BRUNO FACCINI SANTORO

DEVELOPMENT OF A STOCHASTIC MODEL PREDICTIVE CONTROLLER FOR PROCESSES IN THE CHEMICAL INDUSTRY

Tese apresentada à Escola Politécnica da Universidade de São Paulo para obtenção do título de Doutor em Engenharia

São Paulo

2015

BRUNO FACCINI SANTORO

DEVELOPMENT OF A STOCHASTIC MODEL PREDICTIVE CONTROLLER FOR PROCESS IN THE CHEMICAL INDUSTRY

Tese apresentada à Escola Politécnica da Universidade de São Paulo para obtenção do título de Doutor em Engenharia

Área de concentração: Engenharia Química

Orientador: Prof. Dr. Darci Odloak

São Paulo
2015

Este exemplar foi revisado e alterado em relação à versão original, sob responsabilidade única do autor e com a anuência de seu orientador.

São Paulo, 14 de julho de 2015

Assinatura do autor

Assinatura do orientador

Catalogação-na-publicação

Santoro, Bruno Faccini
DEVELOPMENT OF A STOCHASTIC MODEL PREDICTIVE CONTROLLER FOR PROCESSES IN THE CHEMICAL INDUSTRY / B. F. Santoro -- São Paulo, 2014.
143 p.

Tese (Doutorado) - Escola Politécnica da Universidade de São Paulo. Departamento de Engenharia Química.

1.Controle de Processos 2.Controle Preditivo 3.Processos Estocásticos I.Universidade de São Paulo. Escola Politécnica. Departamento de Engenharia Química II.t.

AGRADECIMENTOS

Por muitas vezes nos últimos anos li os agradecimentos das mais diversas pessoas em suas teses e dissertações. É evidente que a maioria delas aplica fórmulas prontas, conforme certos campos de inspiração: religioso ("Obrigado a Deus por tudo"), protocolar ("A todos aqueles que, direta ou indiretamente..."), apaixonado ("Ao amor da minha vida e luz dos meus dias"), culpado ("Desculpas à minha família pelo afastamento"), entre outros. A questão que se coloca é: como demonstrar sinceramente toda a gratidão por chegar ao final deste processo?

Obrigado em primeiro lugar à minha família (aos membros de sangue e à pequena incorporação mais recente) por aceitar, encorajar, alimentar, encantar e acreditar.

Agradeço ao Prof. Darci Odloak, pela oportunidade que me foi oferecida e pelo exemplo durante todo esse período. A contribuição dos membros da banca de qualificação, Prof. Cláudio Garcia, Prof. Luís Cláudio Lopes e Prof. Galo Le Roux também foi muito apreciada.

Gracias a Alejandro y a Antonio, por la buenísima recepción, por el ejemplo de dedicación y por los lisos.

Devo lembrar do mestre Tsunediro Takahashi. Nenhum trabalho deste porte seria possível sem ter garra. Mesmo após muitos anos, o senhor foi uma inspiração quando precisei acreditar no que não parecia possível.

Um reconhecimento sincero a todos os amigos do laboratório: David, por compartilhar sua visão de mundo e estar sempre disposto a ouvir; André, pelas preciosas lições sobre paciência e relaxamento; Zé, por todos os eventos realizados com sucesso; Moisés, pelo inconformismo diante de um mundo que não faz sentido; Márcio, pelas conversas proveitosas e ao John por ter ficado conosco o máximo que pôde. A lembrança se estende a todos os demais: José Otávio, Álvaro, Aldo, Bruno, Misagh, Michele, Patricia, Veronica, Fernando, Mário, entre outros.

Finalmente, cumpre agradecer o apoio financeiro do CNPq através do processo de número 141418/2011-7 e a todos aqueles que, direta ou indiretamente, não atrapalharam o desenvolvimento desta tese.

“Tudo é mentira neste mundo onde se pensam coisas” (Álvaro de Campos)

ABSTRACT

The success of Model Predictive Control (MPC) strategies in industrial and academic environments in the last decades has been remarkable. However, there are many open questions in the area, especially when the simplifying hypothesis of perfect model is dropped. The explicit consideration of uncertainties lead to important progresses in the area of robust control, but it still exhibits a few drawbacks: high computational load and over conservative behavior are issues that may have hindered the application of robust strategies in practice.

The approach of Stochastic Model Predictive Control (SMPC) aims at the reduction of conservativeness due to the incorporation of statistical information about noise. Since processes in chemical industry are always subject to disturbances, resulting from model-plant mismatch or from unmeasured disturbances, this technique is an interesting alternative for future control algorithms.

The main objective of this thesis is the development of SMPC algorithms that take into account some of the specificities of such processes, which have not been adequately handled in the literature so far. The most important contribution is the inclusion of integral action in the controller through a velocity description of the model. Besides, hard input constraints – associated with safety or physical limits – and probabilistic state constraints – usually derived from product specification - are also included in the formulation. Two approaches were followed in this work, the first is more direct and the second provides closed-loop stability guarantee at the price of increased conservativeness.

Another interesting feature that is developed in this thesis is the zone control of systems subject to disturbances. This form of control is often present in industrial arrays due to the lack of degrees of freedom, and the proposed approach is the first to merge zone control and SMPC. Different simulations of all proposed controllers and comparison to literature benchmarks are provided to show the application potential of the developed techniques.

Keywords: Process Control, Model Predictive Control, Stochastic processes

RESUMO

O sucesso de estratégias de controle preditivo baseado em modelo (MPC, na sigla em inglês) tanto em ambiente industrial quanto acadêmico tem sido marcante. No entanto, ainda há diversas questões em aberto na área, especialmente quando a hipótese simplificadora de modelo perfeito é abandonada. A consideração explícita de incertezas levou a importantes progressos na área de controle robusto, mas esta ainda apresenta alguns problemas: a alta demanda computacional e o excesso de conservadorismo são questões que podem ter prejudicado a aplicação de estratégias de controle robusto na prática.

A abordagem de controle preditivo estocástico (SMPC, na sigla em inglês) busca a redução do conservadorismo através da incorporação de informação estatística dos ruídos. Como processos na indústria química sempre estão sujeito a distúrbios, seja devido a diferenças entre planta e modelo ou a distúrbios não medidos, está técnica surge como uma interessante alternativa para o futuro.

O principal objetivo desta tese é o desenvolvimento de algoritmos de SMPC que levem em conta algumas das especificidades de tais processos, as quais não foram adequadamente tratadas na literatura até o presente. A contribuição mais importante é a inclusão de ação integral no controlador através de uma descrição do modelo em termos de velocidade. Além disso, restrições obrigatórias (*hard*) nas entradas – associadas a limites físicos ou de segurança – e restrições probabilísticas nos estados – normalmente advindas de especificações de produtos – também são consideradas na formulação. Duas abordagens foram seguidas neste trabalho, a primeira é mais direta enquanto a segunda fornece garantias de estabilidade em malha fechada, contudo aumenta o conservadorismo.

Outro ponto interessante desenvolvido nesta tese é o controle por zonas de sistemas sujeitos a distúrbios. Essa forma de controle é comum na indústria devido à falta de graus de liberdade, sendo a abordagem proposta a primeira contribuição da literatura a unir controle por zonas e SMPC. Diversas simulações de todos os controladores propostos e comparações com modelos da literatura são exibidas para demonstrar o potencial de aplicação das técnicas desenvolvidas.

Palavras-chave: Controle de processos, Controle preditivo baseado em modelo, processos estocásticos.

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF ACRONYMS

CSTR	Continuous stirred-tank reactor
ISE	Integral Square Error
LB	Lower Bound
LQR	Linear Quadratic Regulator
MPC	Model Predictive Control
MSRCI	Maximum Stochastic Robust controlled invariant (set)
SMPC	Stochastic Model Predictive Control
SRCI	Stochastic Robust controlled invariant (set)
UB	Upper Bound
ZMPC	Zone Model Predictive Control

NOMENCLATURE

0	Null matrix of any dimension
$\mathbf{1}_x$	Vector of ones with x components
A	Generic chemical reactant
A_p	Transition matrix in positional form
A_v	Transition matrix in velocity form
A_Ψ	Transition matrix of the extended state Ψ
A_{CL}	Closed-loop transition matrix
$\bar{A}$	Extended transition matrix (forced component)
$\bar{\bar{A}}$	Extended transition matrix (autonomous component)
B	Generic chemical product
B_p	Input matrix in positional form
B_v	Input matrix in velocity form
B_Ψ	Input matrix of the extended state Ψ
$\bar{B}$	Extended transition matrix (forced component)
$\bar{\bar{B}}$	Extended transition matrix (autonomous component)
$\bar{B}_{i,\Psi}$	Input matrix of the extended state Ψ for predictions i steps ahead
C_p	Output matrix in positional form
C_v	Output matrix in velocity form
$\bar{C}$	Extended output matrix (forced component)
$\bar{\bar{C}}$	Extended output matrix (forced component)
$\tilde{c}$	Vector used in the polyhedral approximation of S_{SRCI}
d	Affine component of the input parameterization
$\bar{D}$	Aggregation of affine components
$\tilde{D}$	Feasible solution of $\bar{D}$
$\mathbb{E}$	Expected value function
F_p	Noise matrix in positional form
F_v	Noise matrix in velocity form
F_Ψ	Noise matrix of the extended state Ψ

$\bar{F}$	Extended transition matrix (forced component)
$\bar{\bar{F}}$	Extended transition matrix (autonomous component)
$\bar{F}_{i,\Psi}$	Noise matrix of the extended state Ψ for predictions i steps ahead
$\mathbb{F}$	Cumulative distribution function
$\mathbb{F}^{-1}$	Left quantile function
g	Affine parameter defining the state constraint
G	Feedback component of the input parameterization
$\bar{G}$	Aggregation of feedback components
$\bar{G}^{+}$	Positive part of $\bar{G}$
$\bar{G}^{-}$	Negative part of $\bar{G}$
$\tilde{G}$	Feasible solution of $\bar{G}$
h	Affine parameter defining the state constraint
I_x	Identity matrix of dimension x
J	Weighted integral square error
k	Time step
K	Feedback gain of the dual controller
K_s	Dual controller feedback gain in the strategy with recursive feasibility guarantee
m	Control horizon
M	Matrix to sum control moves to find input values
$nsim$	Simulation horizon
nu	Number of inputs
nw	Number of process noises
nx	Positional state dimension
ny	Number of outputs
p	Prediction horizon
P	Terminal weight
$\mathbb{P}$	Probability
Q	Penalty matrix corresponding to the outputs
$\bar{Q}$	Extended output penalty matrix (forced component)

$\bar{\bar{Q}}$	Extended output penalty matrix (autonomous component)
$\bar{Q}_u$	Extended input target penalty matrix
Q_λ	Tuning matrix related to the offset function
R	Penalty matrix corresponding to the inputs
$\bar{R}$	Extended input penalty matrix (forced component)
$\bar{\bar{R}}$	Extended input penalty matrix (autonomous component)
S_{SRCI}	Stochastic robust controlled invariant set
$\tilde{S}$	Matrix used in the polyhedral approximation of S_{SRCI}
T_i	i-eth component of the objective function
u	Process input
u_{max}	Input upper bound
U_{max}	Aggregation of input upper bounds
u_{min}	Input lower bound
U_{min}	Aggregation of input lower bounds
u_{tar}	Artificial input target
$\bar{U}_{prev}$	Aggregation of input values in the previous time step
$\bar{U}_{tar}$	Aggregation of artificial input targets
$\mathfrak{U}$	Convex set of input constraints
$\mathrm{V}_{\mathrm{aut}}$	Autonomous component of the objective function
$\mathrm{V}_{\mathrm{for}}$	Forced component of the objective function
V_{O}	Offset function
w	Process noise
$weight$	Normalization weight
w_{max}	Maximum noise value
w_{min}	Minimum noise value
$\bar{W}$	Aggregation of predicted noise
$\bar{W}_{\max}$	Aggregation of noise upper bounds
$\bar{W}_{\min}$	Aggregation of noise upper bounds
x_p	State in positional form
x^{s}	State component corresponding to the predicted steady state

x_{tar}	Artificial state target
x_v	State in velocity form
$x_{v,tar}$	Artificial velocity state target
$\mathfrak{X}$	Set of the previous state observations
$\bar{X}$	Aggregation of future states
y	Process output
y_{max}	Maximum output value defining the zone
y_{min}	Minimum output value defining the zone
y_{sca}	Scaled output
y^{sp}	Set-point
y^{tar}	Artificial set-point
$\bar{Y}_{tar}$	Aggregation of artificial set-points (forced component)
$\bar{\bar{Y}}_{tar}$	Aggregation of artificial set-points (autonomous component)

Symbols including Greek alphabet

α	Confidence level
Γ	Set containing the desirable output values in the zone control
δ_y	Slack variable of the zone MPC
Δu	Input variation (control action)
Δu_{max}	Control action bound
Δw	Noise variation
Δ_w	Upper bound of the 1-norm of w
$\Delta_{\Delta w}$	Upper bound of the 1-norm of Δw
$\Delta\bar{U}$	Aggregation of future control moves (forced component)
$\Delta\bar{\bar{U}}$	Aggregation of future control moves (autonomous component)
$\Delta\bar{W}$	Aggregation of future noise variations (forced component)
$\Delta\bar{\bar{W}}$	Aggregation of future noise variations (autonomous component)
λ	Epigraph of the offset function
μ	Mean value (of output or input)

$\mu^{\Delta w}$	Mean of a particular noise realization
$\mu^{\Delta w \Delta w}$	Covariance of a particular noise realization
$\bar{\bar{\mu}}^{\Delta w \Delta w}$	Covariance of a particular noise realization in the dual controller mode
$\mu_{ij}^{\Delta w \Delta w}$	Subcomponents of $\mu^{\Delta w \Delta w}$ (i,j = 0, 1, 2)
μ_w	Noise mean
π	Generic control policy
σ	Standard deviation (of output or input)
$\Sigma_{\bar{X}}$	Covariance of predicted state
Σ_{UU}	Covariance of control actions in the dual controller mode
Σ_{UX}	Cross covariance of state and control actions in the dual controller mode
Σ_w	Noise covariance
Σ_{WU}	Cross covariance of noise and control actions in the dual controller mode
Σ_{WX}	Cross covariance of noise and state in the dual controller mode
Φ	Feedback function
Ψ	Extended state used in the calculation of the invariant set
Ω	Invariant set

Superscript

^{T}	Matrix transpose

CONTENTS

1. Introduction

1.1. Motivation

In the last 40 years, model predictive control (MPC) has evolved from an industrial heuristic approach to a mature, well-based theory. From a practical standpoint, its use nowadays is widespread in chemical industries, especially oil refining and petrochemicals. On the other hand, from a theoretical point of view, many important stability questions have been addressed and solved (Mayne et al., 2000). Necessary and sufficient conditions for stability of nominal controllers are completely understood.

One immediate consequence of this effort was the idea to extend the stability results to uncertain models, since there is always some degree of plant-model mismatch, which can seriously deteriorate the performance of the closed-loop system. The earliest attempts to improve robustness of MPC date back to the 1990s and usually considered that the parameters describing the plant belong to a bounded set. This hypothesis allows worst-case analyses, which have been used to guarantee stability and feasibility but may be over conservative in terms of performance.

More recently, there is a trend to incorporate statistical knowledge about disturbances in MPC, reducing conservativeness without risking process safety. Most of the literature so far considers only the regulator problem, with the work in Couchman et al. (2006) being an exception, as they present an algorithm to track a set-point when the output is corrupted by noise but state evolution is deterministic. The referred work does not handle the more general problem of tracking when the state is a random variable, which is one of the main targets of the present work.

In Hokayem et al. (2012), it is presented an algorithm to reject disturbances when the state is subject to additive Gaussian noise and there are hard bounds in the inputs. Korda et al. (2011) proposed a method to handle soft constraints with feasibility guarantees.

This thesis is inspired by these two last proposals to address the case of output-tracking through the use of an incremental (velocity) description of the system, including hard bounds in control moves and also dealing systematically with state constraints.

1.2. Objectives

The main objective of the present thesis is to develop a predictive controller, suitable to the needs and reality of the chemical process industry. Some of the requirements of this controller are the following:

- Track different set-points, corresponding to changing operating conditions;
- On-line computation effort compatible with typical process sampling time (approximately 1 minute);
- Stability guarantees in order to ensure safety to the process operation;
- Good performance even when direct state measurement is not possible;
- Relative robustness to model mismatch and to imprecisions in the characterization of noise.

As it will become clear from the bibliographical review, the first point (extension of current algorithms to the output tracking problem) has not been addressed so far in the literature and could be a first major obstacle to the adoption of these techniques in the chemical industry. Associated with this new type of formulation is the question of stability, because it is unclear if known proofs are also valid on this more general framework.

Once these more fundamental questions have been solved, it will be necessary to tackle the barriers that could prevent practical applications. Usually, new algorithms based on more complex mathematics than current MPC technology will not become useful in practice, if their general ideas cannot be mastered by application engineers. Therefore, the development of this work is guided by permanent consideration on what may be achieved in practice, both in terms of modeling precision and computational effort.

1.3. Outline of the work

This thesis is composed of seven chapters, including the present introduction. The structure of the remaining chapters is as follows:

In Chapter 2, we present an overview of the literature of stochastic model predictive control (SMPC).

Chapter 3 contains the development of a SMPC algorithm for systems subject to bounded additive noise. The formulation includes hard constraints over the inputs and soft probabilistic constraint over the state. The main contribution of the formulation in this chapter when compared to the state of the art is the explicit consideration of the set-point tracking problem.

The basic algorithm presented in the Chapter 3 is thoroughly tested via Monte Carlo simulations in Chapter 4. The simulated system is a linearized model representing a distillation column. Symmetric and asymmetric noise distributions are considered, as well as the influence of the probabilistic constraints.

Chapter 5 extends the formulation of Chapter 3 towards a zone control strategy. The relaxation of a fixed set-point into a zone is a convenient strategy in the stochastic framework, since the state is not allowed to rest in any given point due to persistent disturbance. Simulations on the same system of Chapter 4 are also presented.

Chapter 6 provides a theoretical refinement of the controller, indicating how the optimization problem has to be modified in order to guarantee recursive feasibility and stochastic stability. It is shown that under the control law defined by the controller, the system asymptotically converges to a set containing the set-point (or another equilibrium point, when the set-point is not reachable). The controller is applied to control a simulated plant given by the linearization of a CSTR.

Finally, Chapter 7 summarizes the contributions of the thesis and proposes some directions for future works.

2. Literature review

As stated before, Model Predictive Control is a mature technology, both in the nominal and robust cases. The performance of a practical implementation depends heavily on the accuracy of its model when compared to the dynamics of the real plant. It is evident that no model is able to completely represent it, therefore the control framework must be chosen properly to handle the inherent mismatch.

Stochastic optimal control is another branch of theory that was developed long time ago (Åstrom, 2006). Its practical implementation, however, was limited to small systems, because the usual algorithm to solve the resulting optimization problem is dynamic programming, whose complexity grows exponentially with system's size.

More recently, there is a trend to look for a combination of these two techniques, model predictive control and stochastic optimal control, which would have practical advantages. Stochastic predictive control incorporates the information about noise distribution, trying to reduce the conservativeness intrinsic of robust control: large disturbances usually occur with little probability, and the consideration of this fact may lead to more aggressive controllers that rarely disrespect constraints (Couchman et al., 2006).

Since the problem of finding the general optimal solution of stochastic control is intractable, a suboptimal strategy is to consider only certain parameterized inputs and calculate them in a receding horizon style. Most of the literature is concerned with linear parameterization, for the sake of simplicity. Some features make the distinction between the published works: the way disturbances affect the system (additive or multiplicative); noise probability distribution function and whether they are bounded or unbounded; type of feedback (measured or estimated state); presence of constraints in states or inputs and whether they are hard or soft; recursive feasibility and stability guarantees and finally the control objective (regulator, tracking or constraint satisfaction only).

This Chapter highlights some of the most influential works in stochastic predictive control, mainly by identifying key modeling assumptions and compiling theoretical proved statements. After this analysis, it becomes clear that there is a gap concerning possible implementations in chemical industry since there are no

algorithms designed for the tracking problem and with guaranteed performance when subject to hard input bounds.

One of the earliest contributions to stochastic model predictive control is in van Hessem et al. (2001), motivated by the trade-off between constraint violation and profitability that occurs in the selection of the operating point of a non-linear plant faced with random disturbances. The authors propose an algorithm to maximize profit while guaranteeing a bound in the probability that constraints are satisfied. Noise is modeled as additive and Gaussian and the probabilistic constraint is enforced through driving the state to a confidence ellipsoid. However, the required back-off between the operation point and the constraints is translated into a non-linear matrix inequality, which is difficult to solve for large systems. Even if the optimization problem may be exactly solved, it is unclear if closed-loop stability is always achieved.

In Couchman et al. (2006) there is another of the first contributions to accommodate probabilistic information within model predictive control. However, it admits that state evolution is completely deterministic but that matrix C, relating outputs and states, is drawn from a normal distribution. The control objective is to minimize the violation probability of a constraint that bounds the distance between a given output and its set-point. There is also a stochastic constraint regarding the other output, which is constrained to be close to its set-point within a given probability. As is usual in real applications, input constraints are hard, which means that they can never be violated. For this reason, a dual controller is employed, composed of a first mode of free control moves and a second mode, where a static feedback keeps the state around the set-point. Probabilistic constraints are translated in deterministic counterparts through the use of cumulative distribution functions. The main contribution of their work is the calculation of the invariant set Ω where the second control mode is active. The choice of this set is made in order to guarantee recursive feasibility. Stability proof comes from the terminal constraint, as in standard MPC, which is possible because state's evolution is deterministic.

A more general approach is taken in Cannon et al. (2009), where matrices A and B are assumed to be random (multiplicative uncertainty). However, only the regulation problem is considered. A dual controller is also implemented, but then terminal constraints cannot be straightforwardly applied, since it would be necessary

to propagate the effect of disturbances on the predictions and computational complexity would therefore become prohibitive. This issue is addressed by defining an augmented state composed of current state and future control moves. The evolution of the augmented state is autonomous by construction and its prediction in the next time instant is constrained to be in a confidence ellipsoid with probability *p*. Input constraints are merely imposed in probability, which is not suitable for a real implementation. The confidence ellipsoid constraint causes the problem to be formulated as a Linear Matrix Inequality (LMI). The objective function, corresponding to the expected value of the usual quadratic cost of MPC for the regulator problem, is shown to be a bounded supermartingale, therefore convergent, and this guarantees system stability. Another shortcoming of their approach is the assumption that the initial state belongs to the confidence ellipsoid. If this is not the case, the control algorithm is modified to bring the state to the desired region without losing the supermartingality property, but no proof is provided.

Kouvaritakis et al. (2010) present a significant result from a computational point of view, because the optimization problem that must be solved online is just a QP. This is achieved for a simpler situation than in previous works of the same group, in which additive, bounded disturbances are considered. Its main theoretical contribution is a separation between nominal (deterministic) evolution and random disturbance and the propagation of uncertainty is performed based on offline calculations. Only the regulator problem is considered and the objective function is also the expected quadratic cost. No input constraints are enforced and a dual mode controller is used. Since noise is bounded, invariant sets may be calculated as in traditional robust control approaches, providing recursive feasibility.

Finally, Cannon et al. (2012) extend previous results to the output feedback case, incorporating a state estimator to the control loop. Disturbance modeling follows the same lines as in Kouvaritakis et al. (2010). Constraints are more general, including linear combinations of states and inputs – but without rate constraints in the inputs, bounding the difference between consecutive inputs. Nonetheless, all constraints are exclusively probabilistic. The authors claim that their approach is more general than the robust MPC precisely because of the softening of constraints. Most of the controller setup is identical to Kouvaritakis et al. (2010), such as the dual mode, objective function and worst-case approach to the evaluation of the terminal set. The

stability proof is once again based on the feasibility of the problem at the initial time instant. The numerical example in the paper shows a case where the initial state is outside the confidence ellipsoid but very close to it; it is not clear how large is the attractive domain of this class of controllers.

Another line of research is presented in Hokayem et al. (2009), being different from Couchman et al. (2006) and Cannon et al. (2012) specially in considering unbounded noise distributions. Also, the authors insisted that hard input bounds may not be neglected from a practical point of view. Their work addresses only systems with Schur-stable A matrix (all of its eigenvalues are in the interior of the unit circle), therefore mean square stability is not an issue. Their most important result is a proof that the state variance is bounded, which is not trivial when considering unbounded noise and hard input constraints. The innovative idea behind this result is a parameterization of inputs as a function of previous noise (and not previous states). Also, it is necessary to saturate noise before calculating the inputs in order to satisfy their constraints. The online optimization problem is greatly simplified thanks to the analytical calculation of terms related to future noise, in a similar way to Kouvaritakis et al. (2010) and Cannon et al. (2012), instead of using numerical Monte Carlo algorithms to calculate an approximation of the expected values.

Later, in Ramponi et al. (2010), the above method is extended to the case where A is marginally stable (i.e., it may have some eigenvalues equal to 1). The authors explain that this is the most general result achievable with bounded inputs, according to Yang et al. (1995). Nonetheless, their work merely provides a fixed non-linear feedback law that renders state variance bounded; no optimization towards a goal is performed. Hokayem et al. (2012) introduce an input parameterization that may be optimized online to solve the regulator problem without losing the bounded variance result. In Chapter 3 of this thesis, the method is discussed with more details, since it provides one of the bases for our proposed tracking algorithm.

Korda & Cigler (2012) also consider a non-linear feedback based on a saturation that is similar to Hokayem et al. (2012), but they consider the objective function to be the expected value of the p-norm of the states and inputs, with p between 1 and infinity. The authors claim that this generalization of the objective function allows users to follow more closely the economic criteria and also helps to achieve some balance between known properties of controllers based on norms 1 and 2. Since

noise is assumed to be Gaussian, the expected values of the objective function may be analytically calculated. Recursive feasibility of probabilistic constraint in states is proved assuming bounded noise. The authors claim that the results derived for Gaussian noise are useful to define suboptimality bounds, because in reality every noise realization is bounded and therefore not Gaussian. The argument for recursive feasibility relies on a dual controller paradigm, where constraints are strictly enforced in mode 1 and implicitly in mode 2. The strategy is also presented in Korda et al. (2011); this work also contains a constraint handling method based on invariant sets detailed in Chapter 3 of this thesis.

Primbs & Sung (2009) also study the problem of stochastic control for a large class of systems, namely those with multiplicative uncertainty in matrices A and B and any distribution function for noise. They consider the regulation problem with perfect state information. Surprisingly, input and state constraints are enforced solely in mean value, which possibly leads to unrealistic solutions. There is a proof that the objective function behaves as a stochastic Lyapunov function (supermartingale), but this stability result depends on complicated assumptions that the authors do not check even for the small numerical example presented.

An approach that follows a completely different kind of reasoning is presented in Bernardini & Bemporad (2009). They consider linear time-varying systems, where matrices A and B take values in a finite set, and they assume that the probability density function of these random variables is known. In fact, this function may evolve in time, as is the case of a Markov process. Due to the assumption that they assume values in a finite set, it is possible to build a tree of all possible realizations of disturbance through the prediction horizon. For computational efficiency, only some of its nodes are analyzed and the objective function is a weighted average of the standard quadratic cost evaluated at previously selected nodes. The algorithm for this selection is based on a maximum likelihood approach. Finally, stability is enforced through the use of a robust Lyapunov function, valid for all disturbance realizations.

3. Stochastic control for systems with bounded noise

3.1. System description and input parameterization

Consider a linear time-invariant system subject to process noise:

$$\begin{aligned} x_p(k+1) &= A_p x_p(k) + B_p u(k) + F_p w(k) \\ y(k) &= C_p x_p(k) \\ x_p(k) \in \mathbb{R}^{nx}, u(k) \in \mathbb{R}^{nu}, y(k) &\in \mathbb{R}^{ny}, w(k) \in \mathbb{R}^{nw} \end{aligned} \tag{3-1}$$

where {w(k)} is a sequence of independent uniformly distributed random variables with bounded support. Noise mean and covariance are known and equal to μ_w and Σ_w, respectively. Full-state measurement and stabilizability of the pair (A_p, B_p) are assumed.

