned here results in a well-defined periodically stable 3D gait, then the use of better simulators and, eventually, a real robot would present the necessary validations for this method.

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APPENDIX A – LAGRANGIAN METHOD FOR PLANAR SYSTEMS

During this section it is presented the basic concepts about the Lagrangian method restricted to planar open kinematic chains with only rotational joints.

The world frame (a.k.a reference frame) is positioned at the contact of the first joint to the ground (being this contact pinned or not) with the x axis pointing up (anti-parallel to the gravity vector) and with the y axis transversal to the gravity pointing to the right. Because of hypothesis HG4 even when the unpinned model is necessary the stance leg is still in contact with the ground and so the world frame is still completely defined.

A.1 The Energy of a Single Link

Suppose a single link i , with a reference point ${}_0p_i = ({}_0p_i^v, {}_0p_i^h) \in \mathbb{R}^2$ measured according to the world frame and in solidarity with the link, and with a reference frame centered in ${}_0p_i$ with the x axis parallel to the length of the link and the y axis perpendicular to it.

Let θ_i^{abs} be the angle between the world frame x axis, hereinafter referred to as ${}_0x$ axis, and the link frame x axis, hereinafter referred to as ${}_ix$. θ_i^{abs} is called the *absolute angle* of the i -th link since it is measured with reference to the absolute frame. All angles are supposed positive in the counterclockwise direction.

So, any point ${}_ip_k$ referenced in the link frame can be expressed in the inertial frame (${}_0p_k$) according to Equation (A.1).

$$\begin{bmatrix} {}_0p_k^v \\ {}_0p_k^h \end{bmatrix} = \begin{bmatrix} {}_0p_i^v \\ {}_0p_i^h \end{bmatrix} + \underbrace{\begin{bmatrix} \cos(\theta_i^{abs}) & \sin(\theta_i^{abs}) \\ -\sin(\theta_i^{abs}) & \cos(\theta_i^{abs}) \end{bmatrix}}_{R(\theta_i^{abs})} \begin{bmatrix} {}_ip_k^v \\ {}_ip_k^h \end{bmatrix}, \quad (\text{A.1})$$

and its velocity from the link frame to the inertial frame can be expressed by Equation (A.2).

$$\begin{bmatrix} {}_0\ddot{p}_k^v \\ {}_0\ddot{p}_k^h \end{bmatrix} = \begin{bmatrix} {}_0\dot{p}_i^v \\ {}_0\dot{p}_i^h \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} R(\theta_i^{abs}) \begin{bmatrix} {}_i\dot{p}_k^v \\ {}_i\dot{p}_k^h \end{bmatrix} \dot{\theta}_i^{abs} \quad (\text{A.2})$$

If the link is free, its configuration and velocity is said to have three degrees of freedom — ${}_0p_i^v$, ${}_0p_i^h$ and θ_i^{abs} .

Particularly this relations are valid for the center of mass of the link — ${}_ip_{cm,i} \rightarrow {}_0p_{cm,i}$. With the position and speed of a link's center of mass it is possible to calculate its energies. As said, the gravity is supposed pointing downwards and parallel to the x_0 axis, so the potential and kinetic energies of the link are given by Equations (A.3) and (A.4).

$$V_i = m_i \vec{g} \cdot {}_0p_{cm,i}, \quad (\text{A.3})$$

$$K_i = \frac{m_i ({}_0\dot{p}_k^v)^2 + m_i ({}_0\dot{p}_k^h)^2 + J_{cm,i} (\dot{\theta}_i^{abs})^2}{2}, \quad (\text{A.4})$$

where: m_i is the link's mass; $J_{cm,i}$ is its moment of inertia about its center of mass; and $\vec{g}$ is the gravitational vector.

For a set of N links, each one's potential and kinetic energy is summed to the total energy of the system, resulting in a function with $3N$ configuration variables and $3N$ velocities. However, a joint imposes constraints in the configuration variables, meaning that the system's total energy can be expressed with fewer variables.

For example, a single link, when free in space has 3 degrees of freedom, but if pinned to the ground by a rotational joint, both ${}_0p_i^v$ and ${}_0p_i^h$ are constant, meaning that the system's energy can be expressed in terms only of θ_i^{abs} .

A.2 Energy of a Free Open Kinematic Chain

Consider now an open kinematic chain composed by N links and $N - 1$ revolute joints, both enumerated from 1 to N and from 1 to $N - 1$, respectively, and that a given link i has its frame of reference's origin in the $i - 1$ joint and that the first link frame of reference's origin is arbitrary (later on, it will be chosen as the contact point with the ground).