The main objective is to track a (possibly time-varying) set-point y^{sp}, while respecting hard constraints over input values (u) and control moves (Δu) and soft constraints over the state. It is known (Maeder et al., 2009) that one way to achieve offset-free tracking is to use a velocity description of the system, which means that Δu replaces u as the input. Considering an augmented state defined as $x_v(k) = \begin{bmatrix} \Delta x_p(k)^T & y(k)^T \end{bmatrix}^T$, where $\Delta x_p(k) = x_p(k) - x_p(k-1)$, it is clear that (3-1) is equivalent to

$$\begin{aligned} \begin{bmatrix} \Delta x_p(k+1) \\ y(k+1) \end{bmatrix} &= \begin{bmatrix} A_p & 0 \\ C_p A_p & I_{ny} \end{bmatrix} \begin{bmatrix} \Delta x_p(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} B_p \\ C_p B_p \end{bmatrix} \Delta u(k) \\ &\quad + \begin{bmatrix} F_p \\ C_p F_p \end{bmatrix} \Delta w(k) \\ y(k) &= \begin{bmatrix} 0 & I_{ny} \end{bmatrix} \begin{bmatrix} \Delta x_p(k) \\ y(k) \end{bmatrix} \\ &\Leftrightarrow \\ x_v(k+1) &= A_v x_v(k) + B_v \Delta u(k) + F_v \Delta w(k) \\ y(k) &= C_v x_v(k) \end{aligned} \tag{3-2}$$

Note that the proposed incremental form causes $\Delta w(k)$ and $\Delta w(k-1)$ to be function of $w(k-1)$, therefore the process $\{\Delta w(k)\}$ is not composed of independent random variables.

Using this formulation, the problem that we would like to solve is

Problem 1

$$\min_{\Delta u(k),\cdots,\Delta u(k+m-1)} \mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=1}^{p}\left\|y(k+i)-y_{sp}\right\|_Q^2+\sum_{i=0}^{m-1}\left\|\Delta u(k+i)\right\|_R^2\right] \tag{3-3}$$

subject to

$$\left|\Delta u(k+i)\right| \le \Delta u_{\max}\,, i=0,\cdots,m-1 \tag{3-4}$$

$$u_{\min} \le u(k+i) \le u_{\max}, i=0,\cdots,m-1 \tag{3-5}$$

$$\mathbb{P}\left[g_j^T x_v(k+1) \le h_j\right] \ge 1-\alpha_j, \quad j=1,\cdots,r; \tag{3-6}$$

Q and R are symmetric positive-definite tuning matrices, p is the prediction horizon, m is the control horizon, $u_{\min}$, $u_{\max}$ and $\Delta u_{\max}$ are physical constraints of the inputs and control moves. $\mathbb{E}_{\mathfrak{X}(k)}[\bullet]$ is the conditional expectation given $\mathfrak{X}(k)$, which is the set of state observations up to time k, $\mathfrak{X}(k)=\{x_v(0),\cdots,x_v(k)\}$. $\mathbb{P}[\bullet]$ is the probability, g_j^T and h_j are parameters that define r linear constraints over the state, which have to be satisfied at a confidence level of $(1-\alpha_j)$.

Due to the presence of disturbances, an open-loop calculation of future control moves may cause the problem to be infeasible in the presence of state constraints (Scokaert & Mayne, 1998). Also, open-loop control could be excessively conservative when looking for inputs that stabilize the system for all possible disturbance realizations (Mayne et al., 2000). Therefore, it is advantageous to consider causal feedback policies, such as described in Goulart et al. (2006):

$$\begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+m-1) \end{bmatrix} = \begin{bmatrix} \pi_0\left(x_v(k)\right) \\ \pi_1\left((x_v(k), x_v(k+1))\right) \\ \vdots \\ \pi_{m-1}\left(x_v(k), x_v(k+1), \cdots, x_v(k+m-1)\right) \end{bmatrix} \tag{3-7}$$

Problem 1 is generally intractable if the optimization is performed over the whole class of causal policies π. One possible approach to find a suboptimal solution is to parameterize the input as an affine function of the state. As shown in Goulart et al. (2006), this strategy is equivalent to use the parameterization (3-8) concerning the process disturbance. Moreover, the resulting optimization problem is convex and hence tractable, which Goulart et al. (2006) prove that is not the case when a state feedback parameterization is considered.

$$u(k+i) = d(k+i) + \sum_{j=0}^{i-1} G(k+i, k+j) w(k+j) \tag{3-8}$$

The decision variables of the optimization problem are the affine terms $d(k+i) \in \mathbb{R}^{nu}$ and the feedback gains $G(k+i, k+j) \in \mathbb{R}^{nu \times nx}$. A possible physical interpretation of this unconventional parameterization is that the first term drives the input mean value, whereas the feedback is responsible for the variance reduction. This concept may be adapted to the case of incremental input as follows

$$\Delta u(k+i) = d(k+i) + \sum_{j=0}^{i-1} g(k+i, k+j) \Delta w(k+j)$$

$$\begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+m-1) \end{bmatrix} = \begin{bmatrix} d(k) \\ d(k+1) \\ \vdots \\ d(k+m-1) \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ g(k+1,k) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ g(k+m-1,k) & g(k+m-1,k+1) & \dots & g(k+m-1,k+m-2) & 0 \end{bmatrix} \begin{bmatrix} \Delta w(k) \\ \Delta w(k+1) \\ \vdots \\ \Delta w(k+m-1) \end{bmatrix}$$

$$\Delta\bar{U}(k) = \bar{D}(k) + \bar{G}(k)\Delta\bar{W}(k)$$
$$\Delta\bar{U}(k) \in \mathbb{R}^{nu\cdot m}; \bar{D}(k) \in \mathbb{R}^{nu\cdot m}; \bar{G}(k) \in \mathbb{R}^{nu\cdot m \text{ x } nw\cdot m}; \Delta\bar{W}(k) \in \mathbb{R}^{nw\cdot m} \tag{3-9}$$

In section 3.2, the objective function is manipulated in order to achieve a computationally tractable expression.

3.2. Objective function

As it is standard in predictive control, the objective function is the sum of two contributions: the first term weights the distance between predicted outputs and the set-point, whereas the second penalizes the control moves. We consider that the prediction horizon is not necessarily equal to the control horizon, based on a dual controller framework, where the prediction of future inputs after the control horizon *m* is a simple feedback law:

$$\Delta u(k+j) = K\Delta w(k+j-1), \quad m < j < p \tag{3-10}$$

where $K \in \mathbb{R}^{nu\times nx}$ is an off-line calculated matrix such that $\left(A_p + B_p K\right)$ is stable.

Taking advantage of this choice of the second mode controller, it is possible to separate the forced and autonomous contributions to the evolution of the control objective, as follows:

$$\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=1}^{m}\left\|y(k+i)-y_{tar}\right\|_Q^2 + \sum_{i=0}^{m-1}\left\|\Delta u(k+i)\right\|_R^2\right] + \mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=m+1}^{p}\left\|y(k+i)-y_{tar}\right\|_Q^2\right] + \\
&+V_O\left(y_{tar}, y_{sp}\right) \\
&= V_{for}(k) + V_{aut}(k) + V_O\left(y_{tar}, y_{sp}\right)
\end{aligned} \tag{3-11}$$

The term $V_O\left(y_{tar}, y_{sp}\right)$ is an offset function, using the same strategy of Ferramosca et al. (2012) to penalize deviations between the real and artificial set-points. y_{tar} is a new decision variable of the optimization problem, corresponding to an artificial reference that will be steered as close as possible to the true set-point. It

is advantageous to introduce an artificial set-point since the original one may not be reachable when constraints are present and also it could not correspond to a steady state. More details about the calculation of the target and the offset function are provided in Section 3.3.

It is necessary to express the cost as an explicit function of the decision variables, instead of the expected value expressions presented so far. First, it is presented an analysis of the forced component of the cost and then of the autonomous one, which follows the same rationale.

In order to simplify this procedure, it is convenient to collect the terms corresponding to future states, inputs and noise in the vectors $\bar{X}(k)$, $\Delta\bar{U}(k)$ and $\Delta\bar{W}(k)$, respectively, defined as

$$\bar{X}(k)=\begin{bmatrix} x_v(k+1) \\ x_v(k+2) \\ \vdots \\ x_v(k+m) \end{bmatrix};\Delta\bar{U}(k)=\begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+m-1) \end{bmatrix};\Delta\bar{W}(k)=\begin{bmatrix} \Delta w(k) \\ \Delta w(k+1) \\ \vdots \\ \Delta w(k+m-1) \end{bmatrix} \tag{3-12}$$

$$\bar{X}\in\mathbb{R}^{(nx+ny)\cdot m},\Delta\bar{U}\in\mathbb{R}^{nu\cdot m},\Delta\bar{W}\in\mathbb{R}^{nw\cdot m}$$

The dynamics of the extended state is given by

$$\bar{X}(k)=\bar{A}x_v(k)+\bar{B}\Delta\bar{U}(k)+\bar{F}\Delta\bar{W}(k) \tag{3-13}$$

where

$$\bar{A}=\begin{bmatrix} A_v \\ A_v^2 \\ \vdots \\ A_v^m \end{bmatrix};\bar{B}=\begin{bmatrix} B_v & 0 & 0 & 0 \\ A_vB_v & B_v & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_v^{m-1}B_v & \cdots & A_vB_v & B_v \end{bmatrix};\bar{F}=\begin{bmatrix} F_v & 0 & 0 & 0 \\ A_vF_v & F_v & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_v^{m-1}F_v & \cdots & A_vF_v & F_v \end{bmatrix} \tag{3-14}$$

$$\bar{A}\in\mathbb{R}^{(nx+ny)\cdot m\times(nx+ny)},\bar{B}\in\mathbb{R}^{(nx+ny)\cdot m\times nu\cdot m},\bar{F}\in\mathbb{R}^{(nx+ny)\cdot m\times nw\cdot m}$$

Using this notation, the forced component of the objective function is calculated through

$$V_{for}(k) = \mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{C}\bar{X}(k) - \bar{Y}_{tar}\right\|_{\bar{Q}}^2 + \left\|\Delta\bar{U}(k)\right\|_{\bar{R}}^2\right] \tag{3-15}$$

where the other matrices are defined as $\bar{Q} = \text{diag}(Q,\cdots,Q) \in \mathbb{R}^{m\cdot ny \times m\cdot ny}$, $\bar{R} = \text{diag}(R,\cdots,R) \in \mathbb{R}^{m\cdot nu \times m\cdot nu}$, $\bar{C} = \text{diag}(C_v,\cdots,C_v) \in \mathbb{R}^{m\cdot ny \times m\cdot(nx+ny)}$ and finally $\bar{Y}_{tar} = \left[y_{tar}{}^T \quad \cdots \quad y_{tar}{}^T\right]^T \in \mathbb{R}^{m\cdot ny}$.

Expanding the quadratic forms, we have

$$\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k))}\left[\left\|\bar{C}\bar{X}(k) - \bar{Y}_{tar}\right\|_{\bar{Q}}^2 + \left\|\Delta\bar{U}(k)\right\|_{\bar{R}}^2\right] = \\
&\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{C}\bar{X}(k)\right\|_{\bar{Q}}^2 - 2\bar{Y}_{tar}^T\bar{Q}\bar{C}\bar{X}(k) + \left\|\bar{Y}_{tar}\right\|_{\bar{Q}}^2\right] + \mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\Delta\bar{U}(k)\right\|_{\bar{R}}^2\right] = \\
&\underbrace{\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{X}(k)\right\|_{\bar{C}^T\bar{Q}\bar{C}}^2\right]}_{T_1} - 2\bar{Y}_{tar}^T\bar{Q}\bar{C}\underbrace{\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{X}(k)\right]}_{T_2} + \left\|\bar{Y}_{tar}\right\|_{\bar{Q}}^2 + \underbrace{\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\Delta\bar{U}(k)\right\|_{\bar{R}}^2\right]}_{T_3}
\end{aligned} \tag{3-16}$$

Each of the terms labeled, T_1, T_2 and T_3, may be evaluated independently, as presented in Appendix A. The same kind of reasoning may be applied to the autonomous term. First, we define vectors $\bar{\bar{X}}(k)$, $\Delta\bar{\bar{U}}(k)$ and $\Delta\bar{\bar{W}}(k)$, as follows:

$$\bar{\bar{X}}(k) = \begin{bmatrix} x_v(k+m+1) \\ x_v(k+m+2) \\ \vdots \\ x_v(k+p) \end{bmatrix}; \Delta\bar{\bar{U}}(k) = \begin{bmatrix} \Delta u(k+m) \\ \Delta u(k+m+1) \\ \vdots \\ \Delta u(k+p-1) \end{bmatrix}; \Delta\bar{\bar{W}}(k) = \begin{bmatrix} \Delta w(k+m) \\ \Delta w(k+m+1) \\ \vdots \\ \Delta w(k+p-1) \end{bmatrix} \tag{3-17}$$

$\bar{\bar{X}} \in \mathbb{R}^{(nx+ny)\cdot(p-m)}, \Delta\bar{\bar{U}} \in \mathbb{R}^{nu\cdot(p-m)}, \Delta\bar{\bar{W}} \in \mathbb{R}^{nw\cdot(p-m)}$

As before, these extended vectors satisfy the dynamic equation

$$\bar{\bar{X}}(k) = \bar{\bar{A}}x(k+m) + \bar{\bar{B}}\Delta\bar{\bar{U}}(k) + \bar{\bar{F}}\Delta\bar{\bar{W}}(k) \tag{3-18}$$

with

$$\bar{\bar{A}} = \begin{bmatrix} A_v \\ A_v^2 \\ \vdots \\ A_v^{p-m} \end{bmatrix}; \bar{\bar{B}} = \begin{bmatrix} B_v & 0 & 0 & 0 \\ A_v B_v & B_v & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_v^{p-m-1} B_v & \cdots & A_v B_v & B_v \end{bmatrix}; \bar{\bar{F}} = \begin{bmatrix} F_v & 0 & 0 & 0 \\ A_v F_v & F_v & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_v^{p-m-1} F_v & \cdots & A_v F_v & F_v \end{bmatrix} \tag{3-19}$$

$$\bar{\bar{A}} \in \mathbb{R}^{(nx+ny)\cdot(p-m)\times(nx+ny)}, \bar{\bar{B}} \in \mathbb{R}^{(nx+ny)\cdot(p-m)\times nu\cdot(p-m)}, \bar{\bar{F}} \in \mathbb{R}^{(nx+ny)\cdot(p-m)\times nw\cdot(p-m)}$$

Therefore, the autonomous component of the objective function may be written as

$$\begin{aligned} V_{aut}(k) &= \mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=m+1}^{p} \left\| y(k+i) - y_{tar} \right\|_Q^2\right] = \mathbb{E}_{\mathfrak{X}(k)}\left[\left\| \bar{\bar{C}}\bar{\bar{X}}(k) - \bar{\bar{Y}}_{tar} \right\|_{\bar{\bar{Q}}}^2\right] = \\ &\underbrace{\mathbb{E}_{\mathfrak{X}(k)}\left[\left\| \bar{\bar{X}}(k) \right\|_{\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}}^2\right]}_{T_4} - 2\bar{\bar{Y}}_{tar}^T \bar{\bar{Q}}\bar{\bar{C}} \underbrace{\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{\bar{X}}(k)\right]}_{T_5} + \left\| \bar{\bar{Y}}_{tar} \right\|_{\bar{\bar{Q}}}^2 \end{aligned} \tag{3-20}$$

where the other matrices are defined as $\bar{\bar{Q}} = \text{diag}(Q,\cdots,Q) \in \mathbb{R}^{(p-m)ny\times(p-m)ny}$, $\bar{\bar{Y}}_{tar} = \begin{bmatrix} y_{tar}^T & \cdots & y_{tar}^T \end{bmatrix}^T \in \mathbb{R}^{(p-m)ny}$, $\bar{\bar{C}} = \text{diag}(C_v,\cdots,C_v) \in \mathbb{R}^{(p-m)ny\times(p-m)(nx+ny)}$.

The idea that allows the simplification of the objective function is to replace the predicted states by an expression that depends on the current state, future noise and control actions. The current state is known by hypothesis, which greatly simplifies the problem. All details of these calculations and definition of auxiliary matrices are provided in Appendix A. The expression of the cost could be further simplified by disregarding the constants, independent of the decision variables. For completeness, however, we provide its full expression but reinforce that the implementations were performed with the reduced form.

$$
\begin{aligned}
&V(k) = V_{for}(k) + V_{aut}(k) = \\
&\left\|x(k)\right\|^2_{\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{A}} + tr\left(\bar{F}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\mu^{\Delta w\Delta w}(k)\right) \\
&+2x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\left(\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right) \\
&+2tr\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\left(\mu^{\Delta w\Delta w}(k)\bar{G}^T+\mu^{\Delta w}(k)\bar{D}^T\right)\right) \\
&+tr\left(\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{B}+\bar{R}\right)\left(\bar{G}\mu^{\Delta w\Delta w}(k)\bar{G}^T+\bar{G}\mu^{\Delta w}(k)\bar{D}^T+\bar{D}\mu^{\Delta w}(k)^T\bar{G}^T+\bar{D}\bar{D}^T\right)\right) \\
&-2\bar{Y}_{sp}^T\bar{Q}\bar{C}\left(\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)+\left\|\bar{Y}_{tar}\right\|^2_{\bar{Q}} \\
&tr\left(T_m{}^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{A}}T_m\Sigma_{\bar{X}}\right)+tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UU}\right)+tr\left(\bar{\bar{F}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\bar{\bar{\mu}}^{\Delta w\Delta w}\right) \\
&+2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UX}\right)+2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WX}\right)+2tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WU}\right) \\
&-2\bar{\bar{Y}}_{sp}^T\bar{\bar{Q}}\bar{\bar{C}}\left(\bar{\bar{A}}T_m\left(\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)\right)+\left\|\bar{\bar{Y}}_{tar}\right\|^2_{\bar{\bar{Q}}}
\end{aligned}
\tag{3-21}
$$

3.3. Target calculation

When considering the general tracking problem, it is necessary to consider the possibility that the user-provided set-point is not achievable due to input constraints. This is particularly common in hierarchical control structures, since different models are usually employed for real-time optimization and control. It is desirable to replace any unreachable values by the closest feasible (on a least square sense) set-point.

The chosen strategy to circumvent this problem was to incorporate the calculation of an artificial set-point, associated to an artificial state target x_{tar} which is only required to be a steady state. It uses the positional formulation to calculate a steady-state, respecting input constraints as follows:

$$
\begin{aligned}
x_{tar} &= A_p x_{tar} + B_p u_{tar} \\
y_{tar} &= C_p x_{tar} \\
u_{\min} &\le u_{tar} \le u_{\max}
\end{aligned}
\tag{3-22}
$$

In terms of objective function, we have included an offset function as presented in (3-11). There are many possibilities to choose this function, such as infinity norm or 1-norm. In this work we have restrained ourselves to the former. The implementation of the offset function may be done in epigraph form, which preserves linearity. The

next lemma provides a conversion of the offset function to linear constraints for the case of zone control:

Lemma (Ferramosca et al., 2009): The set $\left\{ y_{tar} : \min_{y_{\min} \le y_{sp} \le y_{\max}} \left\| y_{tar} - y_{sp} \right\|_{\infty} \le \lambda \right\}$ is given by

$$\begin{cases} y_{tar} - \lambda \mathbf{1}_{ny} \le y_{\max} \\ -y_{tar} - \lambda \mathbf{1}_{ny} \le -y_{\min} \\ \lambda \ge 0 \end{cases} \tag{3-23}$$

where $\mathbf{1}_{ny}$ is a column vector with *ny* unitary components.

Our formulation exhibits three differences when compared to the one of Ferramosca et al. (2009), motivated by improvements of the numerical performance of the optimization problem. First, we have not considered a uniform value of $\lambda \in \mathbb{R}$ but instead one particular value for each of the outputs, therefore defining a slack variable $\lambda \in \mathbb{R}^{ny}$. Second, we have included a factor $Q_{\lambda} \in \mathbb{R}^{ny}$ for scaling purposes, which allows the objective function to weight better the trade-off between choosing an artificial set-point different from the real one and the other control costs. Finally, this chapter is not dedicated to the zone control strategy, so minimum and maximum output values are actually equal to the real set-point. A more detailed discussion of the zone approach is provided in Chapter 1.

Therefore, the offset function considered in the simulations and the constraints associated with it are, respectively:

$$V_O\left(y_{tar}, y_{sp}\right) = Q_{\lambda}^{T} \lambda \tag{3-24}$$

subject to

$$\begin{aligned} & y_{tar} - \lambda \le y_{sp} \\ & -y_{tar} - \lambda \le -y_{sp} \\ & \lambda \ge 0 \end{aligned} \tag{3-25}$$

Notice that the distance from the artificial to the original set-point could be directly implemented as a weighted norm such as $\left\| y_{tar} - y_{sp} \right\|_{Q_{\lambda}}^{2}$. The epigraph form is

preferable since it is more readily generalizable to the zone control problem described in Chapter 4.

3.4. Constraints

Two kinds of constraints should be considered in this problem. First, (3-4)-(3-5) correspond to physical limits of the plant. Second, (3-6) represent soft constraints over linear combinations of the state, which may be used to model output limits. Moreover, as discussed in section 3.4.2, it is possible to include a kind of probabilistic zone control if (3-6) is modified in order to encompass linear functions of the set-point.

3.4.1. Physical limits

Bounds on the maximum allowed control moves follow naturally from the chosen parameterization, in a similar spirit to Hokayem et al. (2012). Let $w_{\max}$, $w_{\min} \in \mathbb{R}^{nw}$ be respectively the maximum and minimum values of the noise, which are finite due to the bounded support assumption. Note that the absolute value of $w_{\max}$ and $w_{\min}$ may be different, since noise distribution is not necessarily symmetric and zero-mean. Therefore, a conservative version of (3-4) is given by

$$\left|d_i\right| + \left\|G_i\right\|_\infty \left(w_{\max} - w_{\min}\right) \le \Delta u_{\max}, i = 1, \cdots, m \cdot nu \tag{3-26}$$

where d_i is the i-th element of vector $\bar{D}(k)$ and G_i is the i-th row of matrix $\bar{G}(k)$.

Regarding (3-5), it suffices to notice that future input values and control moves are related through

$$\begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+m-1) \end{bmatrix} = \begin{bmatrix} u(k-1) \\ u(k-1) \\ \vdots \\ u(k-1) \end{bmatrix} + \underbrace{\begin{bmatrix} I_{nu} & 0 & \cdots & 0 \\ I_{nu} & I_{nu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I_{nu} & I_{nu} & \cdots & I_{nu} \end{bmatrix}}_{M} \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+m-1) \end{bmatrix} \tag{3-27}$$

Let

$$U_{\max}(k)=\begin{bmatrix} u_{\max}-u(k-1) \\ \vdots \\ u_{\max}-u(k-1) \end{bmatrix};U_{\min}(k)=\begin{bmatrix} u_{\min}-u(k-1) \\ \vdots \\ u_{\min}-u(k-1) \end{bmatrix} \tag{3-28}$$

Then, (3-5) is equivalent to

$$U_{\min}(k)\le M\Delta\bar{U}(k)\le U_{\max}(k) \tag{3-29}$$

Considering the extreme values of noise realizations, (3-29) is assured if

$$\begin{aligned} &M\bar{D}(k)+M\bar{G}^{+}(k)\left|\bar{W}_{\max}\right|+M\bar{G}^{-}(k)\left|\bar{W}_{\min}\right|\le U_{\max}(k) \\ &U_{\min}(k)\le M\bar{D}(k)-M\bar{G}^{+}(k)\left|\bar{W}_{\max}\right|+M\bar{G}^{-}(k)\left|\bar{W}_{\min}\right| \end{aligned} \tag{3-30}$$

where $\bar{G}^{+}(k)$ and $\bar{G}^{-}(k)$ are positive (element-wise) matrices such that $\bar{G}(k)=\bar{G}^{+}(k)-\bar{G}^{-}(k)$, $\bar{W}_{\max}=\begin{bmatrix} w_{\max}{}^{T} & \cdots & w_{\max}{}^{T} \end{bmatrix}^{T}$, $\bar{W}_{\min}=\begin{bmatrix} w_{\min}{}^{T} & \cdots & w_{\min}{}^{T} \end{bmatrix}^{T}\in\mathbb{R}^{m\cdot nw}$.

3.4.2. Probabilistic constraints based on invariant sets

3.4.2.1. Theoretical background

Invariant sets are an important tool to achieve stability results, both in disturbed and undisturbed situations (Blanchini, 1999). The intuitive definition of a positively invariant set is that, if the state is in the set for a given time, then it will remain in the same set for all future time. It is clear that this concept is close, but not equivalent, to stability. For example, if a system is positively invariant in a *bounded* set, this implies the existence of a uniform bound on its evolution, which guarantees Lyapunov marginally stability. On the other hand, the state may evolve in an unbounded invariant set and move arbitrarily far from the origin, in which case it would not correspond to a stable system.

Nominal model predictive control usually relies on invariant sets. On the simplest case, some algorithms force the state to reach the origin after a finite number of steps, because it will remain at this equilibrium point if the system is not disturbed. More generally, the dual controller strategy admits that the state is steered to a larger invariant set, where it would remain if subject to a linear feedback law.

When disturbances are considered in the control formulation, it is necessary to modify the definition of invariant set, by considering that a state belongs to a robust invariant set if there is a feedback law (not necessarily linear) that keeps it in the set for all possible future disturbances. As control inputs are always bounded in practice, this definition is only meaningful if it is also assumed that the disturbance is bounded; otherwise, there would be a sufficiently large disturbance, able to remove the state from the set even if the control action was at its maximum value.

This requirement of invariance when confronted with all possible disturbances is a source of conservativeness. If there is some information about their distributions, then it is possible to enforce invariance with a certain probability. This approach is investigated in Cannon et al. (2009) but it is not the main guideline of further reasoning in the current work. The inspiration comes from Korda et al. (2011), where the problem solved is to synthesize a strongly feasible MPC for linear time-invariant systems, subject to additive noise, hard input constraints and probabilistic constraints over states. The link between probabilistic constraints and invariant sets comes from the method named by the authors as “First-step constraint”.

This first-step approach forces, at each sampling time, that the state in the next instant belongs to a “stochastic robust controlled invariant set”, which is a refinement of robust invariant set: besides being invariant when faced with all possible disturbances, in this set there is also a bound on the probability of satisfying the state constraint for future evolution.

Definition 3-1 [(Korda et al., 2011)]: A set $S_{SRCI} \subset \mathbb{R}^{nx}$ is stochastic robust controlled invariant for a system of the form $x(k+1) = Ax(k) + Bu(k) + w(k)$ subject to the soft constraint

$$\mathbb{P}\left[g_j^T x(k+1) \le h_j \right] \ge 1 - \alpha_j, \quad j = 1, \cdots, r; \; k \in \mathbb{N}^* \tag{3-31}$$

if there is a continuous feedback control law $u(k)=\Phi(x(k))$ such that $u(k)\in\mathfrak{U}$, which assures that S_{SRCI} is positively invariant for the closed loop system, $Ax(k)+B\Phi(x(k))+w\in S_{SRCI}$, and the soft constraint remains feasible,

$$\mathbb{P}\left[g_j^T\left(Ax(k)+B\Phi(x(k))+w\right)\le h_j\right]\ge 1-\alpha_j,\quad j=1,\cdots,r;\ k\in\mathbb{N}^*$$

As can be seen from above, this proposal of an invariant set is aimed only at achieving strong feasibility, i.e., “to guarantee that for every feasible initial state the closed-loop process remains feasible due to any admissible disturbance realization and any sequence of feasible control inputs generated in a receding horizon fashion” (Korda et al., 2011).

In order to create a practical algorithm, (Korda et al., 2011) reformulate the probabilistic constraint through the use of the cumulative distribution $\mathbb{F}_{g_j^T w}(\cdot)$ and the left quantile function $\mathbb{F}^{-1}_{g_j^T w}(\cdot)$ of the random variables $g_j^T w$:

$$\begin{aligned}
&\mathbb{P}\left[g_j^T x(k+1)\le h_j\right]\ge 1-\alpha_j,\quad j=1,\cdots,r;\\
&\Leftrightarrow\\
&\mathbb{F}_{g_j^T w}\left(h_j-g_j^T\left(Ax(k)+Bu(k)\right)\right)\ge 1-\alpha_j,\quad j=1,\cdots,r;\\
&\Leftrightarrow\\
&g_j^T\left(Ax(k)+Bu(k)\right)\le h_j-\mathbb{F}^{-1}_{g_j^T w}\left(1-\alpha_j\right),\quad j=1,\cdots,r;
\end{aligned}\tag{3-32}$$

Since $\mathbb{F}^{-1}_{g_j^T w}\left(1-\alpha_j\right)$ can be calculated offline, for example by Monte Carlo methods, (3-32) is a regular, linear constraint on state and inputs. Therefore, the calculation of the stochastic robust controlled invariant set may be performed with available standard algorithms. In particular, in this work we use the Matlab Invariant Set Toolbox (Kerrigan, 2005).

To achieve the largest possible domain of attraction, Korda et al. (2011) look for the maximum stochastic robust controlled invariant (MSRCI) set, denoted by S^*_{SRCI}, which is the set containing all S_{SRCI}. This set may be of any shape, but available

software provide a polyhedral approximation, with precision (and complexity) arbitrarily large, of the form $S^*_{SRCI} = \left\{ x \in \mathbb{R}^{nx} \mid \tilde{S}x \le \tilde{c} \right\}$.The so-called first-step constraint implies that only the first control move is considered for the problem constraints. Let $u(k) = d(k)$, then the necessary constraints are:

$$\tilde{s}_j^T \left(Ax(k) + Bd(k) \right) \le \tilde{c}_j - \left\| \tilde{s}_j^T \right\|_1 \Delta_w, \quad j = 1, \cdots, \tilde{r}; \tag{3-33}$$

$$g_j^T \left(Ax(k) + Bd(k) \right) \le h_j - F^{-1}_{g_j^T w} \left(1 - \alpha_j \right), \quad j = 1, \cdots, r; \tag{3-34}$$

where $\tilde{s}_j^T$ and $\tilde{c}_j$ are the rows of matrices $\tilde{S} \in \mathbb{R}^{\tilde{r} \times nx}$ and $\tilde{c} \in \mathbb{R}^{\tilde{r}}$. Δ_w is an upper bound of the norm-1 of *w*. Korda et al. (2011) remark that the MSRCI is a superset of the robust controlled invariant set associated with constraints imposed in hard form, i.e., $g_j^T x(k+1) \le h_j$. Additionally, even if the confidence parameter α_j is set to 0 for all *j* they do not coincide, because S^*_{SRCI} is the set containing all points that can reach the robust controlled invariant set in one step.

Since only the first move is constrained by this strategy, there are degrees of freedom corresponding to the future moves. Therefore, the first-step constraint may be perfectly accommodated with the affine parameterization (3-9), which is explicitly done in the next subsection.

3.4.2.2. Tracking formulation

There are three main difficulties that have to be tackled when applying the approach of stochastic invariant sets to the controller framework proposed in this chapter. First, we consider constraints over control moves and input values, whereas Korda et al. (2011) deal only with the latter. Second, the controller considers noise in incremental form, but its most natural description is in positional form. Finally, since the set-point varies, it would be more useful to consider soft constraints including linear combinations of states and the set-point.

As an example of the last observation, soft constraints of the form (3-31) may be used to model probabilistic zone constraints, forcing the state to be close to the set-

point within a certain probability level. A standard zone constraint could be expressed as

$$\left| y(k+1) - y_{tar} \right| \leq h \tag{3-35}$$

Besides, it is possible to soften (3-35) into probabilistic constraints,

$$\begin{aligned} &\mathbb{P}\left[y(k+1) - y_{tar} \leq h \;\; | \, x(k) \right] \geq 1 - \alpha_1; \;\; k \in \mathbb{N}^* \\ &\mathbb{P}\left[y_{tar} - y(k+1) \leq h \;\; | \, x(k) \right] \geq 1 - \alpha_2; \;\; k \in \mathbb{N}^* \end{aligned} \tag{3-36}$$

It is clear that the structure of (3-36) is close to the form (3-31), as the probabilistic constraints are imposed over linear combinations of the state and the artificial set-point. Therefore, it is advantageous to define an extend state comprised of the real state and the artificial set-point and express the probabilistic constraint over the augmented state.

Additionally, in order to accommodate all three requirements defined in the beginning of this subsection, let Ψ be an augmented state such that $\Psi(k) = \left[x_v(k)^T \quad u(k)^T \quad w(k-1)^T \quad y_{tar}{}^T \right]^T$. Notice that linear constraints over Ψ include input limits and also soft constraints like (3-36). Moreover, the inclusion of previous noise realization $w(k-1)$ enables one to recast the problem of calculating invariant sets in the traditional framework of positional noise.

In order to use standard results on invariant sets, it is also necessary to define a dynamic evolution for this augmented state. By resuming to the notation of incremental state introduced in Section3.1, the dynamics is given by

$$\begin{bmatrix} x_v(k+1) \\ u(k+1) \\ w(k) \\ y_{tar}(k+1) \end{bmatrix} = \begin{bmatrix} A_v & 0 & -F_v & 0 \\ 0 & I_{nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{ny} \end{bmatrix} \begin{bmatrix} x_v(k) \\ u(k) \\ w(k-1) \\ y_{tar}(k) \end{bmatrix} + \begin{bmatrix} B_v \\ I_{nu} \\ 0 \\ 0 \end{bmatrix} \Delta u(k) + \begin{bmatrix} F_v \\ 0 \\ I_{nw} \\ 0 \end{bmatrix} w(k) \tag{3-37}$$

$$\Psi(k+1) = A_\Psi \Psi(k) + B_\Psi \Delta u(k) + F_\Psi w(k)$$
$$\Psi \in \mathbb{R}^{nx+2*ny+nu+nw}$$

Notice that $x_v(k)$ and $y_{tar}(k)$ are related through constraints (3-36) but not through their time evolution, since $y_{tar}(k)$ assumes a constant, independent value at each sample time. Moreover, since we are assuming measured state, the value of $w(k-1)$ is known at time *k*. General probabilistic constraints over linear combinations of Ψ are given by

$$\mathbb{P}_k\left[g_j^T\Psi(k+1)\le h_j\right]\ge 1-\alpha_j,\quad j=1,\cdots,r;\ k\in\mathbb{N}^* \tag{3-38}$$

As previously, let Δ_w be an upper bound of the 1-norm of *w* and consider the conservative approach taken by Korda et al. (2011), where constraints (3-38) are replaced with

$$\mathbb{P}\left[g_j^T\Psi(k+1)\le h_j \mid \Psi(k)\right]\ge 1-\alpha_j,\quad j=1,\cdots,r;\ k\in\mathbb{N}^* \tag{3-39}$$

By partitioning g_j^T as $\begin{bmatrix} g_{1,j}^T & g_{2,j}^T & g_{3,j}^T & g_{4,j}^T\end{bmatrix}$, where each component corresponds to the four distinct elements of Ψ, it is clear that (3-36) is equivalent to (3-40)

$$\begin{aligned}
&\mathbb{P}\left[g_j^T\Psi(k+1)\le h_j \mid \Psi(k)\right]\ge 1-\alpha_j\\
&\Leftrightarrow\\
&\mathbb{P}\left[g_{1,j}^T x_v(k+1)+g_{2,j}^T u(k)+g_{3,j}^T w(k)+g_{4,j}^T y_{tar}(k)\le h_j \mid \Psi(k)\right]\ge 1-\alpha_j\\
&\Leftrightarrow\\
&\mathbb{P}\begin{bmatrix} g_{1,j}^T\left(A_v x_v(k)+B_v\Delta u(k)-F_v w(k-1)+F_v w(k)\right)+g_{2,j}^T\left(u(k-1)+\Delta u(k)\right)\\ +g_{3,j}^T w(k)+g_{4,j}^T y_{tar}(k)\le h_j \mid \Psi(k)\end{bmatrix}\ge 1-\alpha_j\\
&\Leftrightarrow\\
&g_{1,j}^T\left(A_v x_v(k)+B_v\Delta u(k)-F_v w(k-1)\right)+g_{2,j}^T\left(u(k-1)+\Delta u(k)\right)+g_{4,j}^T y_{tar}(k)\\
&\le h_j-\mathbb{F}^{-1}_{\left(g_{1,j}^T F_v+g_{3,j}^T\right)w}\left(1-\alpha_j\right)
\end{aligned} \tag{3-40}$$

It is now possible to look for the MSRCI of system (3-37), subject to constraints (3-40). The result may be a polyhedron or, if it is not the case, a polyhedral approximation of the real set. As previously discussed, it may be represented by

$S^*_{SRCI} = \left\{ \Psi \in \mathbb{R}^{2nx} \mid \tilde{S}\Psi \le \tilde{c} \right\}$ and matrix $\tilde{S}$ may be decomposed as $\tilde{S} = \left[\tilde{S}_1 \quad \tilde{S}_2 \quad \tilde{S}_3 \quad \tilde{S}_4 \right]$. Finally, since the first control move is not subject to feedback, the invariance constraint sufficient to guarantee that $\Psi(k+1) \in S^*_{SRCI}$ for all possible noise realization is

$$\begin{aligned} &\tilde{s}^T_{1,j}\left(A_v x_v(k) + B_v \Delta u(k) - F_v w(k-1)\right) + \tilde{s}^T_{2,j}\left(u(k-1) + \Delta u(k)\right) + \tilde{s}^T_{4,j} y_{tar}(k) \\ &\le \tilde{c}_j - \left(\left\| \tilde{s}^T_{1,j} \right\|_\infty \left\| F_v \right\|_\infty - \left\| \tilde{s}^T_{3,j} \right\|_\infty \right) \Delta_w, \quad j = 1, \cdots, \tilde{r}; \end{aligned} \tag{3-41}$$

where $\tilde{s}^T_{i,j}$ are the rows of $\tilde{S}_i$ (i = 1,…, 4), respectively.

Notice that (3-41) is linear with respect to all decision variables, there are no concerns about convexity issues and indeed the resulting problem reduces to a QP. To summarize, the complete optimization corresponding to the stochastic controller for tracking with hard input bounds and soft state constraints is given in two forms: the first containing a generic description of the optimization problem defined by (3-42)-(3-47) and another with all constraints fully developed as necessary in a computational implementation, presented in (3-48)-(3-61):

Generic definition:

$$\min \mathbb{E}_{\mathfrak{X}(k)} \left[\sum_{i=1}^{p} \left\| y(k+i) - y_{tar} \right\|^2_Q + \sum_{i=0}^{m-1} \left\| \Delta u(k+i) \right\|^2_R \right] + V_O\left(y_{tar}, y_{sp}\right) \tag{3-42}$$

subject to

$$\begin{aligned} x_{tar} &= A_p x_{tar} + B_p u_{tar} \\ y_{tar} &= C_p x_{tar} \\ u_{\min} &\le u_{tar} \le u_{\max} \end{aligned} \tag{3-43}$$

$$\left| \Delta u(k+i) \right| \le \Delta u_{\max}, i = 0, \cdots, m-1 \tag{3-44}$$

$$u_{\min} \le u(k+i) \le u_{\max}, i = 0, \cdots, m-1 \tag{3-45}$$

$$\mathbb{P}\left[g_j^T x_v(k+1) \le h_j \right] \ge 1 - \alpha_j, \quad j = 1, \cdots, r; \tag{3-46}$$

$$x_v(k+1) \in S^*_{SRCI}, \forall w(k) \tag{3-47}$$

Detailed expression:

$$\min V(k)+V_O\left(y_{tar},y_{sp}\right) \tag{3-48}$$

subject to

$$\begin{aligned}
&V(k)=\left\|x(k)\right\|^2_{\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{A}}+tr\left(\bar{F}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\mu^{\Delta w\Delta w}(k)\right)\\
&+2x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\left(\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)\\
&+2tr\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\left(\mu^{\Delta w\Delta w}(k)\bar{G}^T+\mu^{\Delta w}(k)\bar{D}^T\right)\right)+\\
&tr\left(\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{B}+\bar{R}\right)\left(\bar{G}\mu^{\Delta w\Delta w}(k)\bar{G}^T+\bar{G}\mu^{\Delta w}(k)\bar{D}^T+\bar{D}\mu^{\Delta w}(k)^T\bar{G}^T+\bar{D}\bar{D}^T\right)\right)\\
&-2\bar{Y}_{sp}^T\bar{Q}\bar{C}\left(\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)+\left\|\bar{Y}_{tar}\right\|^2_{\bar{Q}}\\
&tr\left(T_m{}^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{A}}T_m\Sigma_{\bar{X}}\right)+tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UU}\right)+tr\left(\bar{\bar{F}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\bar{\bar{\mu}}^{\Delta w\Delta w}\right)\\
&+2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UX}\right)+2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WX}\right)+2tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WU}\right)\\
&-2\bar{\bar{Y}}_{sp}^T\bar{\bar{Q}}\bar{\bar{C}}\left(\bar{\bar{A}}T_m\left(\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)\right)+\left\|\bar{\bar{Y}}_{tar}\right\|^2_{\bar{\bar{Q}}}
\end{aligned} \tag{3-49}$$

$$V_O\left(y_{tar},y_{sp}\right)=Q_\lambda{}^T\lambda \tag{3-50}$$

$$x_{tar}=A_px_{tar}+B_pu_{tar} \tag{3-51}$$

$$y_{tar}=C_px_{tar} \tag{3-52}$$

$$u_{\min}\le u_{tar}\le u_{\max} \tag{3-53}$$

$$y_{tar}-\lambda\le y_{sp} \tag{3-54}$$

$$-y_{tar}-\lambda\le -y_{sp} \tag{3-55}$$

$$\lambda\ge 0 \tag{3-56}$$

$$\Delta u(k+i)=d(k+i)+\sum_{j=0}^{i-1}G(k+i,k+j)\Delta w(k+j),\ i=0,\cdots,m-1 \tag{3-57}$$

$$\left|d_i\right|+\left\|G_i\right\|_\infty\left(w_{\max}-w_{\min}\right)\le\Delta u_{\max},i=1,\cdots,m\cdot nu \tag{3-58}$$

$$U_{\min}(k)\le M\Delta\bar{U}(k)\le U_{\max}(k) \tag{3-59}$$

$$\begin{aligned}
&g_{1,j}^T\left(A_vx_v(k)+B_v\Delta u(k)-F_vw(k-1)\right)+g_{2,j}^T\left(u(k-1)+\Delta u(k)\right)+g_{4,j}^Ty_{tar}(k)\\
&\le h_j-\mathbb{F}^{-1}_{\left(g_{1,j}^TF_v+g_{3,j}^T\right)w}\left(1-\alpha_j\right),\quad j=1,\cdots,r;
\end{aligned} \tag{3-60}$$

$$\begin{aligned}
&\tilde{s}_{1,j}^T\left(A_vx_v(k)+B_v\Delta u(k)-F_vw(k-1)\right)+\tilde{s}_{2,j}^T\left(u(k-1)+\Delta u(k)\right)+\tilde{s}_{4,j}^Ty_{tar}(k)\\
&\le\tilde{c}_j-\left(\left\|\tilde{s}_{1,j}^T\right\|_\infty\left\|F_v\right\|_\infty-\left\|\tilde{s}_{3,j}^T\right\|_\infty\right)\Delta_w,\quad j=1,\cdots,\tilde{r};
\end{aligned} \tag{3-61}$$

It is worthwhile noticing that this formulation still considers a worst-case value for the disturbance in order to find the robust invariant set. The extra degrees of freedom when compared to standard robust MPC come from the fact that constraint (3-40) only implies (3-35) with probability p. In comparison to nominal MPC, the main advantage is the explicit inclusion of the noise contribution to the prediction of future outputs, generating a kind of feedforward controller.

4. Case Studies

This chapter provides numerical examples of the proposed controller's performance in comparison to other possible control strategies, namely a finite horizon MPC and a truncated LQR (Linear Quadratic Regulator). All controllers were implemented in Matlab® R2012a in Windows® platform with Intel® Core i5-2400 processor at 3.1 GHz and 8 GB of RAM. For the stochastic controller, YALMIP parser (Löfberg, 2004) has been used in problem formulation with GUROBI® (Gurobi Optimization, 2013) as the solver.

The implemented finite horizon MPC controller is based on the same incremental description of the system as the stochastic controller, but ignores disturbances when calculating the predictions. The complete optimization problem is given by (4-1)-(4-5). It was solved using Matlab's built-in routine for quadratic programming (*quadprog*).

$$\min \sum_{i=1}^{p} \left\| y_{MPC}(k+i) - y_{sp} \right\|_Q^2 + \sum_{i=0}^{m-1} \left\| \Delta u(k+i) \right\|_R^2 \tag{4-1}$$

subject to

$$x_{MPC}(k+1) = A_v x_{MPC}(k) + B_v \Delta u(k), \quad k = 0, \cdots, p-1 \tag{4-2}$$

$$y_{MPC}(k) = C_v x_{MPC}(k), \quad k = 1, \cdots, p \tag{4-3}$$

$$u_{\min} \le u(k+i) \le u_{\max}, \; i = 0, \cdots, m-1 \tag{4-4}$$

$$\left| \Delta u(k+i) \right| \le \Delta u_{\max}, \; i = 0, \cdots, m-1 \tag{4-5}$$

Regarding the LQR controller, it considered a steady state gain K_{LQR} calculated from matrices *Q* and *R* and using Matlab's function *dlqr*. Since we consider tracking problems, the feedback used was

$$u(k) = K_{LQR}\left(x(k) - x_{tar} \right) + u_{tar} \tag{4-6}$$

where targets x_{tar} and u_{tar} were calculated using (3-22). Saturation of input values and control moves was introduced after the calculation of *u* according to (4-6). It is important to mention that such truncated controller presents neither optimality nor stability guarantees.

The structure of this Chapter is as follows: first, the system chosen as a case study is described, then two types of simulation results are given, for symmetric zero-mean and non-symmetric non-zero mean noise distributions. The methodology of the simulation consists of 1000 Monte Carlo repetitions, with the same noise realization being provided to three controllers.

Mean values for each sampling time are plotted to show that the mean of the outputs calculated by the stochastic controller converge to the set-point. Moreover, histograms of variables distributions are presented to indicate that the proposed controller achieves lower output variances with less intense control moves.

It was indeed expected that the proposed stochastic controller would outperform the others in both cases, since it explicitly considers noise distribution and system constraints simultaneously. The feedforward effect of the inclusion of disturbances is the main responsible for the reduction of variances.

4.1.1. System description

The system shown below is an example adapted from Ogunnaike et al. (1983), representing the control of a binary ethanol-water distillation column. The system model is composed of first and second order transfer functions plus time delays, identified from experiments in a real column. Since our approach relies on invariant set construction, it is not computationally feasible to consider systems with large state space descriptions. Therefore, we have simplified the original model by omitting time delays. Three controlled variables are subject to three manipulated variables and to two disturbances, as presented in Table 4-1:

Table 4-1: Variables of the ethanol-water system

Variable	Description
y_1	Overhead ethanol mole fraction
y_2	Side stream ethanol mole fraction
y_3	Temperature of tray number 19 (ºC)
u_1	Reflux flow rate (gpm)
u_2	Side stream product flow rate (gpm)
u_3	Reboiler stream pressure (psig)
w_1	Feed flow rate (gpm)
w_2	Feed temperature (ºC)

The system description without time delays is given next, with time constants in minutes:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{bmatrix} = \begin{bmatrix} \dfrac{0.66}{6.7s+1} & \dfrac{-0.61}{8.64s+1} & \dfrac{-0.0049}{9.06s+1} \\ \dfrac{1.11}{3.25s+1} & \dfrac{-2.36s}{5s+1} & \dfrac{-0.0012}{7.09s+1} \\ \dfrac{-34.68}{8.15s+1} & \dfrac{46.2}{10.9s+1} & \dfrac{0.87(11.61s+1)}{(3.89s+1)(18.8s+1)} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \end{bmatrix} + \\ + \begin{bmatrix} \dfrac{0.14}{6.2s+1} & \dfrac{-0.0011(26.32s+1)}{(7.85s+1)(14.63s+1)} \\ \dfrac{0.53}{6.9s+1} & \dfrac{-0.0032(19.62s+1)}{(7.29s+1)(8.94s+1)} \\ \dfrac{-11.54}{7.01s+1} & \dfrac{0.32}{7.76s+1} \end{bmatrix} \begin{bmatrix} w_1(s) \\ w_2(s) \end{bmatrix} \tag{4-7}$$

The model was discretized using a sampling time of 1 minute. Ogunnaike et al. (1983) do not provide information regarding noise distribution, since the work only presents minimum and maximum disturbance values. Therefore, we have chosen distributions within the same order of magnitude but with various shapes.

Regarding system constraints, Ogunnaike et al. (1983) only consider input bounds. In addition to these limits, maximum control moves were here imposed over the system, as presented in Table 4-2:

Table 4-2: Input and control moves bounds

Input	Minimum value (u_{min})	Maximum value (u_{max})	Maximum change (Δu_{max})
u_1	0.068 gpm	0.245 gpm	0.02 gpm
u_2	0.00694 gpm	0.1 gpm	0.02 gpm
u_3	15.6 psig	34 psig	5 psig

Also regarding the constraints, two situations were considered: in the first, there are no state constraints, whereas in the second a soft constraint was added, forcing the overhead ethanol mole fraction (y_1) to be greater than its artificial set-point with 90% of probability. The goal of the first scenario is to show that the mean of the

system output with the proposed controller is able to track the set-point without offset. The second one demonstrates the application of the soft constraint strategy based on invariant sets but it induces a back-off of $y_1$from its set-point. This result should not be confused with a typical offset, since it is intentionally produced by the controller in order to respect the violation probability.

In terms of set-point tracking, the system was to be kept around the origin for the first 100 minutes and at another steady state for the second part. Initial conditions and set-points are presented in Table 4-3:

Table 4-3: Simulation operating points

System initial conditions		Output set-points	
Inputs	Outputs	$0 \le k \le 100$	$101 \le k \le 200$
0.18 gpm	0.7	0.7	0.64
0.046 gpm	0.52	0.52	0.4
20 psig	91 ºC	92 ºC	94 ºC

The set-points considered in Ogunnaike et al. (1983) are between 0.65 and 0.7 for y_1, 0.45 and 0.53 for y_2, 92ºC and 97ºC for y_3.

Since the order of magnitude of the variables spans a considerable range, a normalization procedure was used. The definition of scaled variables is given by

$$y_{sca} = \frac{y}{weight} \tag{4-8}$$

Scaling weights are 10^{-1} for both mole fractions (y_1 and y_2)and 10 for the temperature y_3. Tuning parameters given in next sections correspond to the scaled variables, where previously discussed bounds and transfer functions are given in terms of engineering units.

4.1.2. Truncated Gaussian noise

In this first example, we generate random noise according to distributions with zero-mean and standard deviation of 0.07 gpm for the feed flow rate and 8ºC for the temperature. The distributions are truncated at 3 standard deviations, providing bounded symmetric noise. These values of standard deviation have been chosen to

approximately match the disturbances ranges described in Ogunnaike et al. (1983), which are 0.2 gpm and 20ºC respectively.

The relevant tuning parameters for the three controllers are presented in Table 4-4. The same weights Q and R were used for all controllers, to standardize the relative importance of the variables.

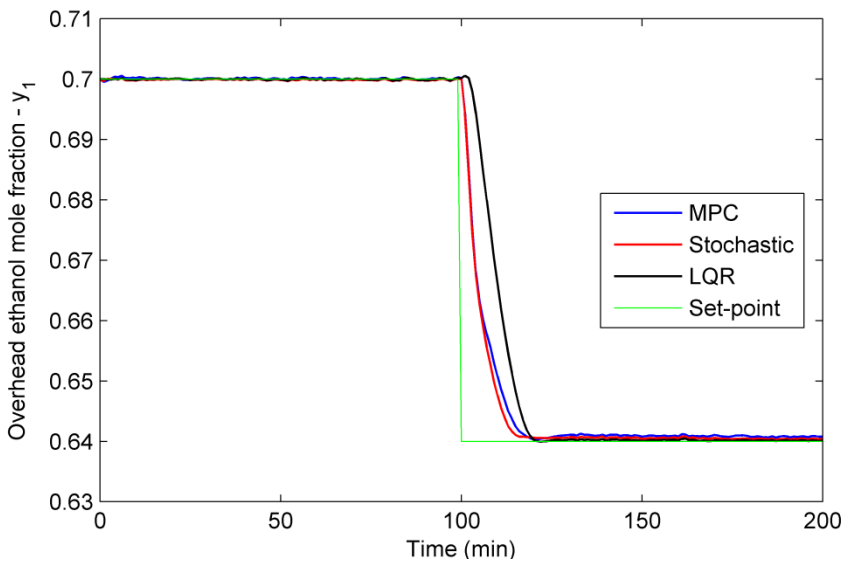
Table 4-4: Tuning parameters – symmetric noise

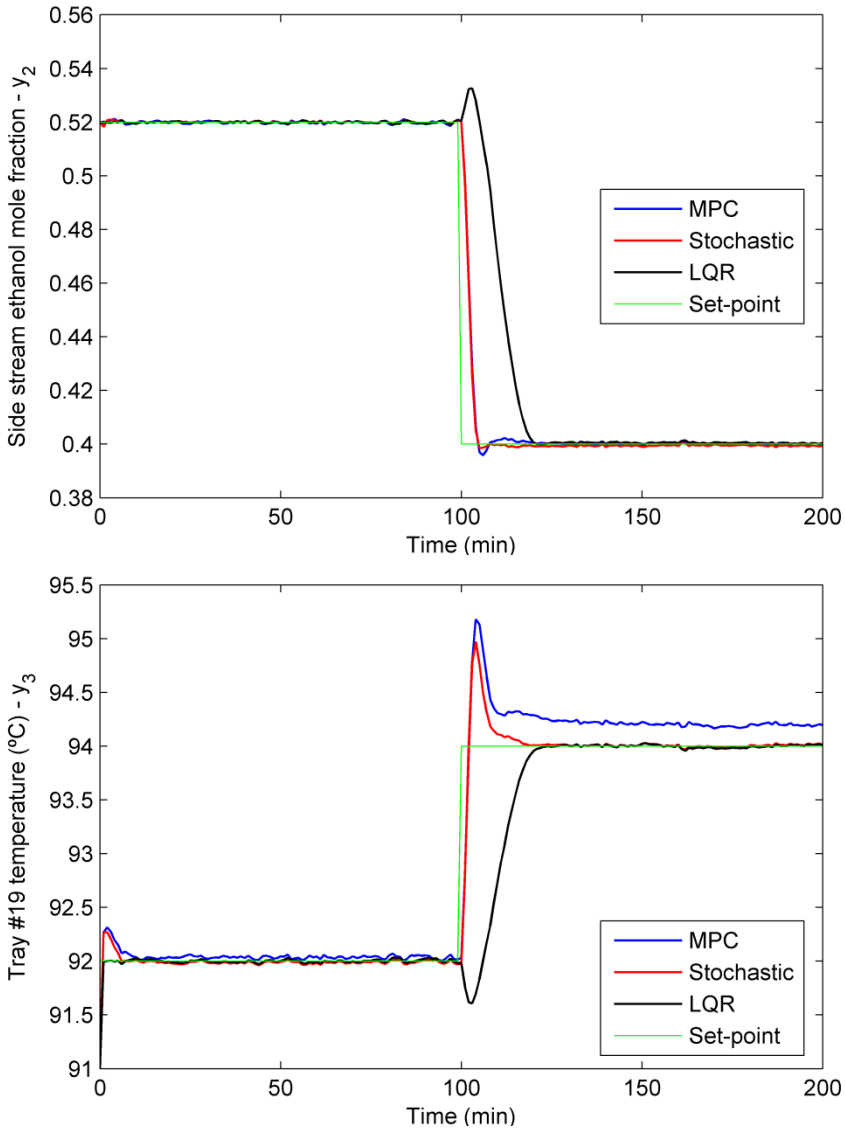
Parameter	Value
Q	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}$
R	$\begin{bmatrix}10^{-1} & 10^{-1} & 10^{-3}\end{bmatrix}$
Q_{λ}	$\begin{bmatrix}10^{4} & 10^{4} & 10^{4}\end{bmatrix}$
p	30
m	3

4.1.2.1. Simulation without state constraints

Figure 4-1 shows the mean values of system outputs. Since they are disturbed by symmetric zero-mean noise, it is clear that on average the effect of a particular noise realization tends to be cancelled out by the contribution of other realizations.