Consider also that for a given joint k , ${}_jp_k$ is the position of this joint with respect to

link j . Also, being i and j two consecutive links, i.e., two links sharing the same joint k , then Equation (A.5) can be written.

$$\begin{bmatrix} {}_0p_i^v \\ {}_0p_i^h \end{bmatrix} + R(\theta_i^{abs}) \begin{bmatrix} {}_ip_k^v \\ {}_ip_k^h \end{bmatrix} = \begin{bmatrix} {}_0p_j^v \\ {}_0p_j^h \end{bmatrix} + R(\theta_j^{abs}) \begin{bmatrix} {}_jp_k^v \\ {}_jp_k^h \end{bmatrix} \quad (\text{A.5})$$

Equation (A.5) imposes two constrains on the system and it can be written for each joint, resulting in $2N - 2$ constrains to the original $3N$ variables system. Therefore, imposing the joints as constrains, the resulting equations have $N + 2$ variables — each link's angle and the first joint frame's origin.

Therefore, any point in a given link's coordinate frame can be expressed in the world frame by a function of these $N + 2$ variables. Furthermore, for any joint i and link j , the center of mass of link j expressed in its frame (${}_jp_{cm,j}$) can be expressed in the world reference frame according to equation (A.6)

$$\begin{bmatrix} {}_0p_{cm,j}^v \\ {}_0p_{cm,j}^h \end{bmatrix} = \begin{bmatrix} {}_0p_i^v \\ {}_0p_i^h \end{bmatrix} + \Phi(\theta_1^{abs}, \theta_2^{abs}, \dots, \theta_N^{abs}) = {}_0p_{cm,j}(\theta_1^{abs}, \theta_2^{abs}, \dots, \theta_N^{abs}, {}_0p_i^v, {}_0p_i^h), \quad (\text{A.6})$$

where Φ is the recurrent function that changes the center of mass from link j to link $i + 1$. Being able to calculate every joint's center of mass, it is immediate the calculation of the potential energy of the system presented in Equation (A.7)

$$V = \sum_{i=1}^N V_i = \sum_{i=1}^N m_i \vec{g} \cdot {}_0p_{cm,i}^v \quad (\text{A.7})$$

For the system's kinetic energy, consider $\theta = [\theta_1^{abs}, \theta_2^{abs}, \dots, \theta_N^{abs}, {}_0p_i^v, {}_0p_i^h]^\top$ the generalized coordinates vector, then:

$$\begin{bmatrix} {}_0\dot{p}_{cm,j}^v \\ {}_0\dot{p}_{cm,j}^h \end{bmatrix} = \left(\frac{\partial}{\partial \theta} \begin{bmatrix} {}_0p_{cm,j}^v \\ {}_0p_{cm,j}^h \end{bmatrix} \right) \dot{\theta}$$

And so, the system's kinetic energy is given by Equation (A.8)

$$K = \sum_{i=1}^N K_i = \sum_{i=1}^N \left(\frac{m_i({}_0\dot{p}_{cm,i}^v)^2 + m_i({}_0\dot{p}_{cm,i}^h)^2 + J_{cm,i}(\dot{\theta}_i^{abs})^2}{2} \right) \quad (\text{A.8})$$

Note that if the system is a pinned open kinematic chain, the process is the same with

the addition of two extra constraints that fixes the values of ${}_0p_i$ and so results in a system with only N variables.

For more details about dynamic modeling of multibody systems please refer to (CRAIG, 2005) or (WESTERVELT et al., 2007) appendix B.

A.3 The Lagrangian

The Lagrangian is the function defined by the difference between the kinetic and the potential energy of a system, as presented in Equation (A.9).

$$\mathcal{L}(q, \dot{q}) := K(q, \dot{q}) - V(q), \quad (\text{A.9})$$

where $q \in \mathbb{R}^N$ and $\dot{q} \in \mathbb{R}^N$ are the system's configuration variables and velocities respectively. Then, using the Lagrange's equation, it is possible to calculate the expression of the robot's generalized torques and forces as in Equation (A.10)

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \Gamma, \quad (\text{A.10})$$

where Γ is the vector of generalized forces and torques. With this expression, it is possible to write this equation in the reduced form as shown in (A.11).

$$D(q)\ddot{q} + C(q, \dot{q}) + G(q) = \Gamma \quad (\text{A.11})$$

With this form, it is simple to write the system in the state space form following Equation (A.12).

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ f &= \begin{bmatrix} \dot{q} \\ D(q)^{-1}(-C(q, \dot{q}) - G(q)) \end{bmatrix} \\ g &= \begin{bmatrix} 0_{N \times M} \\ D(q)^{-1}B(q) \end{bmatrix} \\ B(q)u &= \Gamma \end{aligned} \quad (\text{A.12})$$

where $u \in \mathbb{R}^M$ is the vector of control inputs, $B \in \mathbb{R}^{N \times M}$ is the matrix that relates the

control input to the generalized torques and forces and $x = [q; \dot{q}]$ is the state space vector.

With this method the pinned and unpinned models of the 3-link robot was calculated.