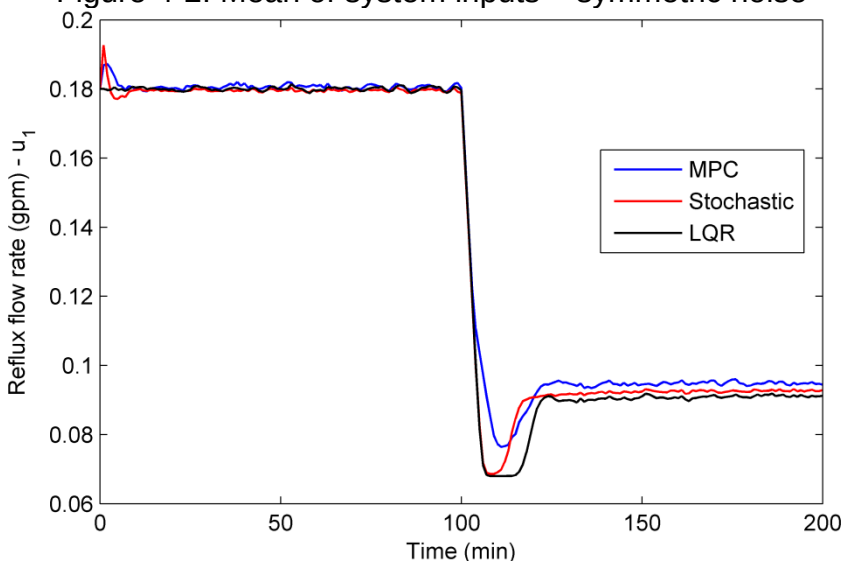
Figure 4-1: Mean of system outputs – symmetric noise

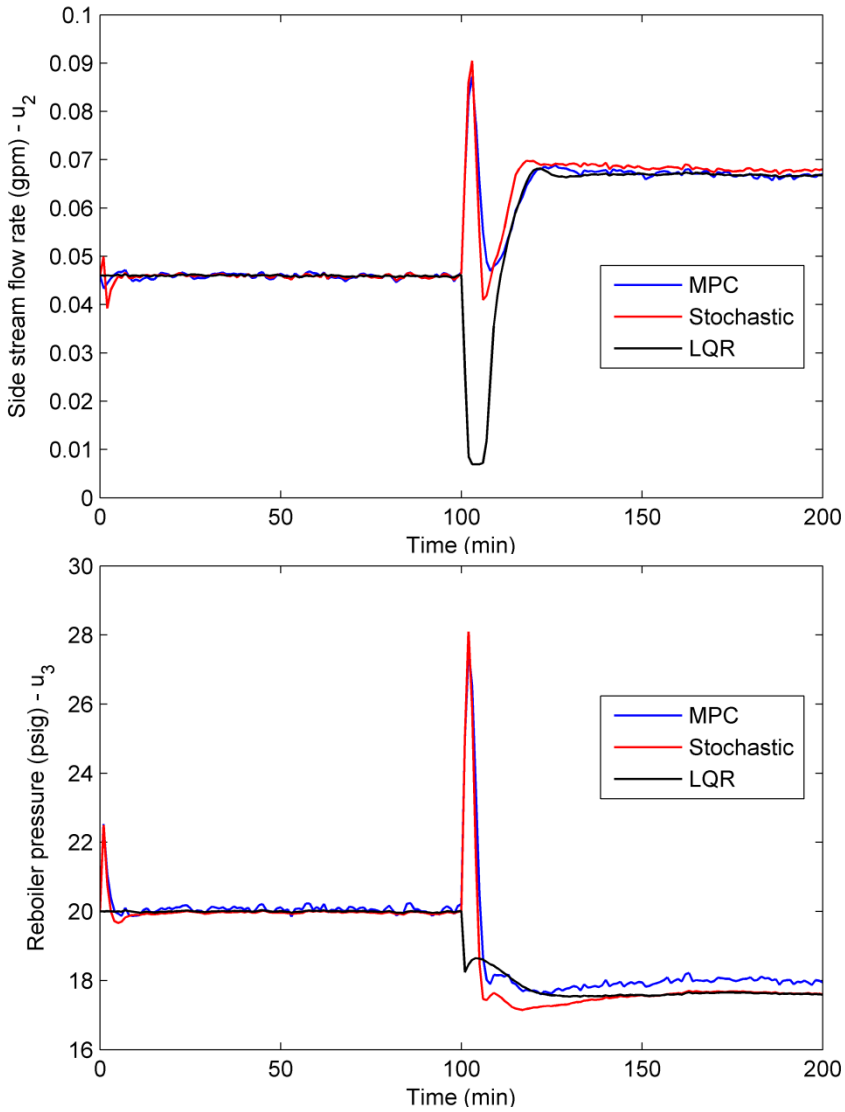




Source: own elaboration

Figure 4-2: Mean of system inputs – symmetric noise





Source: own elaboration

Regarding Figure 4-2 and the inputs, we note that the saturation of the truncated LQR is active on average after the set-point change at 100 min. This implies that at each realization it became active and therefore optimality of this controller was indeed lost. The fact of having active input constraints is naturally handled by the predictive controllers, causes them to achieve better performances in terms of achieving the set-point faster. The difference between the stochastic and standard predictive controllers is more significant regarding y_3, whose average presents an offset in the second part of the simulation.

In order to compare quantitatively the results obtained by each controller, the usual approach is to look at the corresponding objective functions and determine which one is better. It is not possible to follow this procedure in the present case

since each controller considers different contributions, for instance, standard MPC disregards noise whereas the proposed stochastic controller is based on expected values. One alternative to overcome this problem is to use a metric such as the Integral Square Error (ISE). We propose to use a weighted version of this criterion, defined as

$$J = \sum_{k=0}^{nsim} \left\| y(k) - y_{sp}(k) \right\|_Q^2 \tag{4-9}$$

where J is the weighted total error and *nsim* is the simulation horizon. Note that the set-point is explicitly allowed to change during this horizon. If all controllers are designed using the same matrix Q, then the errors are comparable. Finally, since this expression is evaluated after the simulations, the error is computed from the actual outputs obtained with each given noise, avoiding the evaluation of any expected values.

The most direct way to analyze data from all 1000 Monte Carlo repetitions is to compare the means of the weighted integral square errors in all realizations. This comparison is presented in Table 4-5

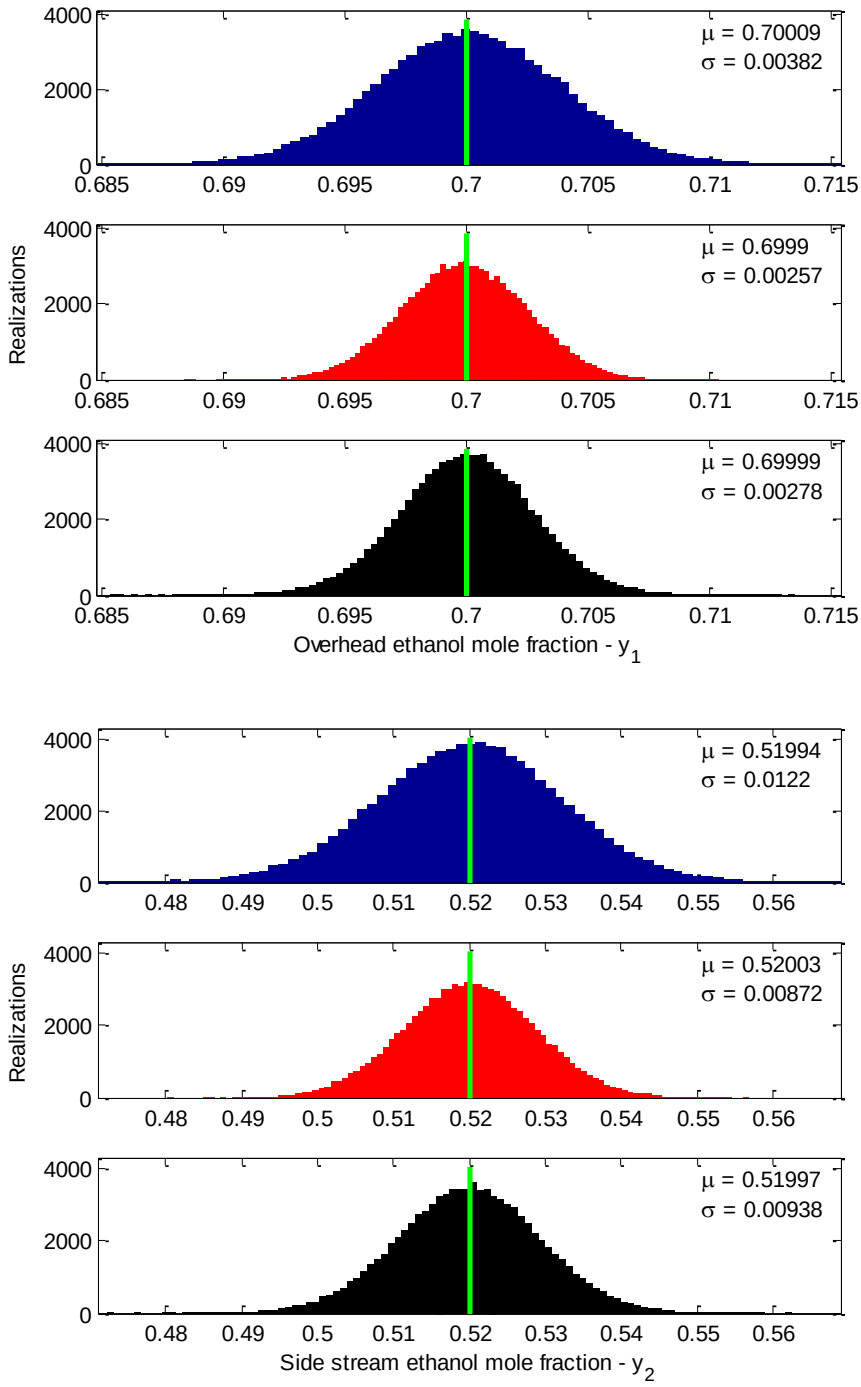
Table 4-5: Average weighted integral square error – symmetric noise

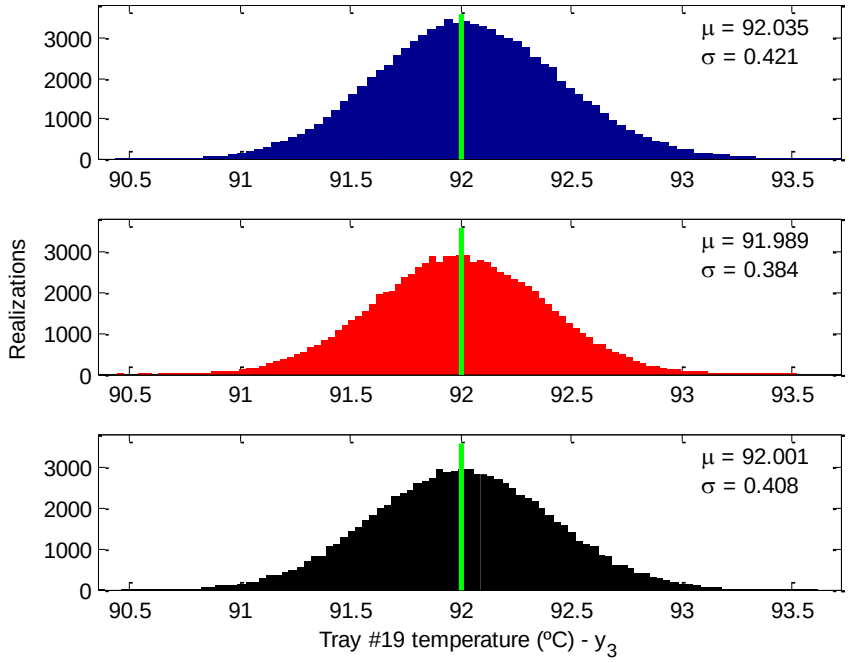
Standard MPC J_{MPC}	Stochastic controller J_{Sto}	Truncated LQR J_{LQR}
7.93	6.13	20.72

Since mean output values are similar for the first two controllers, Figure 4-1may be not enough to explain the better performance revealed by Table 4-5, even though the transient corresponding to the stochastic controller is the fastest. The main reason of this improvement is the variance reduction obtained with the proposed controller. Figures 4-3 and 4-4 show this effect by comparing the distribution of the outputs for all 1000 Monte Carlo repetitions. To improve readability, each figure is dedicated to one steady state of the simulation. The results obtained with each controller are presented in separated histograms, using the same color pattern of previous graphs. Also, the set-point corresponding to that part of the simulation is marked in green. Mean (μ) and standard deviation (σ) values are given for the

distribution obtained with each controller, allowing a quantitative comparison: ideally, the standard deviation should be as small as possible, whereas the mean should be equal to the set-point in this case without state constraints.

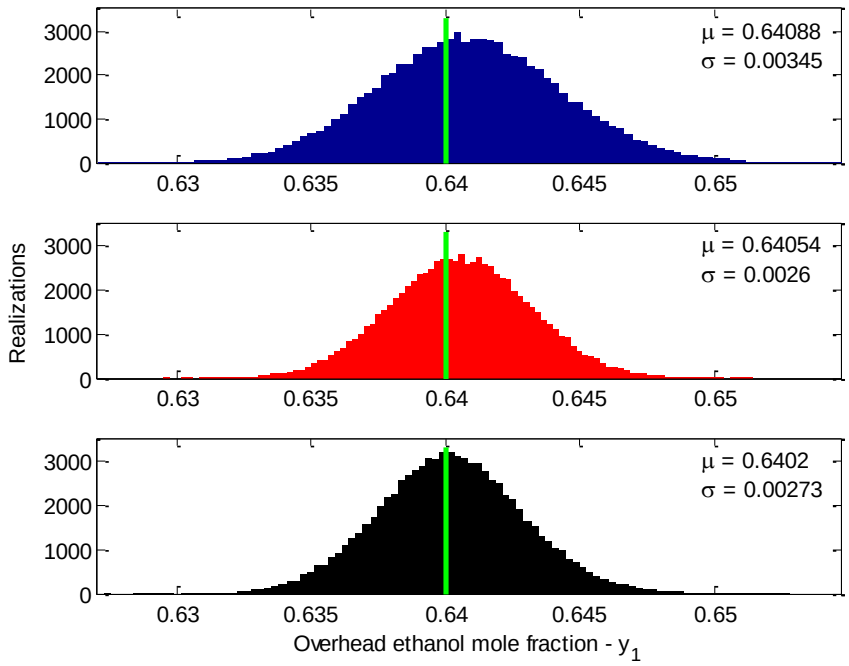
Figure 4-3: Distribution of system outputs between 10 and 100 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green

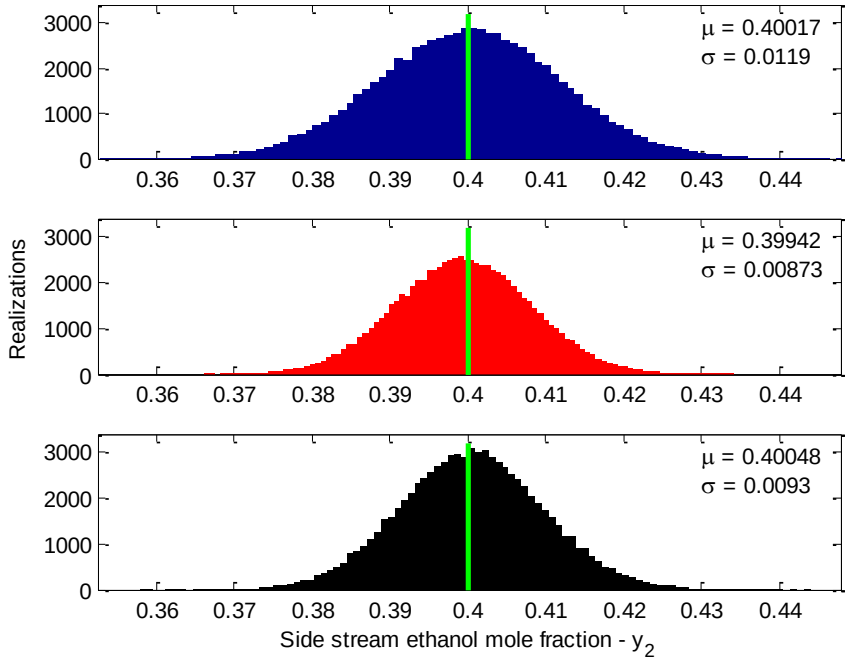


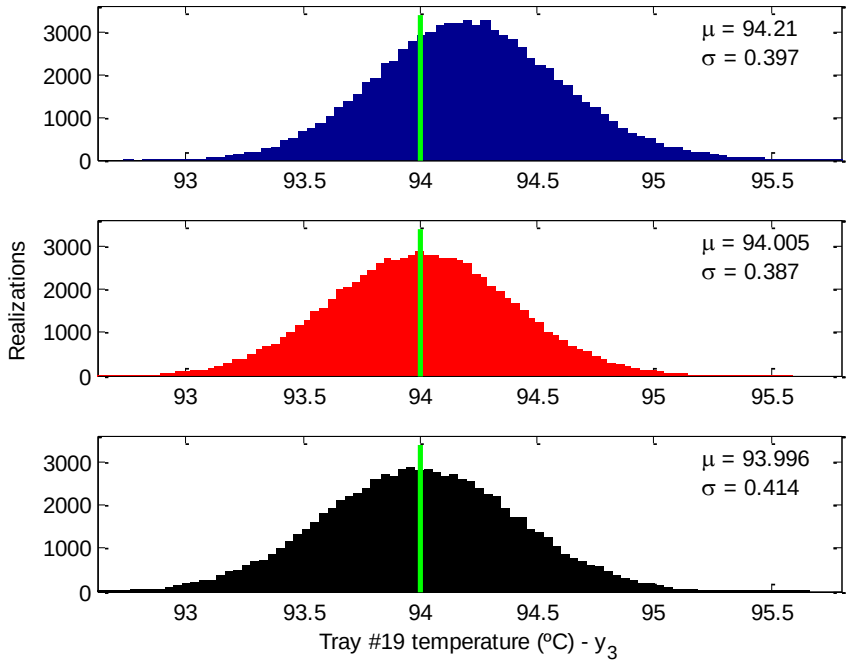


Source: own elaboration

Figure 4-4: Distribution of system outputs between 120 and 200 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green







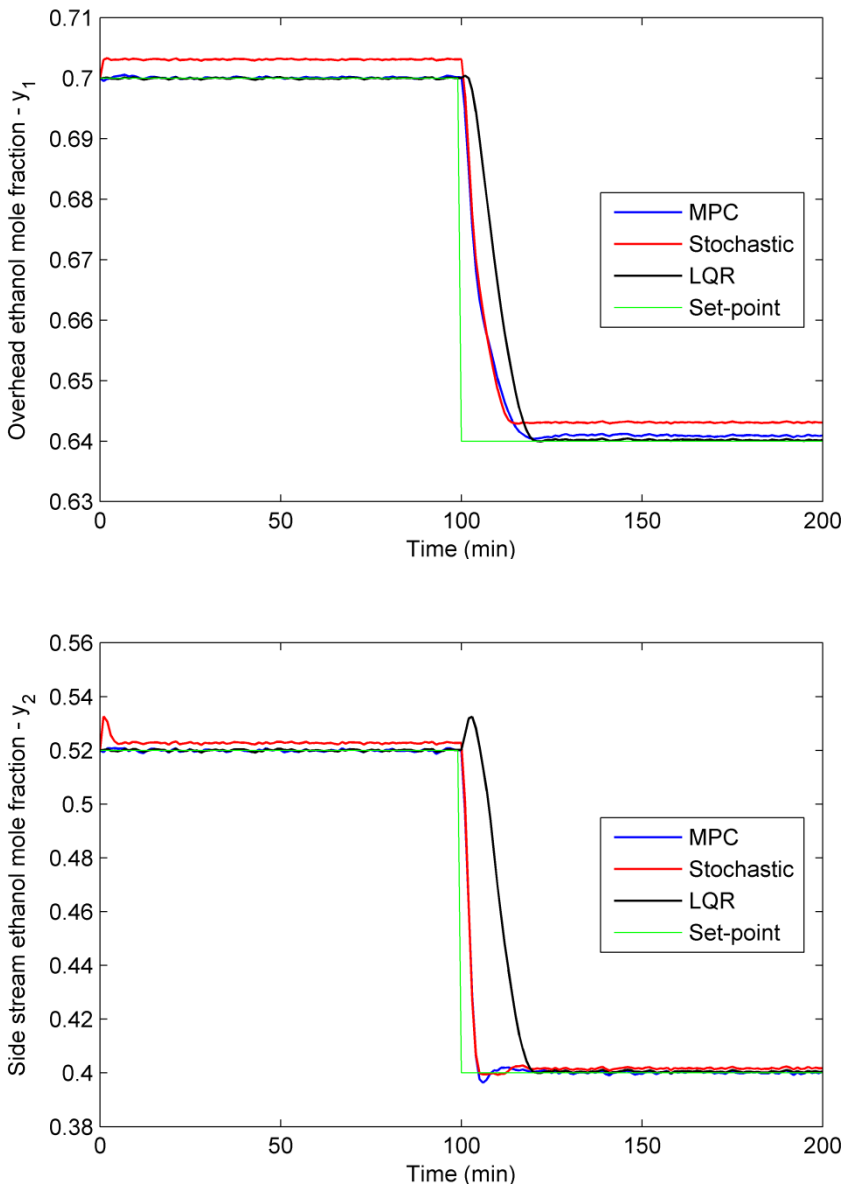
Source: own elaboration

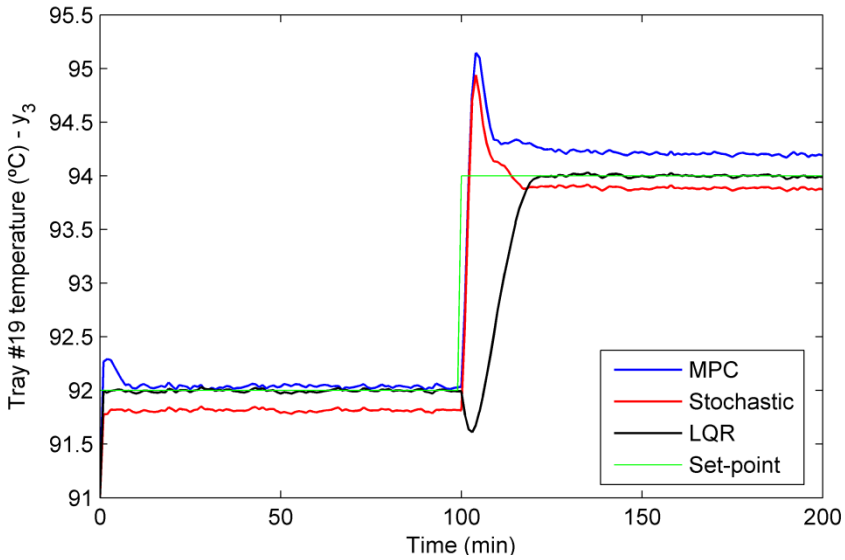
From Figures 4-3 and 4-4 it is clear that the proposed controller is able to reduce output variances for all cases, even when compared to the truncated LQR at steady state, where the probability of activating the constraints is lower. Besides, they

evidence that the mean value is very close to the corresponding set-point for all outputs in both steady states, highlighting its set-point tracking capability.

4.1.2.2. Simulation with state constraints

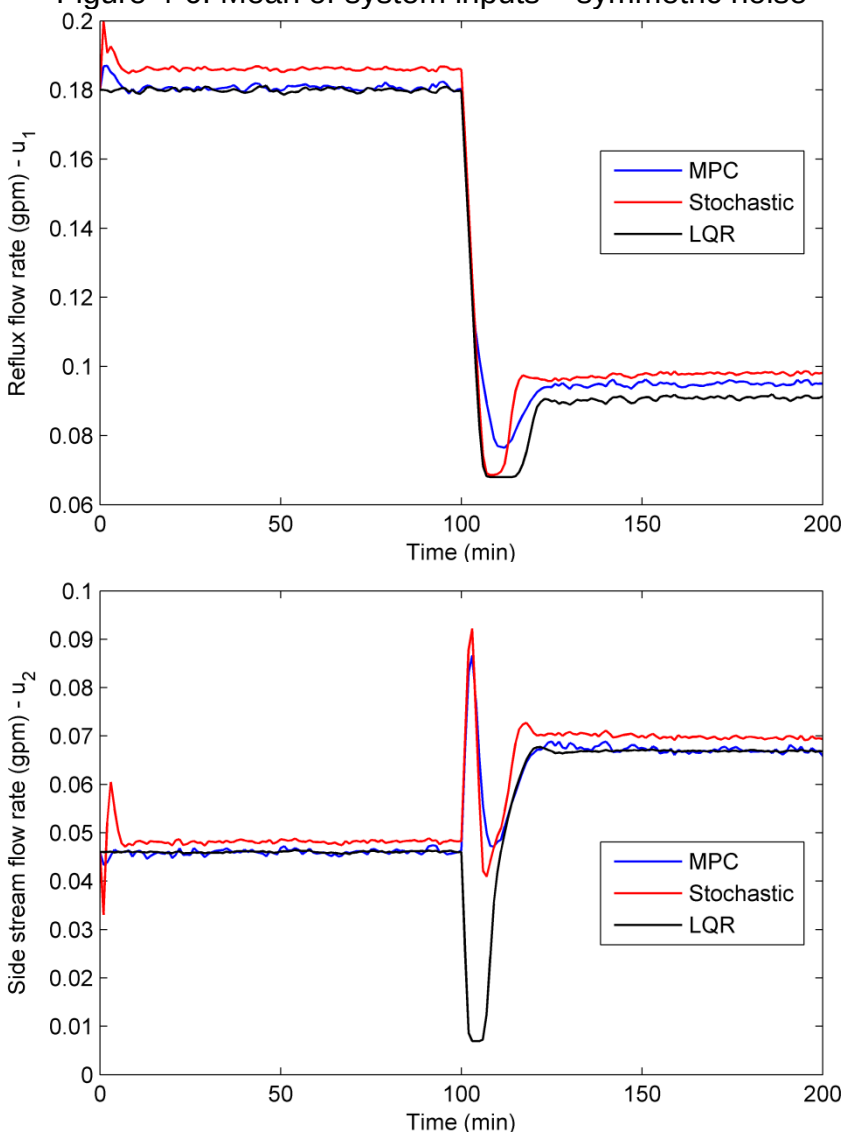
Figure 4-5: Mean of system outputs – symmetric noise

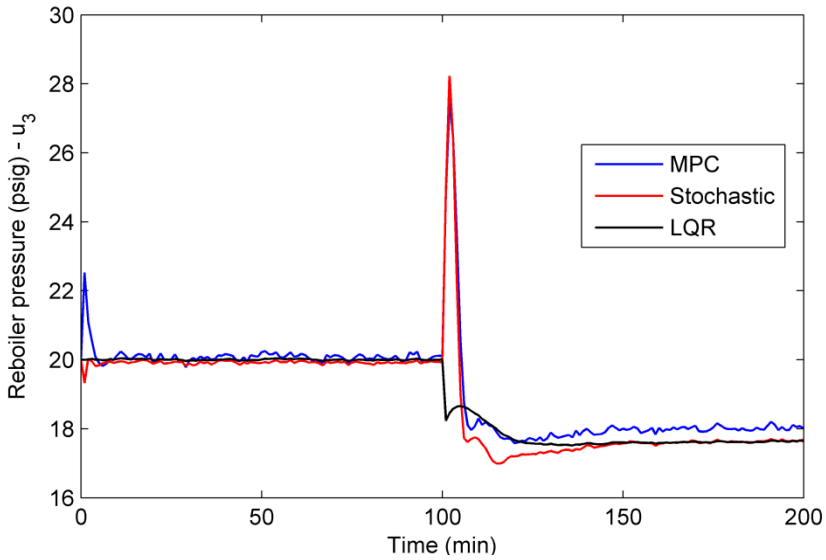




Source: own elaboration

Figure 4-6: Mean of system inputs – symmetric noise





Source: own elaboration

As previously stated, noise realizations compensate each other since their distribution is symmetric. However, in contrast to section 4.1.2.1, the main difference between the results of the proposed controller and literature benchmarks is the presence of a back-off in the overhead ethanol mole fraction (y_1) in relation to its set-point. It is a consequence of the soft constraint: to guarantee that y_1 is greater than the set-point for 90% of the time, it is necessary that its mean value is also greater than the set-point. A compromise must be found, since the objective function drives the outputs towards the set-points but the soft constraint tries to avoid this approximation. Table 4-6 subsumes the effect of this bound in each of the two steady-states of the simulation: from 10 to 100 min and from 120 to 200 min. Since the violation percentage of the stochastic controller is close to the prescribed value of 10%, it shows that the approach is not excessively conservative.

Table 4-6: Percentage of time steps with y_1 lower than its set-point for each controller – symmetric noise

First steady-state ($10 \leq k \leq 100$)		Second steady-state ($120 \leq k \leq 200$)	
Controller	Violations	Controller	Violations
MPC	49,13%	MPC	39,93%
Stochastic	9,66%	Stochastic	9,91%
LQR	49,88%	LQR	47,33%

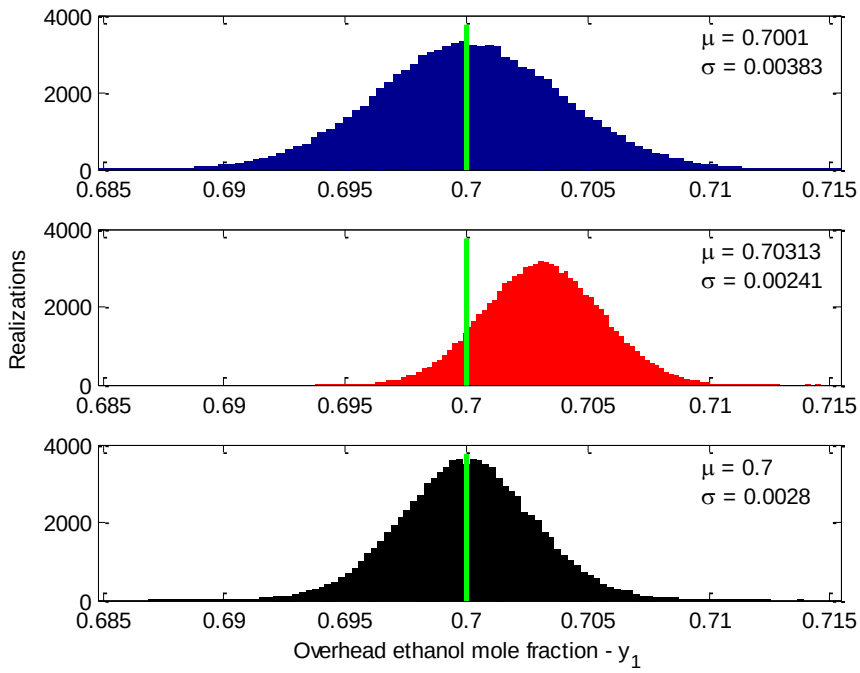
It has been shown in Figure 4-5 that the inclusion of the soft constraint moves the system away from the set-point. However, when using the index defined in (4-9) as a measure of control performance, the following results are obtained:

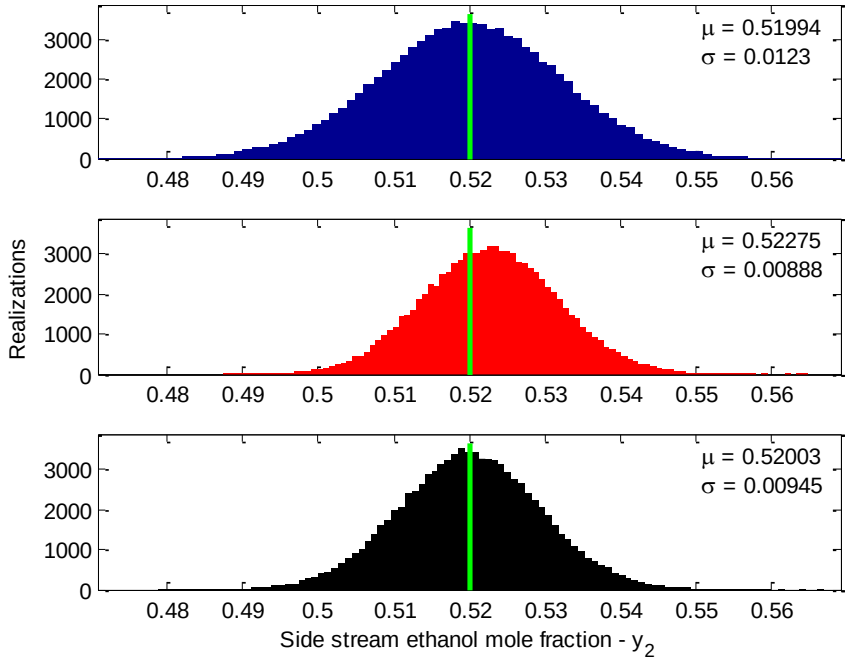
Table 4-7: Average weighted integral square error – symmetric noise

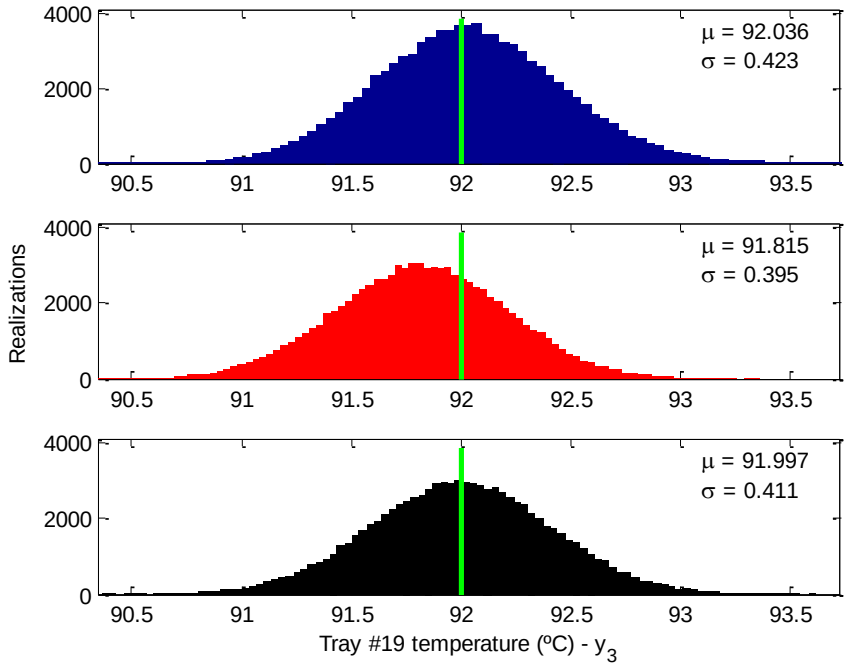
Standard MPC J_{MPC}	Stochastic controller J_{Sto}	Truncated LQR J_{LQR}
7.96	6.90	20.81

Comparing the values of Table 4-7 and Table 4-5, for the proposed controller there is an increase of 12.5% (from 6.13 to 6.9) due to the inclusion of the soft constraint. Nonetheless, it still outperforms the standard MPC that is oblivious to this constraint. Small fluctuations of the average errors for the MPC and LQR between these two tables, in the order of 0.4%, are a consequence of the use of a different random seed to generate noise in each of the examples.

Figure 4-7: Distribution of system outputs between 10 and 100 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green

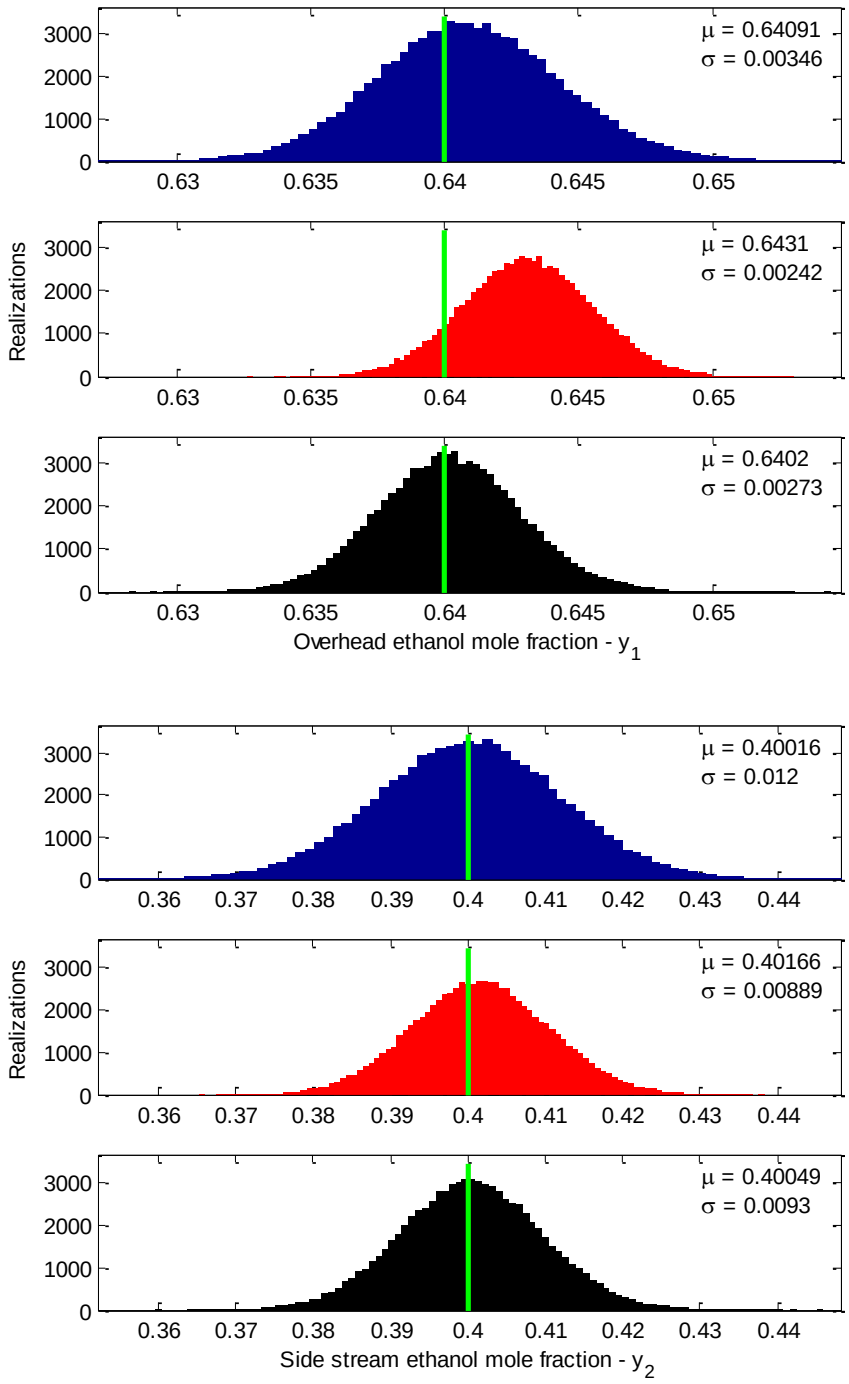


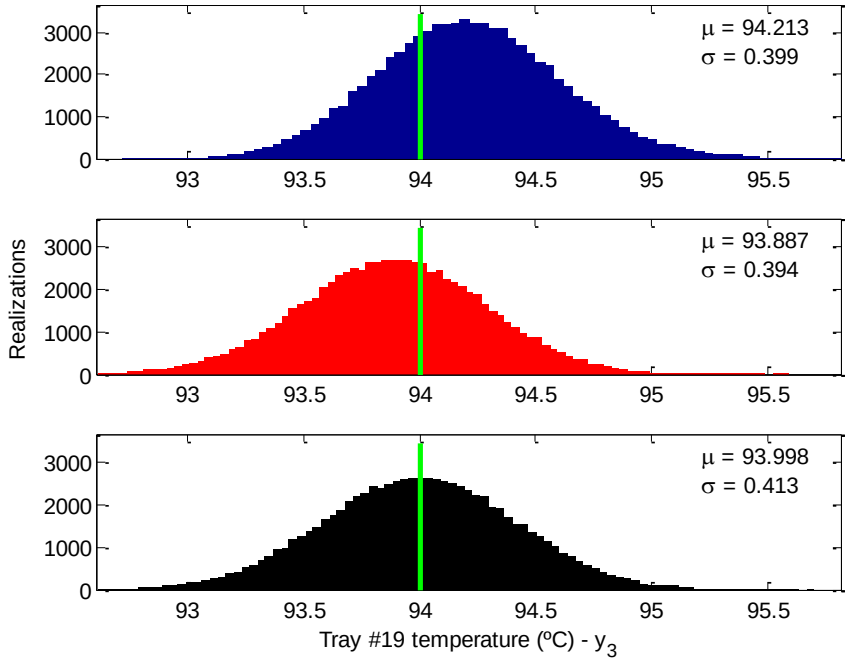




Source: own elaboration

Figure 4-8: Distribution of system outputs between 120 and 200 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green





Source: own elaboration

Comparing the results obtained with the proposed controller in the situations with and without state constraints, it is noticeable with the state constraint the mean is shifted for all outputs because a different steady state with greater value of $y_1$is sought, but this displacement is as small as possible. Tables 4-8 and 4-9 are presented to compile the results dispersed in previous figures. Since MPC and LQR are oblivious to the state constraint, their results should ideally be the same in both circumstances, but some random deviation is inevitable. However, we note that there is no reason to expect the values to be equal at each operating point (first and second steady state).

Table 4-8: Output standard deviations in the first steady state– symmetric noise

	Without state constraint			With state constraint		
Output	MPC	Stochastic	LQR	MPC	Stochastic	LQR
y_1	0.0038	0.0026	0.0028	0.0038	0.0024	0.0028
y_2	0.0122	0.0087	0.0094	0.0123	0.0089	0.0095
y_3	0.421	0.384	0.408	0.423	0.395	0.411

Table 4-9: Output standard deviations in the second steady state– symmetric noise

	Without state constraint			With state constraint		
Output	MPC	Stochastic	LQR	MPC	Stochastic	LQR
y_1	0.0035	0.0026	0.0027	0.0035	0.0024	0.0027
y_2	0.0119	0.0087	0.0093	0.012	0.0089	0.0093
y_3	0.397	0.387	0.414	0.399	0.394	0.413

The proposed controller systematically achieves lower standard deviations in all cases when compared to the benchmark controllers. The presence of the state constraint produces a further reduction of the variance of the first output, which is also expected: if the shape of the distribution of y_1 were the same but only shifted to the right to satisfy the constraint, then it would cause the cost to increase significantly due to the right tail of the distribution being too far away from the set-point. On the other hand, the other outputs are allowed to vary in a larger interval and as a consequence: it would not be consistent if the constrained case achieved an overall smaller variation.

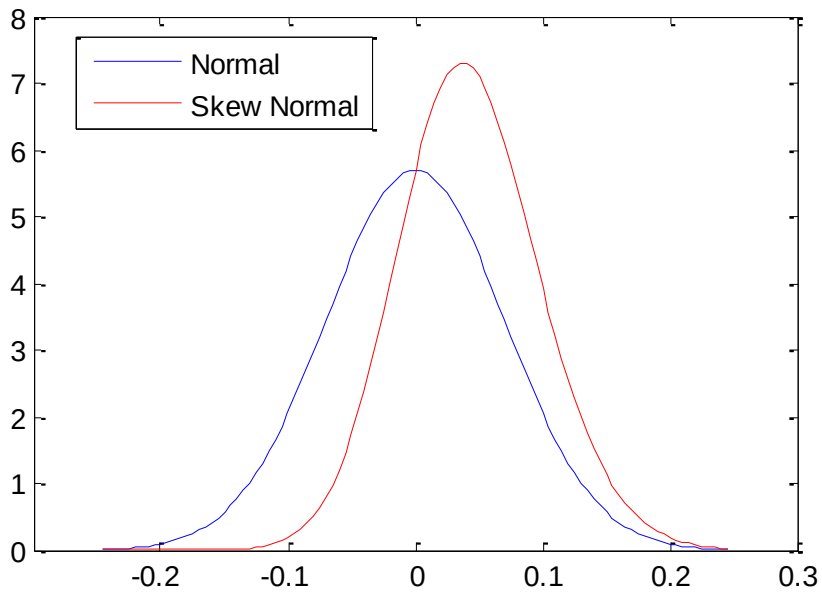
4.1.3. Skew normal noise

In order to develop the controller formulation for a generic noise distribution, we have chosen to consider skew normal distributions (Azzalini, 1985). Contrary to the standard normal distributions, which is completely defined with two parameters (mean and standard deviation), three parameters are necessary in this case. Location and scale parameters are similar, in some sense, to the mean and deviation since they may be used to shift or to flatten the distribution, respectively. The third parameter is called shape and is responsible for the skewness: if it is equal to 0, then the standard normal is recovered; otherwise, the absolute value of the skewness increases as the absolute value of the shape increases, and the sign of the skewness is equal to the sign of the shape.

In all simulations of this section, location and shape were the same for both disturbances (0 and 1.2, respectively) whereas the scales were taken as the standard deviation values of the preceding section, i.e., 0.07 gpm for the feed flow rate and 8ºC for the temperature. For illustration purposes, Figure 4-9 presents a

comparison of probability density function of normal and skew normal distributions considered for the feed flow rate in the simulations.

Figure 4-9: Comparison of density functions



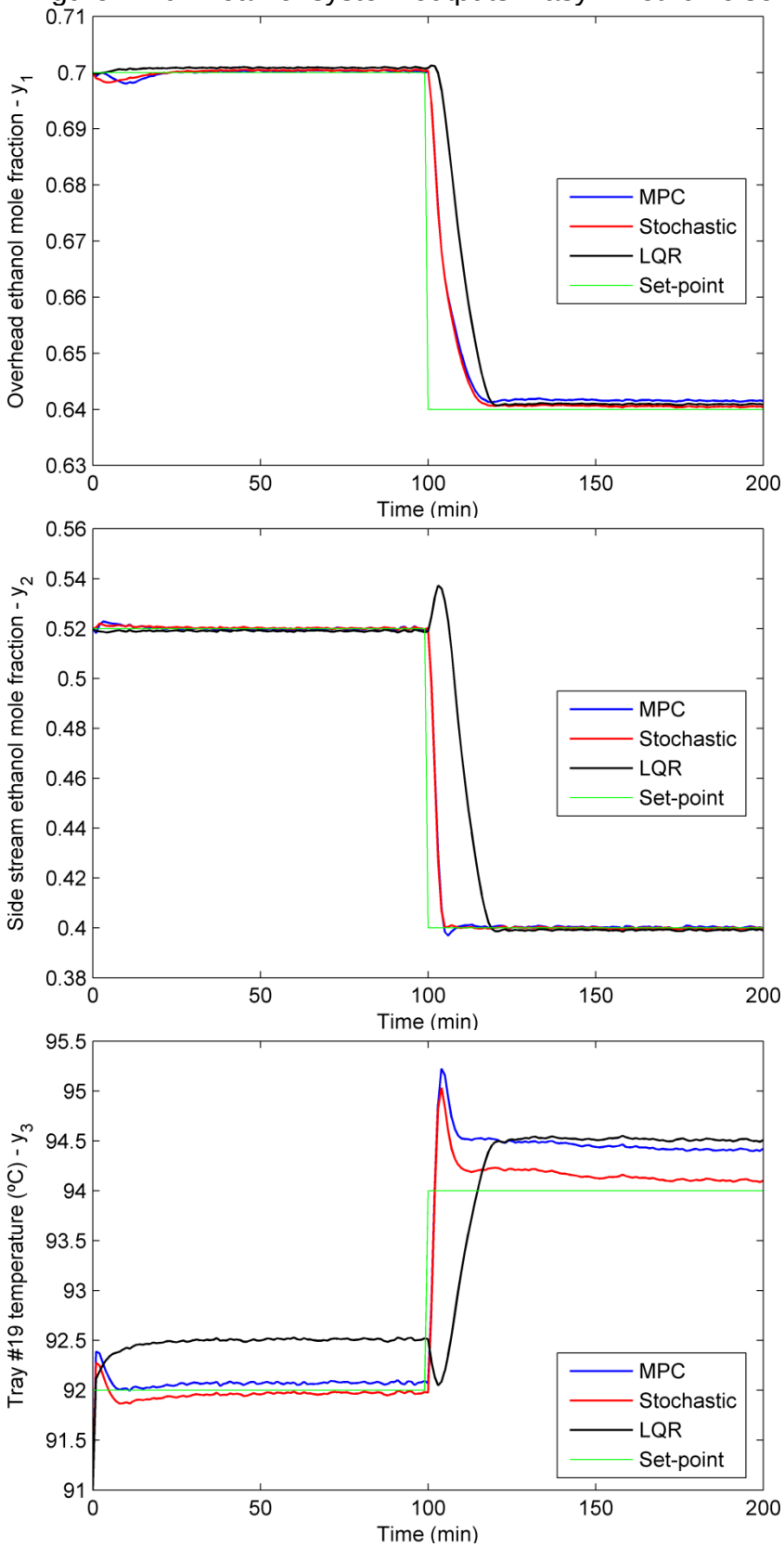
Source: own elaboration

Since distributions of this family are also unbounded, we have decided to truncate it in a similar manner to what was presented in the previous section. Minimum and maximum thresholds were established as the points corresponding to 0.5% and 99.5% of the cumulative distribution. In the referred example of the feed flow rate distribution, the limits are -0.0906 and 0.197 gpm. It is worthwhile to mention that these values correspond to deviation variables relative to a steady state of 0.8 gpm. For the feed temperature, the steady state is 78ºC and the limits of the disturbance in deviation form are -10.35 and 22.46ºC.

In a similar spirit to section 4.1.2, we present mean results and also distribution histograms for the three controllers after 1000 Monte Carlo repetitions, first for the simulation without state constraints and then to the constrained case.

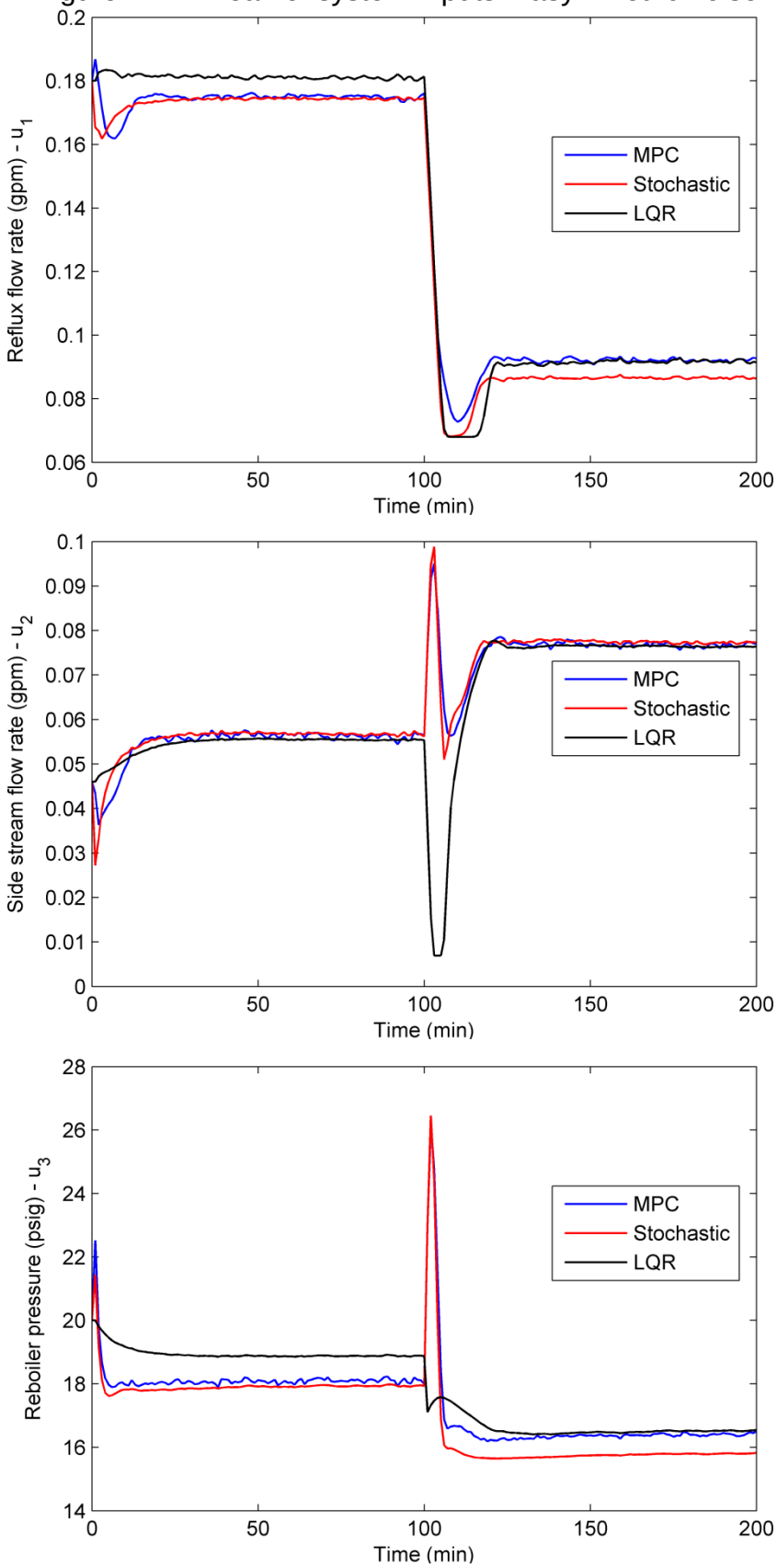
4.1.3.1. Simulation without state constraints

Figure 4-10: Mean of system outputs – asymmetric noise



Source: own elaboration

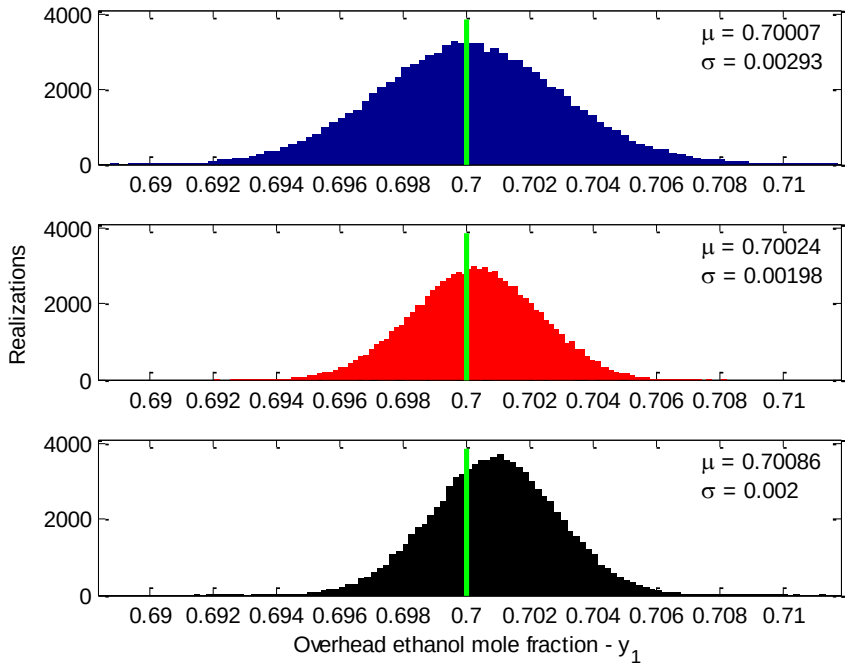
Figure 4-11: Mean of system inputs – asymmetric noise

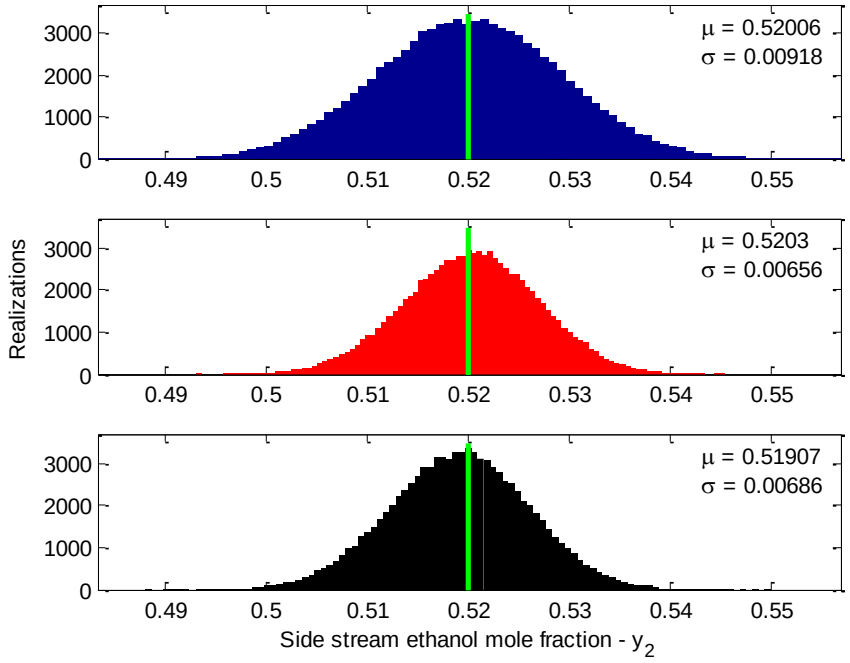


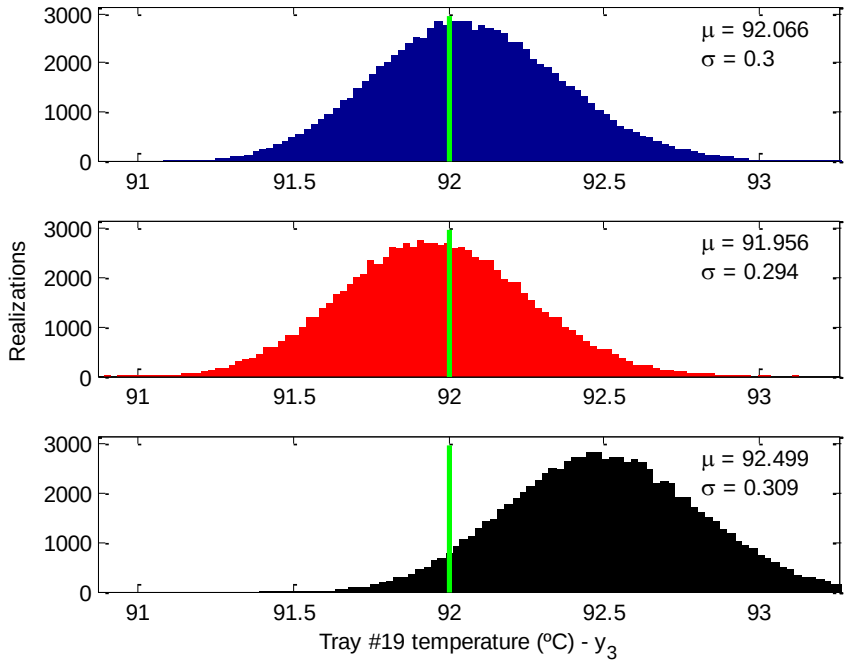
Source: own elaboration

Figures 4-10 and 4-11 show that the truncated LQR also struggles at the moment of the set-point change because the first input saturates. Once again, the most significant difference between the standard MPC and the proposed stochastic controller is in y_3. The situation where the mean value of the disturbances is no longer zero is more challenging and for this reason the mean offset value of the MPC is larger. As previously, it is necessary to compare the distribution of the results to assess how each controller is able to shape it.

Figure 4-12: Distribution of system outputs between 10 and 100 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green.

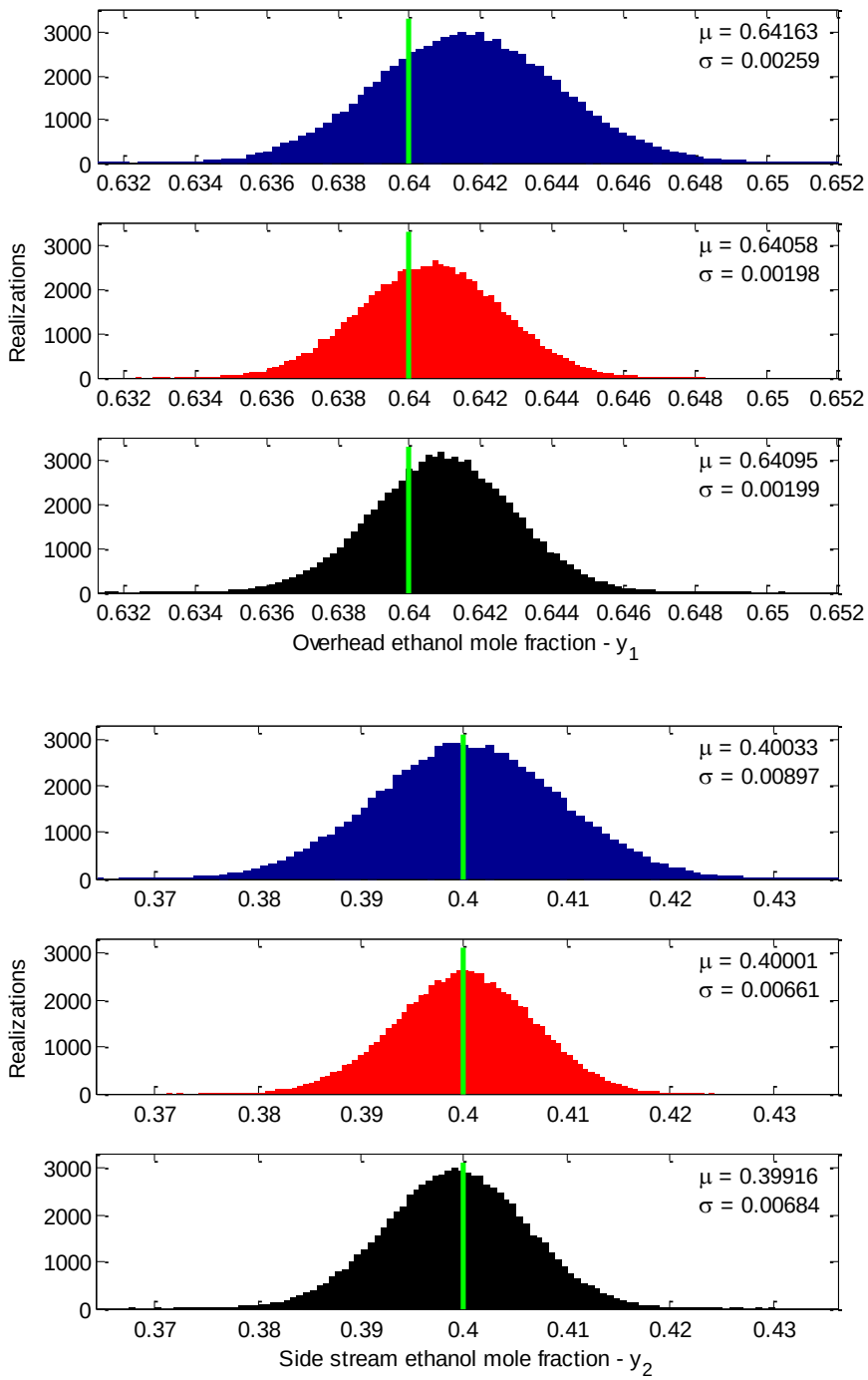


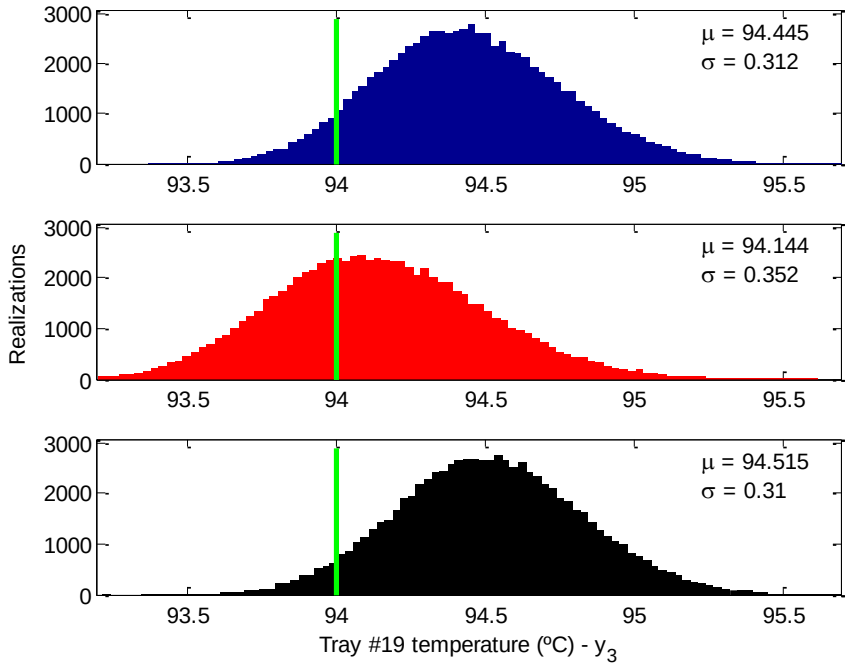




Source: own elaboration

Figure 4-13: Distribution of system outputs between 120 and 200 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green





Source: own elaboration

Figures 4-12 and 4-13 show that the skewness of the disturbances is only modestly present in the outputs, being more evident in y_3, which is not surprising due to direct relation between the feed temperature and the temperature of the trays. The mean value of the outputs obtained with the proposed controller is closer to the set-point in all cases. In terms of the standard deviations, in the case of asymmetric noise it is no longer true that the stochastic controller outperforms the benchmark, with the exception being y_3 in the second part of the simulation. However, we notice that the standard deviation of y_3 in the first part was smaller than the corresponding with the other controllers. The explanation is that y_3 is not at steady state during all the second part of the simulation, but rather drifts from a greater value towards the set-point.

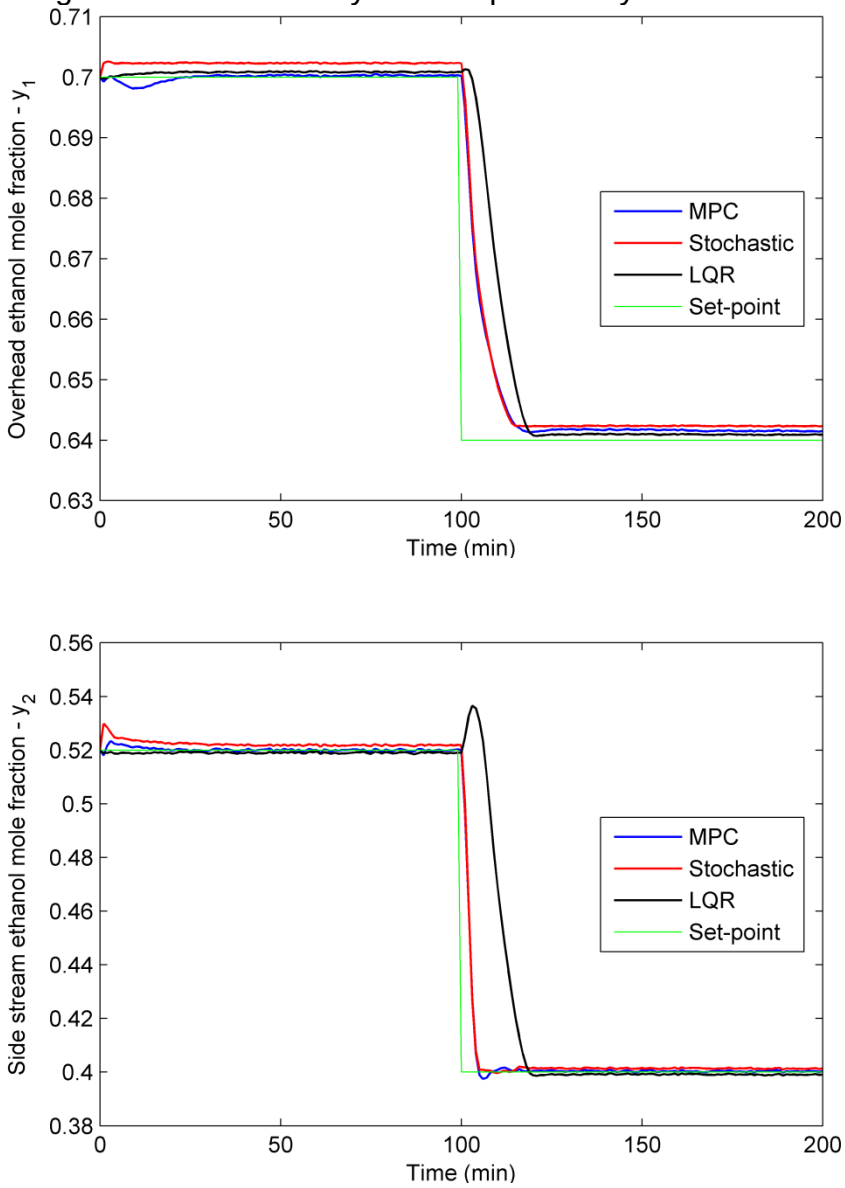
In general, the cost associated with each controller follows the same pattern previously observed, with the reduction of outputs' variances being the responsible for the better performance of the proposed controller, as presented in Table 4-10:

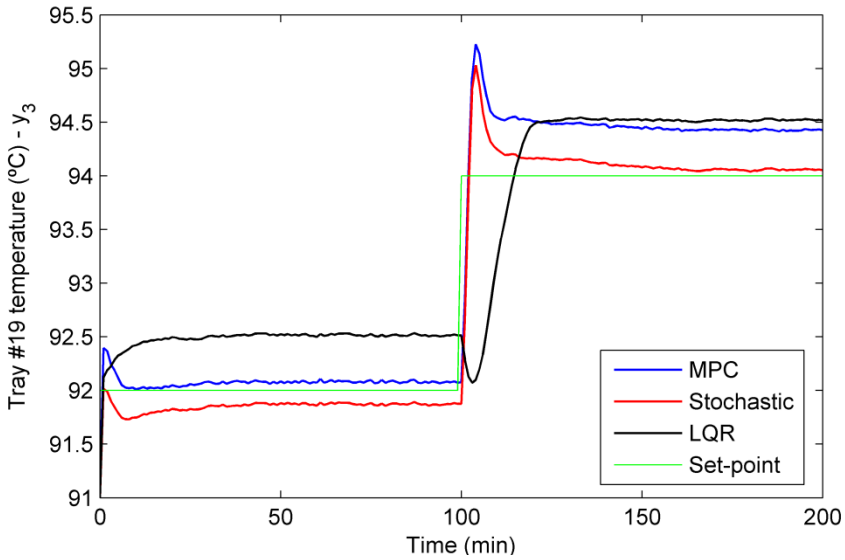
Table 4-10: Average weighted integral square error – asymmetric noise

Standard MPC J_{MPC}	Stochastic controller J_{Sto}	Truncated LQR J_{LQR}
6.55	5.41	21.06

4.1.3.2. Simulation with state constraints

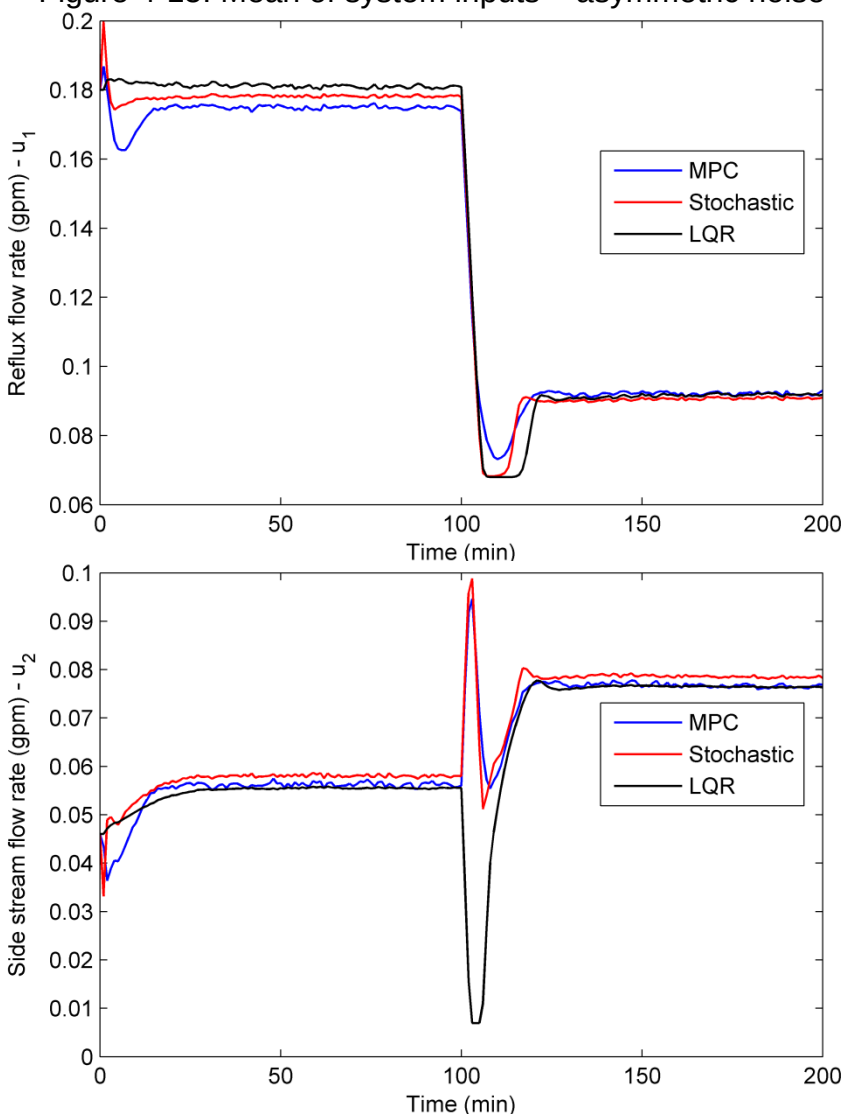
Figure 4-14: Mean of system outputs – asymmetric noise

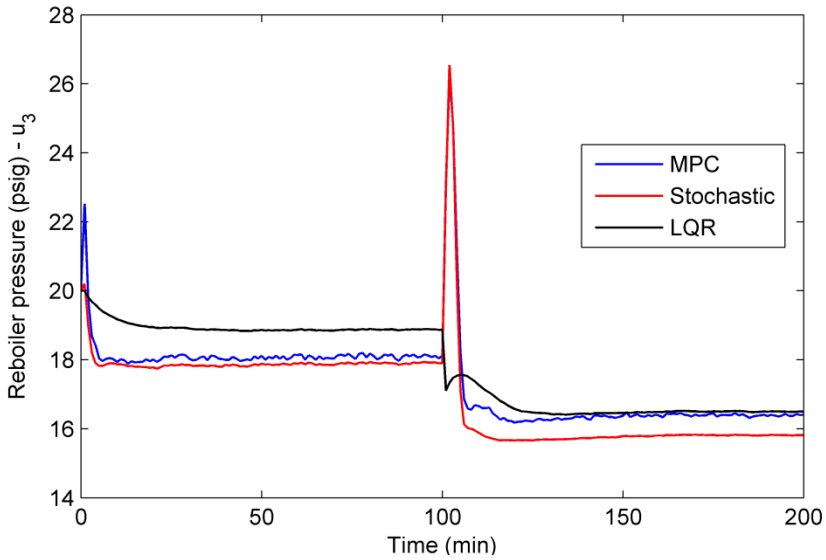




Source: own elaboration

Figure 4-15: Mean of system inputs – asymmetric noise

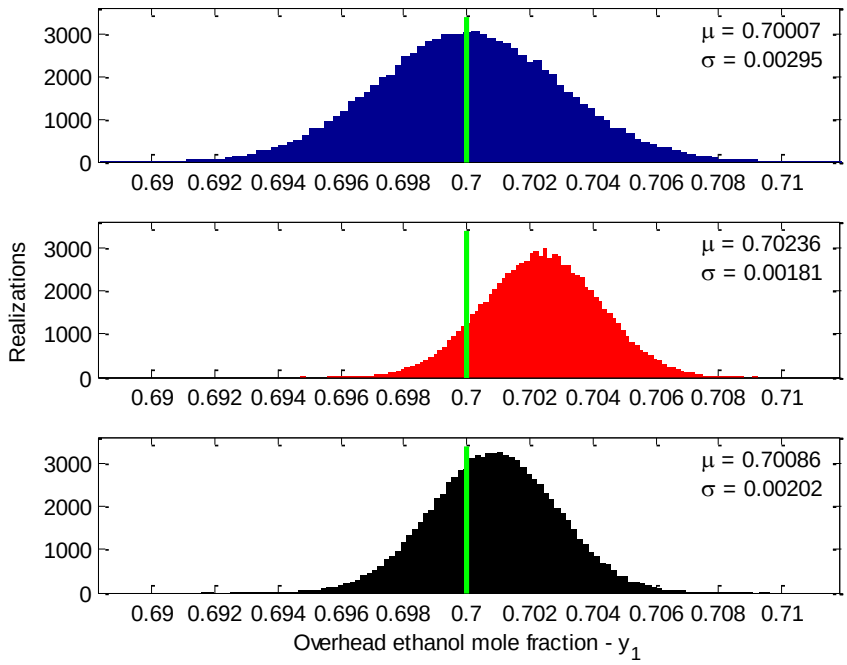


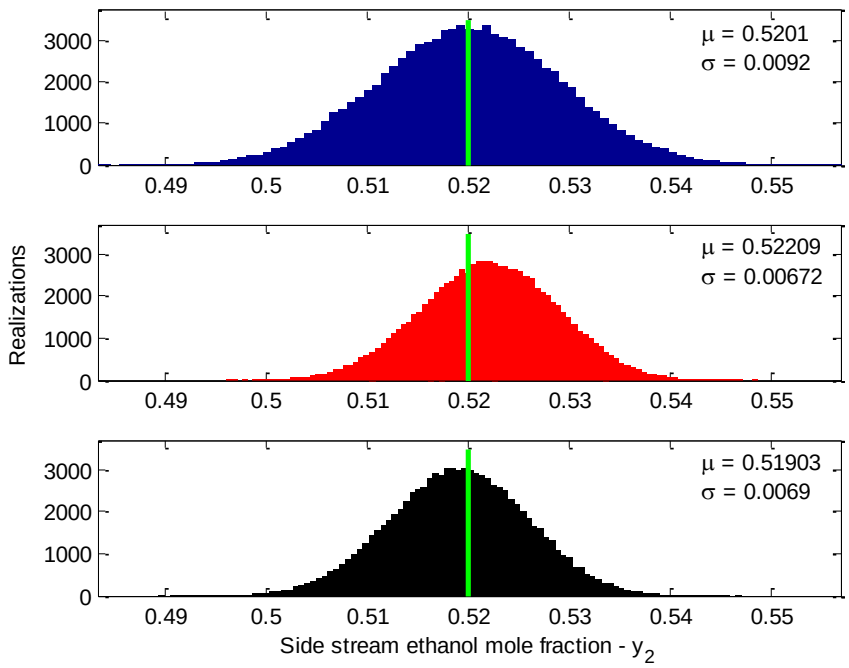


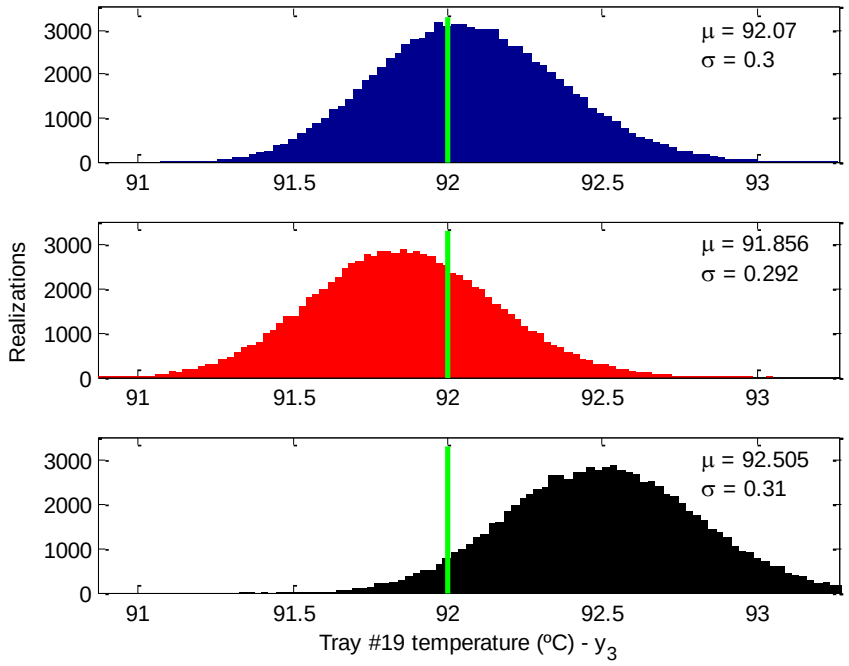
Source: own elaboration

The dynamics represented in Figures 4-14 and 4-15 is similar to the one obtained in the case of symmetric noise. The state constraint induces a back off in y_1, which causes that system to operate at a different steady state and for this reason the other outputs also exhibit an offset. The inclusion of the state constraint does not change the mean time of stabilization of the system in closed loop, therefore we observe the same drift towards a steady state in the second part of the simulation for y_3.

Figure 4-16: Distribution of system outputs between 10 and 100 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green

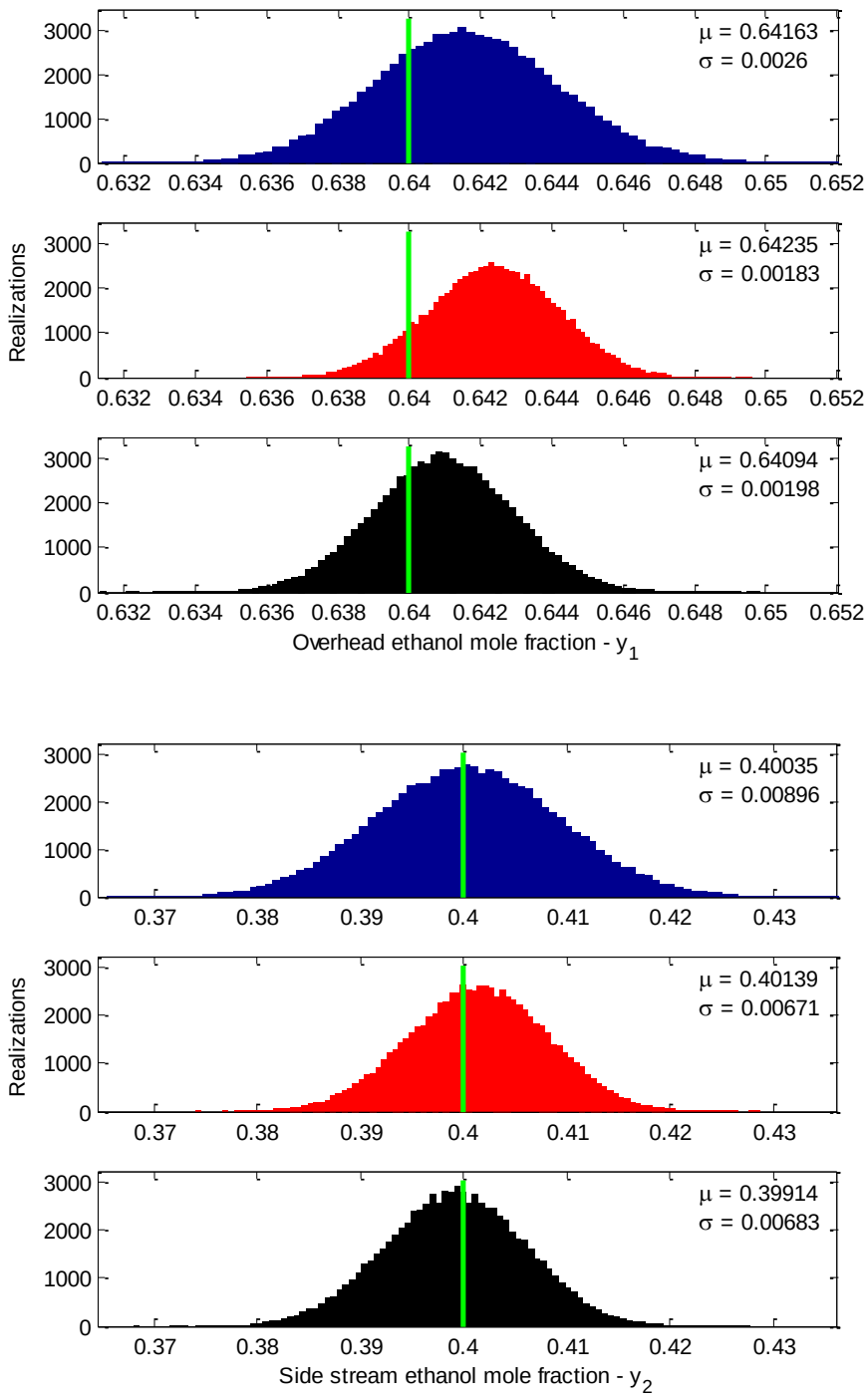


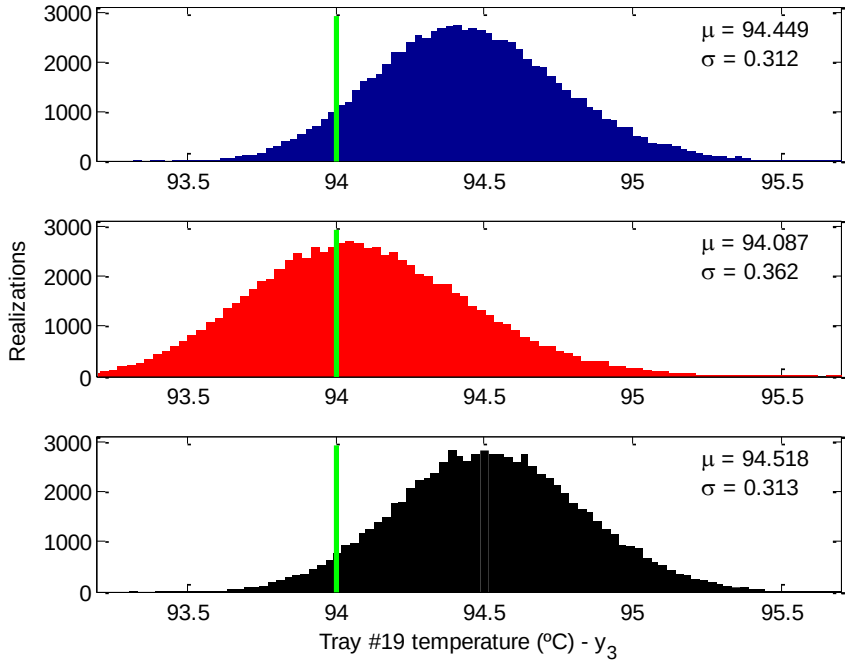




Source: own elaboration

Figure 4-17: Distribution of system outputs between 120 and 200 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green





Source: own elaboration

The histograms of outputs in Figures 4-16 and 4-17 show that the distribution of y_1 is shifted to the right and its variance is reduced in comparison to the case without state constraint, in order to satisfy it. Once again, the skewness is more pronounced in y_3 in the second part of the simulation due to the transient characteristic of this variable. The comparison between average errors of the simulations with and without state constraints is similar to the one described in Section 4.1.2. The inclusion of the soft constrained increases the cost associated with the stochastic controller in 7.8% (from 5.41 to 5.83), but it still outperforms the benchmark controllers as seen in Table 4-11:

Table 4-11: Average weighted integral square error – asymmetric noise

Standard MPC J_{MPC}	Stochastic controller J_{Sto}	Truncated LQR J_{LQR}
6.52	5.83	20.99

Regarding the implementation of the state constraint for the case of non-symmetric noise, results of the percentage of violations are shown in Table 4-12:

Table 4-12: Percentage of time steps with y_1 lower than its set-point for each controller – asymmetric noise

First steady-state ($10 \le k \le 100$)		Second steady-state ($120 \le k \le 200$)	
Controller	Violations	Controller	Violations
MPC	49,34%	MPC	26,77%
Stochastic	9,98%	Stochastic	10,26%
LQR	33,28%	LQR	31,51%

For the first steady-state there is a confirmation of the low conservativeness of the considered approach. However, it is interesting to note that the violation percentage seems to exceed the specified limit of 10% for the second steady-state. A more detailed analysis of the formulation helps to explain this case: the soft constrained is imposed over the artificial set-point, which may be different from its original value. In the presence of significant disturbances, such behavior may emerge as a means to keep feasibility. If calculations of Table 4-12 were performed considering the artificial set-point, then the violation percentage would be lower than 9,6% for both cases. Therefore, we may conclude that the proposed approach achieves, in practice, a high level of constraint satisfaction, regardless of its indirect nature.

The comparison of the standard deviation of all variables in the different conditions is presented in Tables 4-13 and 4-14. The standard deviation of the other controllers (MPC and LQR) is due to the different random seed and may be neglected since it is smaller than 0.5%. We notice the same result previously discussed of the reduced standard deviation of the constrained output with the corresponding increase of the others deviations, in both operating conditions.

Table 4-13: Output standard deviations in the first steady state– asymmetric noise

	Without state constraint			With state constraint		
Output	MPC	Stochastic	LQR	MPC	Stochastic	LQR
y_1	0.0038	0.0026	0.0028	0.0038	0.0024	0.0028
y_2	0.0122	0.0087	0.0094	0.0123	0.0089	0.0095
y_3	0.421	0.384	0.408	0.423	0.395	0.411

Table 4-14: Output standard deviations in the second steady state– asymmetric noise

	Without state constraint			With state constraint		
Output	MPC	Stochastic	LQR	MPC	Stochastic	LQR
y_1	0.0035	0.0026	0.0027	0.0035	0.0024	0.0027
y_2	0.0119	0.0087	0.0093	0.012	0.0089	0.0093
y_3	0.397	0.387	0.414	0.399	0.394	0.413

5. Zone control for systems with bounded noise

This chapter extends the results of Chapter 3 to accommodate zone control strategies. The motivation of this approach is to design a predictive controller that is able to filter disturbances with high frequency and low amplitude: as well as to force the system to remain in the specified zone, no significant control actions should be taken to counteract the disturbance.

In comparison to the method proposed in Chapter 3, it is expected that the zone controller achieves significant lower input variances at the expense of greater output variances. Its main advantage when compared to literature zone control strategies is the ability to account for the expected influence of the disturbances, which translates into keeping the system inside the zones more often.

Section 5.1 provides the mathematical details of the zone control formulation. Next, Section 5.2 presents some case studies using the same system as in Chapter3, comparing the performance of the proposed controller to a literature approach in the area of zone control.

5.1. Problem reformulation

The same state space description and input parameterization of Chapter 3 are considered in this section. In terms of objective function, there are two main changes: the offset function calculation has to be changed to deal with zone control, and input targets are included to guide the system to the interior of the zone.

The modified definition of the offset function is closer to the Lemma of Ferramosca et al. (2009)previously stated. Let the zone be described as the set $\Gamma = \left\{ y \in \mathbb{R}^{ny} \mid y_{\min} \leq y \leq y_{\max} \right\}$. Then, the offset function is redefined in terms of the distance between the artificial target y_{tar} and the set Γ.

Considering the choice of the offset function as the infinity norm and the referred lemma, it may be replaced by the following terms:

$$V_O\left(y_{tar},\Gamma\right)=Q_\lambda^{\ T}\lambda \tag{5-1}$$

subject to

$$\begin{aligned} &y_{tar}-\lambda\le y_{\max}\\ &-y_{tar}-\lambda\le -y_{\min}\\ &\lambda\ge 0 \end{aligned} \tag{5-2}$$

Notice that $V_O\left(y_{tar},\Gamma\right)=0$ if $y_{tar}\in\Gamma$ but the points in the interior of Γ are indistinguishable. For this reason, if the system is evolving in the interior of the zone, the controller is able to simply recalculate the artificial target and do not perform severe control actions. This behavior is appropriate in the context of zone control, however it could lead to situations where the set-point is close to the limit of the zone and disturbances would remove it from the desired region. Therefore, it is advantageous to include input targets that guide the system towards regions with less probability of escaping the zone.

The modified control problem including input targets is:

$$\min \mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=1}^{p}\left\|y(k+i)-y_{tar}\right\|_Q^2+\sum_{i=0}^{m-1}\left\|u(k+i)-u_{tar}\right\|_{Q_u}^2+\sum_{i=0}^{m-1}\left\|\Delta u(k+i)\right\|_R^2\right]+V_O\left(y_{tar},y_{sp}\right) \tag{5-3}$$

The evaluation of the term $\mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=0}^{m-1}\left\|u(k+i)-u_{tar}\right\|_{Q_u}^2\right]$ follows the same reasoning presented in Appendix A. Let $\bar{U}(k)=\left[u(k)^T\quad u(k+1)^T\quad\cdots\quad u(k+m-1)\right]$,then

$$\mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=0}^{m-1}\left\|u(k+i)-u_{tar}\right\|_{Q_u}^2\right]=\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{U}(k)-\bar{U}_{tar}\right\|_{\bar{Q}_u}^2\right] \tag{5-4}$$

where $\bar{U}_{tar}=\left[u_{tar}^{\ T}\quad\cdots\quad u_{tar}^{\ T}\right]^T\in\mathbb{R}^{m\cdot nu}$ and $\bar{Q}_u=\mathrm{diag}(Q_u,\cdots,Q_u)\in\mathbb{R}^{m\cdot nu\times m\cdot nu}$.

To simplify the notation, let $\bar{U}_{prev}(k)=\left[u(k-1)^T\quad\cdots\quad u(k-1)^T\right]^T\in\mathbb{R}^{m\cdot nu}$ be a vector aggregating the input value in the previous time step. Note that $\bar{U}_{prev}(k)\ne\bar{U}(k-1)$ since $\bar{U}(k)$ is a vector of predicted inputs.

According to the notation of (3-27), we have $\bar{U}(k)=\bar{U}_{prev}(k)+M\Delta\bar{U}(k)$. Using the expressions for expected values developed in the Appendix A ((A-3) and (A-11)), the additional term of the cost is given by

$$\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=0}^{m-1}\left\|u(k+i)-u_{tar}\right\|_{Q_u}^2\right]= \\
&tr\left(\bar{Q}_u\left(\bar{U}_{prev}(k)\bar{U}_{prev}(k)^T+2\bar{U}(k-1)\left(\bar{G}\mu^{\Delta w}+D\right)^T M^T\right)\right) \\
&+tr\left(\bar{Q}_u\left(M\left(\bar{G}\mu^{\Delta w\Delta w}\bar{G}^T+\bar{G}\mu^{\Delta w}D^T+D\mu^{\Delta wT}\bar{G}^T+DD^T\right)M^T\right)\right) \\
&-2\bar{U}_{tar}{}^T\bar{Q}_u\left(\bar{U}_{prev}(k)+M\left(\bar{G}\mu^{\Delta w}+D\right)\right)+\bar{U}_{tar}{}^T\bar{Q}_u\bar{U}_{tar}
\end{aligned} \tag{5-5}$$

Following the same structure of Chapter 3, the complete formulation of the stochastic zone controller is presented in a more generic description followed by the detailed expression implemented for simulation :

Generic definition:

$$\min \mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=1}^{p}\left\|y(k+i)-y_{tar}\right\|_Q^2+\sum_{i=0}^{m-1}\left\|\Delta u(k+i)\right\|_R^2\right]+V_O\left(y_{tar},\Gamma\right) \tag{5-6}$$

subject to

$$\begin{aligned}
&x_{tar}=A_p x_{tar}+B_p u_{tar} \\
&y_{tar}=C_p x_{tar} \\
&u_{\min}\le u_{tar}\le u_{\max}
\end{aligned} \tag{5-7}$$

$$\left|\Delta u(k+i)\right|\le \Delta u_{\max}\,, i=0,\cdots,m-1 \tag{5-8}$$

$$u_{\min}\le u(k+i)\le u_{\max}, i=0,\cdots,m-1 \tag{5-9}$$

$$\mathbb{P}\left[g_j^T x_v(k+1)\le h_j\right]\ge 1-\alpha_j,\quad j=1,\cdots,r; \tag{5-10}$$

$$x_v(k+1)\in S_{SRCI}^*, \forall w(k) \tag{5-11}$$

Detailed expression:

$$\min V(k)+V_O\left(y_{tar},\Gamma\right) \tag{5-12}$$

subject to

$$
\begin{aligned}
&V(k)=\left\|x(k)\right\|^2_{\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{A}}+tr\left(\bar{F}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\mu^{\Delta w\Delta w}(k)\right)\\
&+2x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\left(\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)\\
&+2tr\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\left(\mu^{\Delta w\Delta w}(k)\bar{G}^T+\mu^{\Delta w}(k)\bar{D}^T\right)\right)\\
&+tr\left(\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{B}+\bar{R}\right)\left(\bar{G}\mu^{\Delta w\Delta w}(k)\bar{G}^T+\bar{G}\mu^{\Delta w}(k)\bar{D}^T+\bar{D}\mu^{\Delta w}(k)^T\bar{G}^T+\bar{D}\bar{D}^T\right)\right)\\
&-2\bar{Y}_{sp}^T\bar{Q}\bar{C}\left(\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)+\left\|\bar{Y}_{tar}\right\|^2_{\bar{Q}}\\
&tr\left({T_m}^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{A}}T_m\Sigma_{\bar{X}}\right)+tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UU}\right)+tr\left(\bar{\bar{F}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\bar{\bar{\mu}}^{\Delta w\Delta w}\right)\\
&+2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UX}\right)+2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WX}\right)+2tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WU}\right)\\
&-2\bar{\bar{Y}}_{sp}^T\bar{\bar{Q}}\bar{\bar{C}}\left(\bar{\bar{A}}T_m\left(\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)\right)+\left\|\bar{\bar{Y}}_{tar}\right\|^2_{\bar{\bar{Q}}}\\
&+tr\left(\bar{Q}_u\left(\bar{U}_{prev}(k)\bar{U}_{prev}(k)^T+2\bar{U}(k-1)\left(\bar{G}\mu^{\Delta w}+D\right)^T M^T\right)\right)\\
&+tr\left(\bar{Q}_u\left(M\left(\bar{G}\mu^{\Delta w\Delta w}\bar{G}^T+\bar{G}\mu^{\Delta w}D^T+D\mu^{\Delta wT}\bar{G}^T+DD^T\right)M^T\right)\right)\\
&-2{\bar{U}_{tar}}^T\bar{Q}_u\left(\bar{U}_{prev}(k)+M\left(\bar{G}\mu^{\Delta w}+D\right)\right)+{\bar{U}_{tar}}^T\bar{Q}_u\bar{U}_{tar}
\end{aligned}
\tag{5-13}
$$

$$V_O\left(y_{tar},\Gamma\right)={Q_\lambda}^T\lambda \tag{5-14}$$

$$x_{tar}=A_p x_{tar}+B_p u_{tar} \tag{5-15}$$

$$y_{tar}=C_p x_{tar} \tag{5-16}$$

$$u_{\min}\le u_{tar}\le u_{\max} \tag{5-17}$$

$$y_{tar}-\lambda\le y_{\max} \tag{5-18}$$

$$-y_{tar}-\lambda\le -y_{\min} \tag{5-19}$$

$$\lambda\ge 0 \tag{5-20}$$

$$\Delta u(k+i)=d(k+i)+\sum_{j=0}^{i-1}G(k+i,k+j)\Delta w(k+j)\,,\ i=0,\cdots,m-1 \tag{5-21}$$

$$\left|d_i\right|+\left\|G_i\right\|_\infty\left(w_{\max}-w_{\min}\right)\le\Delta u_{\max},i=1,\cdots,m\cdot nu \tag{5-22}$$

$$U_{\min}(k)\le M\Delta\bar{U}(k)\le U_{\max}(k) \tag{5-23}$$

$$
\begin{aligned}
&g_{1,j}^T\left(A_v x_v(k)+B_v\Delta u(k)-F_v w(k-1)\right)+g_{2,j}^T\left(u(k-1)+\Delta u(k)\right)+g_{4,j}^T y_{tar}(k)\\
&\le h_j-\mathbb{F}^{-1}_{\left(g_{1,j}^T F_v+g_{3,j}^T\right)w}\left(1-\alpha_j\right),\quad j=1,\cdots,r;
\end{aligned}
\tag{5-24}
$$

$$
\begin{aligned}
&\tilde{s}_{1,j}^T\left(A_v x_v(k)+B_v\Delta u(k)-F_v w(k-1)\right)+\tilde{s}_{2,j}^T\left(u(k-1)+\Delta u(k)\right)+\tilde{s}_{4,j}^T y_{tar}(k)\\
&\le\tilde{c}_j-\left(\left\|\tilde{s}_{1,j}^T\right\|_\infty\left\|F_v\right\|_\infty-\left\|\tilde{s}_{3,j}^T\right\|_\infty\right)\Delta_w,\quad j=1,\cdots,\tilde{r};
\end{aligned}
\tag{5-25}
$$

5.2. Case studies

The system considered in the following simulations is the same as in Section 4.1.2, with truncated Gaussian noise. In order to evaluate the performance of the proposed controller, it is necessary to confront it with other literature proposals. There are not many benchmarks options in this specific area. The work of Ferramosca et al. (2012) is an interesting option since it deals with additive noise in a worst-case basis to guarantee closed-loop stability. However, their formulation does not include input targets, which greatly change system dynamics. For this reason, we chose to compare the proposed controller with the one presented in González & Odloak (2009).

Nonetheless, there are two drawbacks in the choice of this controller as benchmark: it considers an infinite horizon and it is oblivious to disturbances. Since the design of Ferramosca et al. (2012) calculates the objective function based only on a nominal output predictions, it is conceivable that the comparison with González & Odloak (2009)is more meaningful. A simplified version of its optimization problem is as follows:

$$\min \sum_{i=1}^{\infty} \left\| y(k+i) - y_{tar} - \delta_y \right\|_Q^2 + \sum_{i=0}^{m-1} \left\| u(k+i) - u_{des} \right\|_{Q_u}^2 + \sum_{i=0}^{m-1} \left\| \Delta u(k+i) \right\|_R^2 + \left\| \delta_y \right\|_{S_y}^2 \tag{5-26}$$

subject to

$$x^s(k+m) - y_{tar} - \delta_y = 0 \tag{5-27}$$

$$u_{\min} \le u(k+i) \le u_{\max}, \; i = 0, \cdots, m-1 \tag{5-28}$$

$$\left| \Delta u(k+i) \right| \le \Delta u_{\max}, \; i = 0, \cdots, m-1 \tag{5-29}$$

$$y_{\min} \le y_{sp} \le y_{\max} \tag{5-30}$$

The term $x^s(k+m)$ in constraint (5-27) is one of the state components and it is equivalent to the prediction of outputs steady state at the end of the control horizon. This constraint forces the steady state prediction to be equal to the artificial set-point y_{tar} whenever possible, because it is relaxed with the slack variable δ_y. It

should be noticed that the state space representation considered in that work is derived from the step response, being different from the choice of the stochastic controller. Detailed explanations on this subject may be found in the referred paper.

There is one minor difference between (5-26)-(5-30) and the formulation in González & Odloak (2009). In the original work, they enforce the inputs to be equal to its target whenever possible, similarly to constraint (5-27). We have decided to abandon this additional requirement to render the benchmark controller closer to our new proposal. The same Monte Carlo simulation methodology adopted in Chapter 3 is considered here, with 1000 noise realizations generated and provided to each controller. Regarding controller tuning, there are some additional parameters: Q_u for both controllers and S_y exclusively to the zone MPC (ZMPC). They are presented together with other parameters in Table 5-1:

Table 5-1: Tuning parameters – zone control

Parameter	Value	Parameter	Value
Q	$\begin{bmatrix}10 & 1 & 10\end{bmatrix}$	S_y	$\begin{bmatrix}10^7 & 10^6 & 10^7\end{bmatrix}$
R	$\begin{bmatrix}10^{-1} & 10^{-1} & 10^{-3}\end{bmatrix}$	p	30
Q_λ	$\begin{bmatrix}10^4 & 10^4 & 10^4\end{bmatrix}$	m	3
Q_u	$\begin{bmatrix}1 & 10 & 10^{-3}\end{bmatrix}$		

The proposed simulation scenario is composed of 210 minutes, divided in three parts: in the first two, there are zones corresponding to regions encompassing the origin and the other set-point of Section 4.1.2, respectively. Finally, for the third part, the zones collapse into a new set-point, illustrating that the controller of this Chapter may be seen as an extension of the one in Chapter 3. Output zones and input targets are defined in Tables 5-2 and 5-3 as follows:

Table 5-2: Output zones (LB stands for Lower bound and UB for upper bound)

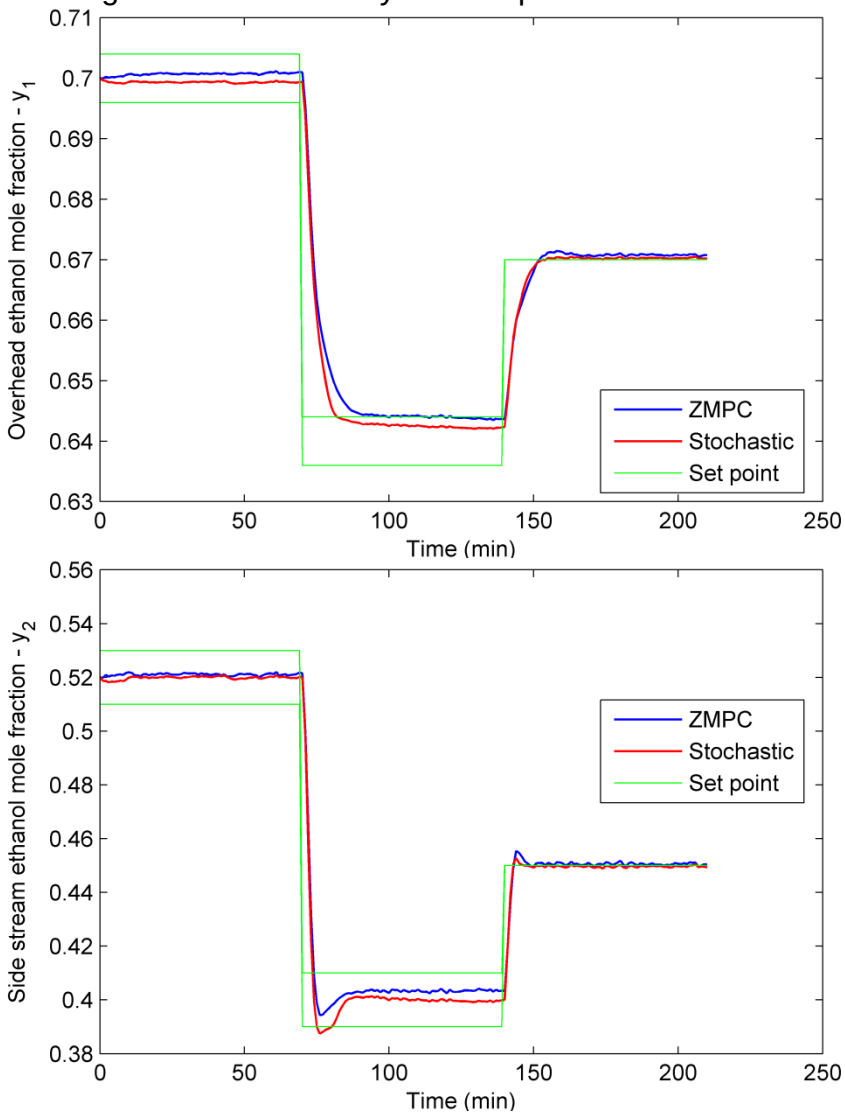
$0 \le k \le 70$		$71 \le k \le 140$		$141 \le k \le 210$	
LB	UB	LB	UB	LB	UB
0.697	0.703	0.637	0.643	0.67	0.67
0.51	0.53	0.39	0.41	0.45	0.45
91.5 ºC	92.5 ºC	93.5 ºC	94.5 ºC	93 ºC	93 ºC

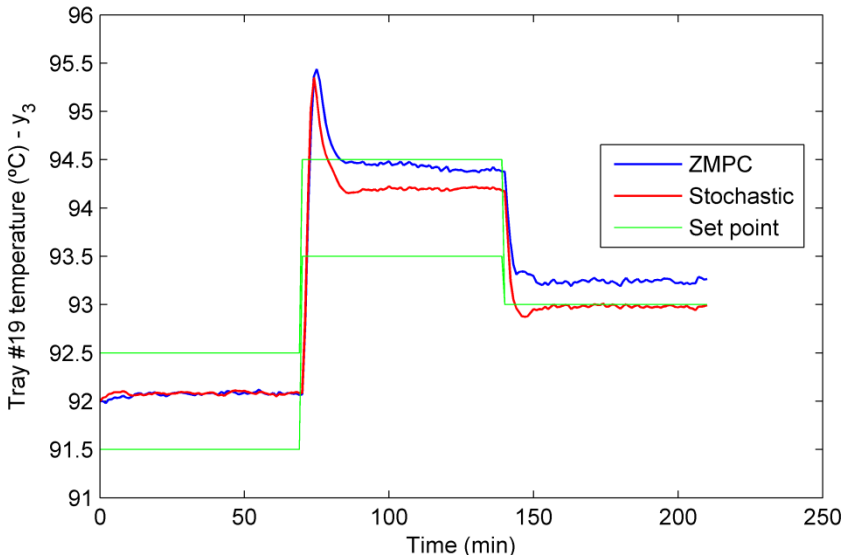
Table 5-3: Input targets

$0 \le k \le 70$	$71 \le k \le 140$	$141 \le k \le 210$
0.18 gpm	0.095 gpm	0.14 gpm
0.046 gpm	0.07 gpm	0.065 gpm
20 psig	17.5 psig	18.7 psig

The mean result of the 1000 repetitions is presented in Figures 5-1 and 5-2:

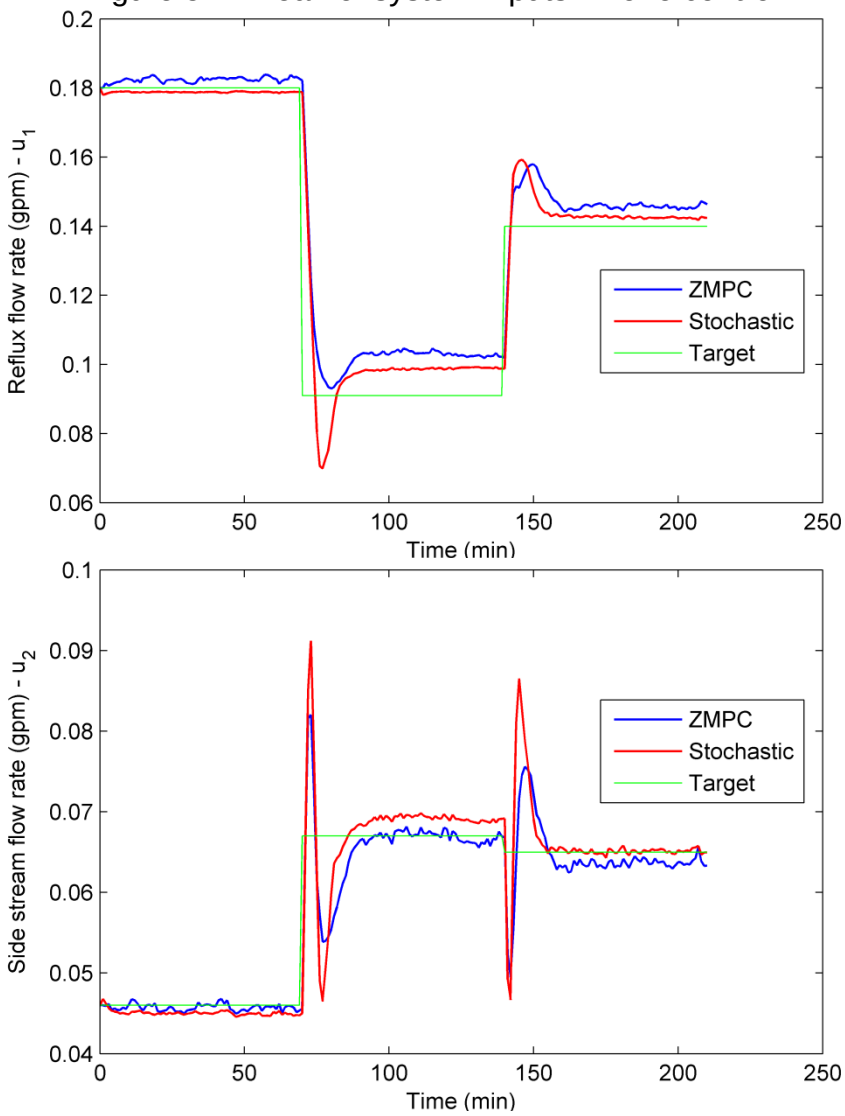
Figure 5-1: Mean of system outputs – zone control

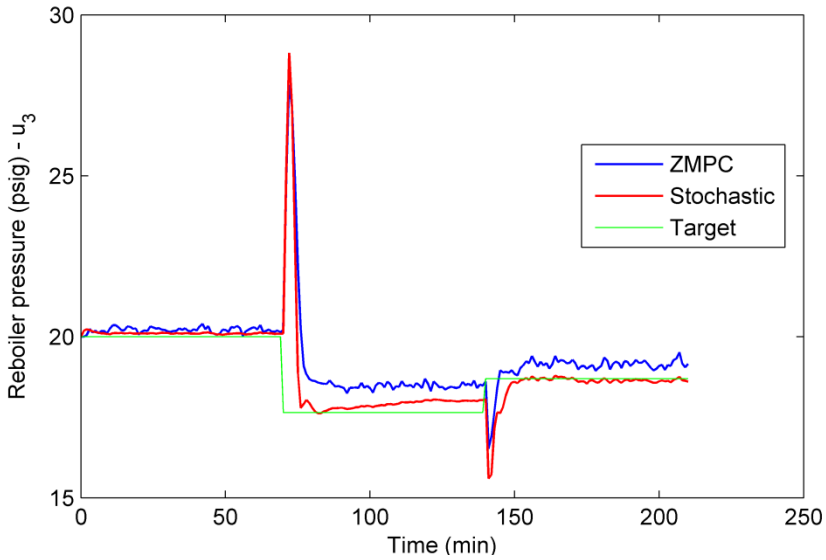




Source: own elaboration

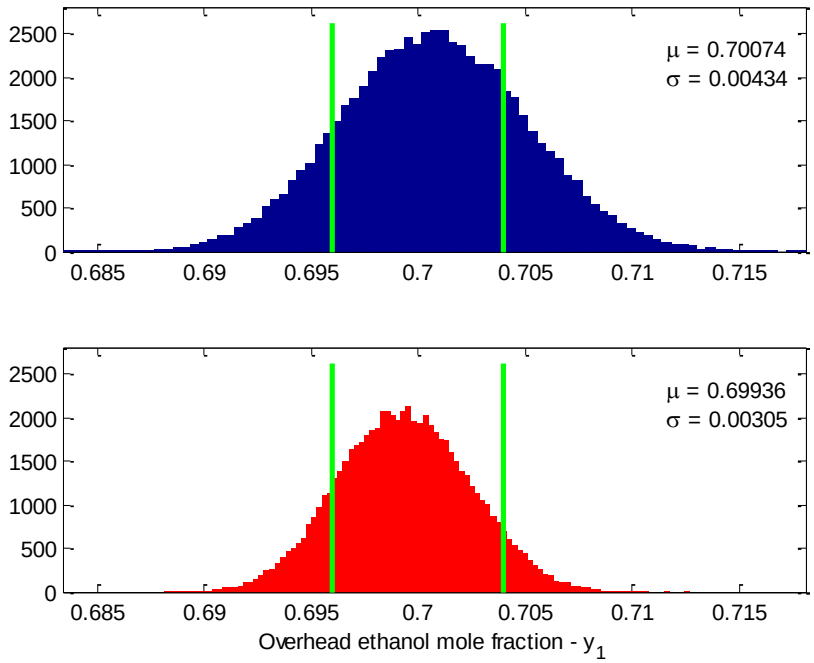
Figure 5-2: Mean of system inputs – zone control

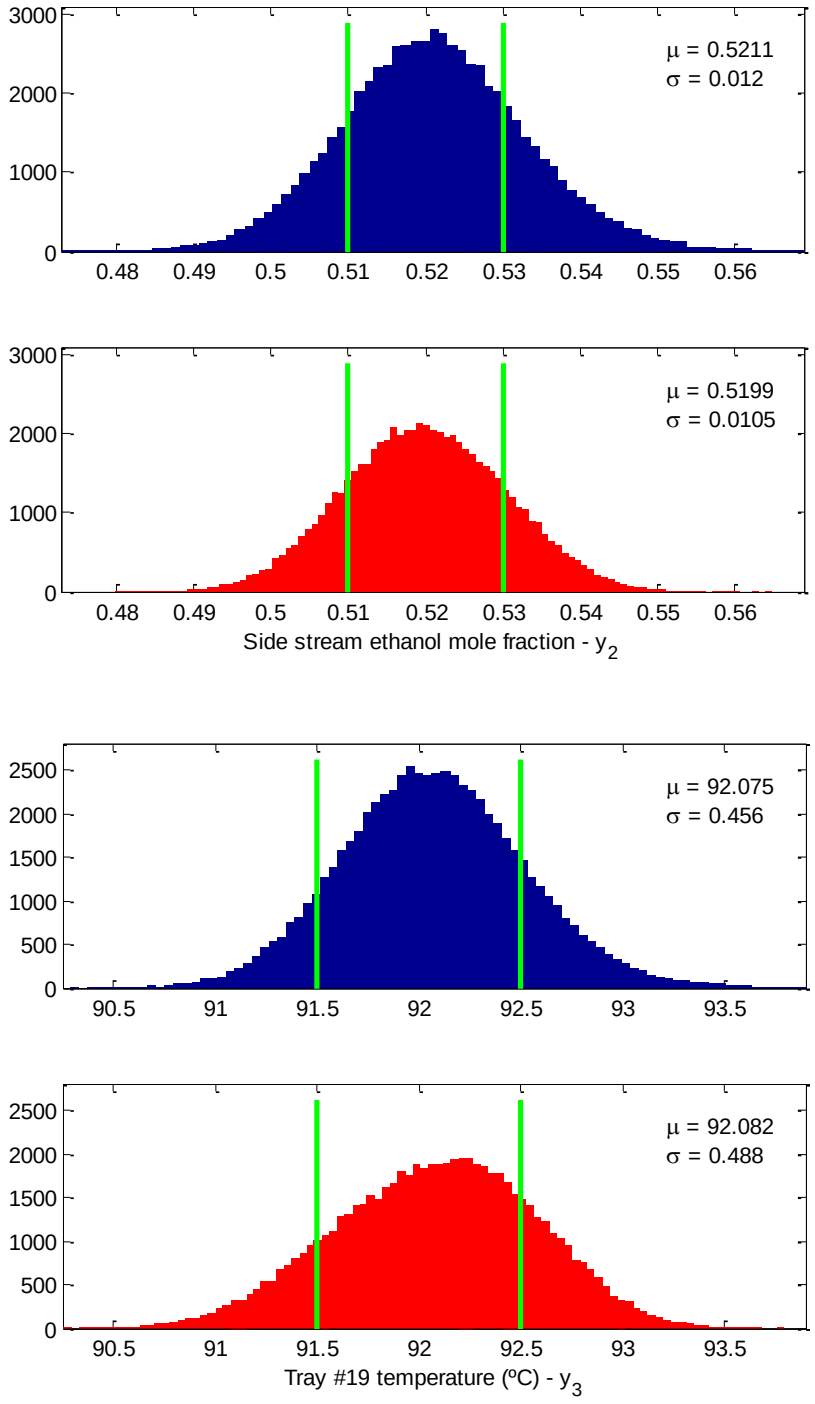




Source: own elaboration

Figure 5-3: Distribution of system outputs between 5 and 70 min for Zone MPC (blue) and the Stochastic controller (red). Zone limits in green





Source: own elaboration

We notice that outputs' variances obtained with the proposed controller are no longer significantly lower than the corresponding values with the standard MPC benchmark. The shape of the distributions are similar in the three parts of the

simulations, for this reason it is sufficient to analyze only the standard deviations as presented in Table 5-4:

Table 5-4: Output standard deviations in each steady state

	First steady state		Second steady state		Third steady state	
Output	ZMPC	Stochastic	ZMPC	Stochastic	ZMPC	Stochastic
y_1	0.00434	0.00305	0.00413	0.00255	0.00413	0.00256
y_2	0.012	0.0105	0.0133	0.00927	0.0134	0.0087
y_3	0.456	0.488	0.454	0.422	0.529	0.349

In fact, the standard deviation is greater with the proposed controller than with the benchmark for one controlled variable (y_3) in the first part of the simulation. However, as stated in the introduction of this section, this is not completely undesirable when zone control is considered, showing that the controller is taking full advantage of defining different set-points at each sampling time.

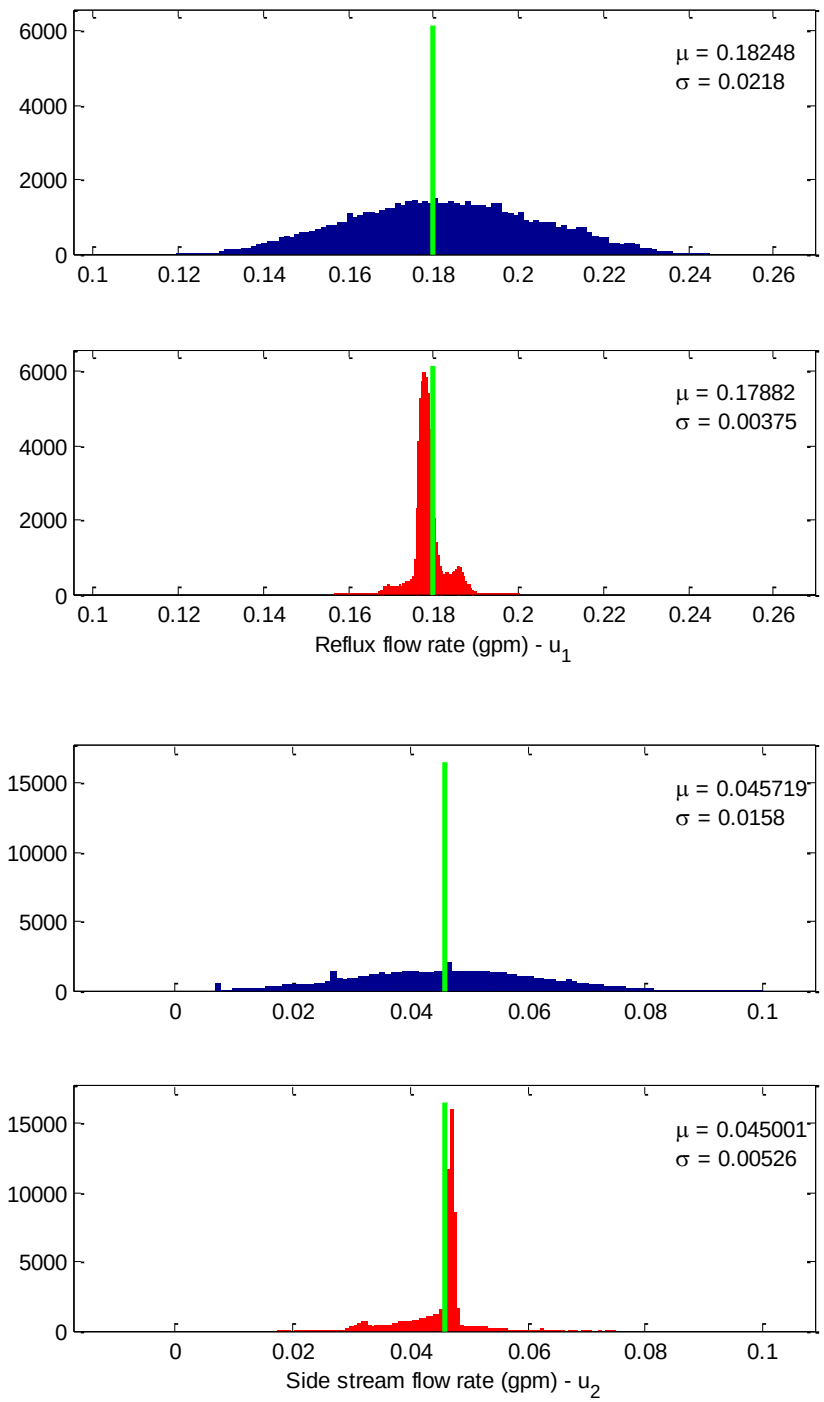
To properly quantify the performance, the integral square error proposed in Section 4.1.2 has to be modified in order to consider the distance to the zone and no longer to the fixed set-point. Therefore,

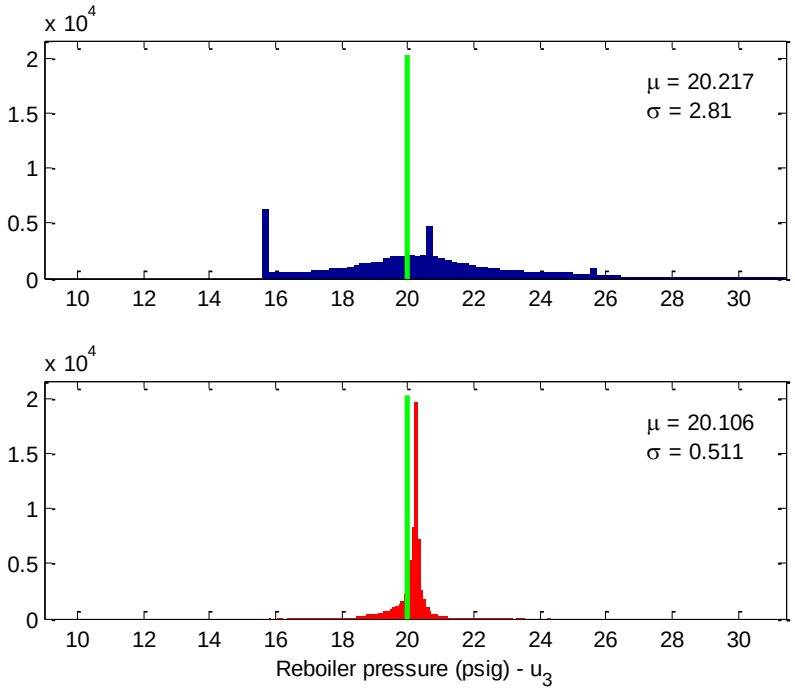
$$J = \sum_{k=0}^{nsim} \min_{y_{\min} \le y_{sp}(k) \le y_{\max}} \left\| y(k) - y_{sp}(k) \right\|_Q^2 \tag{5-31}$$

It must be understood that $y_{sp}(k)$ in equation (5-31) is defined after the simulation has taken place and should not be confused with the artificial target. Actually, it stands only to an auxiliary variable that is equal to $y_{\max}$ if a given output is greater than the upper bound of the zone and $y_{\min}$ if the output is lower. If the variable is inside the zone, then $y(k) = y_{sp}(k)$ and the cost contribution is null. According to this metric, the performance index of the proposed controller is 4.8064, compared to 6.6238 of the benchmark, which is 27.4% lower.

Finally, other useful metric to compare controllers' performances is the distribution of system inputs as presented in Figures 5-4 and 5-5:

Figure 5-4: Distribution of system inputs between 5 and 70 min for Zone MPC (blue) and the Stochastic controller (red). Set-point in green





Source: own elaboration

The difference between both controllers is strikingly clear in terms of inputs' distributions, with the stochastic controller achieving standard deviations smaller by up to one order of magnitude. This is precisely the main advantage of the proposed technique in comparison to state of the art alternatives, since lower input variance without deterioration of the outputs performance represents the possibility of significant savings in terms of actuators' maintenance and substitution.

6. Stochastic controller with stability guarantee

This chapter is concerned in providing formal performance guarantees to the stochastic controller formulation. The chosen approach to tackle this problem is to rewrite the soft constraint using a method proposed in Korda et al. (2011) that, in addition to the strongly feasibility obtained by the first-step constraint of Chapter 3, assures recursive feasibility. Once that a recursive solution is available, the standard method to prove Lyapunov stability of predictive controllers may be used. However, when the disturbance is persistent as in the case considered throughout this work, the cost decreases only until the state achieves a given set and it does not converges to zero.

6.1. Control formulation

As defined in Section 3.4.2, the controller formulation already presented imposes that the state in the immediately next sampling time is included in a robust positive invariant set, as expressed in (3-41). The predicted input values of subsequent time instants are not constrained regarding the invariant set. Therefore, when the optimization problem has to be solved again at the next time step, it is possible that a shifted input sequence – obtained with the standard approach of using the previously optimal solution completed with a local controller for the last term – would not be feasible.

The alternative approach to handle the soft constraints stems from the assumption of bounded noise support, which allows one to use traditional robust control approaches to guarantee recursive feasibility. The control actions are constrained to assure that the state is kept at the invariant set for all possible disturbances, thus the shifted solution is certainly feasible.

As previously, the controller is based on a dual mode strategy, but its components are different. In the first mode, a deterministic counterpart of the soft constraint is imposed explicitly over the control actions. In the second mode, the feedback law becomes

$$\Delta u(k+i) = K_s\left(x_v(k+i) - x_{v,tar}\right) \tag{6-1}$$

where $x_{v,tar}$ is the artificial target associated with the velocity description. Associated with the second mode, it is defined a terminal cost of the form $\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|y(k+m) - y_{tar}\right\|_P^2\right]$, where matrix P is calculated following the commentary in Assumption 1:

Assumption 1: There is a constant feedback K_s corresponding to the law $\Delta u(k) = K_s\left(x(k-1) - x_{v,tar}\right)$ associated with a terminal weight P that is the solution of the following Lyapunov equation,

$$\left(A_v + B_v K_s\right)^T C_v{}^T P C_v \left(A_v + B_v K_s\right) + C_v{}^T Q C_v + K_s{}^T R K_s = C_v{}^T P C_v \tag{6-2}$$

The closed-loop transition matrix $A_{CL} = A_v + B_v K_s$ is strictly stable, i.e., all its eigenvalues are inside the unity circle.

Let us now retake the notation of the augmented state defined in Chapter 3 to develop the expression of the constraints that actually replace the probabilistic expression.

$$\Psi(k) = \left[x_v(k)^T \quad u(k)^T \quad w(k-1)^T \quad x_{v,tar}(k)^T\right]^T,$$

$$\begin{bmatrix} x_v(k+1) \\ u(k+1) \\ w(k) \\ x_{v,tar}(k+1) \end{bmatrix} = \begin{bmatrix} A_v & 0 & -F_v & 0 \\ 0 & I_{nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{nx} \end{bmatrix} \begin{bmatrix} x_v(k) \\ u(k) \\ w(k-1) \\ x_{v,tar}(k) \end{bmatrix} + \begin{bmatrix} B_v \\ I_{nu} \\ 0 \\ 0 \end{bmatrix} \Delta u(k) + \begin{bmatrix} F_v \\ 0 \\ I_{nw} \\ 0 \end{bmatrix} w(k) \tag{6-3}$$

$$\Psi(k+1) = A_\Psi \Psi(k) + B_\Psi \Delta u(k) + F_\Psi w(k)$$

The evolution of the augmented state *i* steps ahead is given by

$$\Psi(k+i) = A_\Psi{}^i \Psi(k) + \bar{B}_{i,\Psi}\left(\bar{D} + \bar{G}\Delta\bar{W}\right) + \bar{F}_{i,\Psi}\bar{W} \tag{6-4}$$

with $\bar{B}_{i,\Psi} = \begin{bmatrix} A_\Psi{}^{i-1}B_\Psi & \cdots & B_\Psi & 0 & \cdots & 0 \end{bmatrix}$, $\bar{F}_{i,\Psi} = \begin{bmatrix} A_\Psi{}^{i-1}F_\Psi & \cdots & F_\Psi & 0 & \cdots & 0 \end{bmatrix}$ and $\bar{W}$ is the aggregation of future noise values in positional form, $\bar{W}(k) = \begin{bmatrix} w(k)^T & \cdots & w(k+m)^T \end{bmatrix}^T$.

Recall that the probabilistic constraint written in terms of the augmented state is given by:

$$\mathbb{P}\left[g_j^T \Psi(k+i) \le h_j \mid \Psi(k+i-1)\right] \ge 1-\alpha_j, \quad j = 1,\cdots,r;\ i \in \mathbb{N}^* \tag{6-5}$$

For (6-5) to be valid, it is necessary that

$$g_j^T\left(A_\Psi \Psi(k+i) + B_\Psi \Delta u(k+i)\right) \le h_j - \mathbb{F}^{-1}_{g_j^T F_\Psi \Delta w}\left(1-\alpha_j\right), \quad j = 1,\cdots,r;\ i = 1,\cdots,m-1 \tag{6-6}$$

Defining $h'_j = h_j - \mathbb{F}^{-1}_{g_j^T F_\Psi \Delta w}\left(1-\alpha_j\right)$, (6-6) may be expressed as a function of the decision variables as follows:

$$\begin{aligned} &g_j^T\left(A_\Psi\left(A_\Psi{}^i \Psi(k) + \bar{B}_{i,\Psi}\left(\bar{D} + \bar{G}\Delta\bar{W}\right) + \bar{F}_{i,\Psi}\bar{W}\right) + B_\Psi \Delta u(k+i)\right) \le h'_j \\ &g_j^T\left(A_\Psi{}^{i+1}\Psi(k) + \bar{B}_{i+1,\Psi}\left(\bar{D} + \bar{G}\Delta\bar{W}\right) + A_\Psi \bar{F}_{i,\Psi}\bar{W}\right) \le h'_j \end{aligned} \tag{6-7}$$

To encompass the difference between $\bar{W}$ and $\Delta\bar{W}$, let us first define $\bar{W}_{prev}(k) = \begin{bmatrix} w(k-1)^T & \cdots & w(k-1)^T \end{bmatrix}^T \in \mathbb{R}^{m \cdot nw}$ and $M_w = \begin{bmatrix} I_{nw} & 0 & \cdots & 0 \\ I_{nw} & I_{nw} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I_{nw} & I_{nw} & \cdots & I_{nw} \end{bmatrix}$. Then, it follows that $\bar{W} = \bar{W}_{prev} + M_w \Delta\bar{W}$.

Therefore, considering a worst-case approach for noise realization until time *i*,

$$g_j^T\left(A_\Psi{}^{i+1}\Psi(k) + \bar{B}_{i+1,\Psi}\bar{D} + A_\Psi \bar{F}_{i,\Psi}\bar{W}_{prev}(k)\right) \le h'_j - \left\| g_j^T\left(\bar{B}_{i+1,\Psi}\bar{G} + A_\Psi \bar{F}_{i,\Psi} M_w\right)\right\|_\infty \Delta_{\Delta w} \tag{6-8}$$

where $\Delta_{\Delta w}$ is the bound on Δw, which is equal to $w_{\max} - w_{\min}$.

In the second mode, the state is constrained to belong to an invariant set defined as the maximum robust invariant subset of the stochastic feasibility set of(6-5).The feedback law $\Delta u(k+i) = K_s\left(x_v(k+i) - x_{v,tar}\right)$ may be summarized as $\Delta u(k+i) = K_\Psi \Psi(k+i)$, where $K_\Psi = \begin{bmatrix} K_s & 0 & 0 & -K_s \end{bmatrix}$.

More precisely, let $X_r^{K_s}$ be the set such that for all $\Psi \in X_r^{K_s}$:

$$\left(A_\Psi + B_\Psi K_\Psi\right)\Psi + F_\Psi w \in X_r^{K_s} \quad ,\forall w \tag{6-9}$$

$$\left\|K_\Psi \Psi(k+i)\right\|_\infty \le \Delta u_{\max} \tag{6-10}$$

$$u_{\min} \le \begin{bmatrix} 0 & I_{nu} & 0 & 0 \end{bmatrix}\Psi \le u_{\max} \tag{6-11}$$

$$g_j^T\left(A_\Psi + B_\Psi \mathrm{K}_\Psi\right)\Psi \le h'_j,\ j = 1,\cdots,r;\ i = 1,\cdots,m-1 \tag{6-12}$$

$X_r^{K_s}$ may be described by a polyhedral approximation of the form $\tilde{S}\Psi \le \tilde{c}$. Consequently, the constraint that has to be added to the optimization problem in order to ensure that the state reaches the terminal set at the end of the control horizon is analogous to (6-8), replacing the vectors that describe the constraints and the time step to *m*:

$$\begin{aligned} &\tilde{s}_j^T\left(A_\Psi{}^m \Psi(k) + \bar{B}_{m,\Psi}\bar{D} + A_\Psi \bar{F}_{m,\Psi}\bar{W}_{prev}(k)\right) \le \tilde{c}_j - \left\|\tilde{s}_j^T\left(\bar{B}_{m,\Psi}\bar{G} + A_\Psi \bar{F}_{m,\Psi} M_w\right)\right\|_\infty \Delta_{\Delta w}, \\ &j = 1,\cdots,r; \end{aligned} \tag{6-13}$$

Finally, the optimization problem that has to be solved at each time step is presented in its generic and detailed expressions:

Problem 1 – abstract definition:

$$\min \mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=0}^{m-1}\left\|y(k+i) - y_{tar}\right\|_Q^2 + \left\|\Delta u(k+i)\right\|_R^2 + \left\|y(k+m) - y_{tar}\right\|_P^2\right] + V_O\left(y_{tar}, y_{sp}\right) \tag{6-14}$$

subject to

$$x_{v,tar} = A_v x_{v,tar}; y_{tar} = C_v x_{v,tar} \tag{6-15}$$

$$\left|\Delta u(k+i)\right| \le \Delta u_{\max}\,, i = 0,\cdots,m-1 \tag{6-16}$$

$$u_{\min} \le u(k+i) \le u_{\max}, i = 0,\cdots,m-1 \tag{6-17}$$

$$\mathbb{P}\left[g_j^T x_v(k+1) \le h_j\right] \ge 1-\alpha_j, \quad j = 1,\cdots,r; \tag{6-18}$$

$$x_v(k+m) \in X_{Ks}^r, \forall w(k),\cdots,w(k+m-1) \tag{6-19}$$

Problem 1 – complete definition:

$$\min_{\bar{D},\bar{G},x_{v,tar}} V(k) = \mathbb{E}_{\mathfrak{X}(k)}\left[\sum_{i=0}^{m-1}\left\|y(k+i)-y_{tar}\right\|_Q^2 + \left\|\Delta u(k+i)\right\|_R^2 + \left\|y(k+m)-y_{tar}\right\|_P^2\right] + V_O\left(y_{tar}, y_{sp}\right) \tag{6-20}$$

s.t.

$$\begin{aligned} x_v(k+1) &= A_v x_v(k) + B_v \Delta u(k) + F_v \Delta w(k) \\ y(k) &= C_v x_v(k) \end{aligned} \tag{6-21}$$

$$x_{v,tar} = A_v x_{v,tar}; y_{tar} = C_v x_{v,tar} \tag{6-22}$$

$$u_{\min} \le u(k+i) \le u_{\max}\,, \; i = 0,\cdots,m-1 \tag{6-23}$$

$$\left|\Delta u(k+i)\right| \le \Delta u_{\max}\,, \; i = 0,\cdots,m-1 \tag{6-24}$$

$$\begin{aligned} \Delta u(k+i) &= d(k+i) + \sum_{j=0}^{i-1} G(k+i,k+j)\Delta w(k+j) \qquad , i = 0,\cdots,m-1 \\ \Delta u(k+i) &= K_s\left(x_v(k+i) - x_{v,tar}\right) \qquad , \; i \ge m \end{aligned} \tag{6-25}$$

$$\begin{aligned} & g_j^T\left(A_\Psi{}^{i+1}\Psi(k) + \bar{B}_{i+1,\Psi}\bar{D} + A_\Psi \bar{F}_{i,\Psi}\bar{W}_{prev}(k)\right) \le h'_j - \left\|g_j^T\left(\bar{B}_{i+1,\Psi}\bar{G} + A_\Psi \bar{F}_{i,\Psi} M_w\right)\right\|_\infty \Delta_{\Delta w}, \\ & j = 1,\cdots,r; \;\; i = 1,\cdots,m-1 \end{aligned} \tag{6-26}$$

$$\begin{aligned} & \tilde{s}_j^T\left(A_\Psi{}^m\Psi(k) + \bar{B}_{m,\Psi}\bar{D} + A_\Psi \bar{F}_{m,\Psi}\bar{W}_{prev}(k)\right) \le \tilde{c}_j - \left\|\tilde{s}_j^T\left(\bar{B}_{m,\Psi}\bar{G} + A_\Psi \bar{F}_{m,\Psi} M_w\right)\right\|_\infty \Delta_{\Delta w}, \\ & j = 1,\cdots,r; \end{aligned} \tag{6-27}$$

The main result of this thesis is stated and proved in the next two theorems: first recursive feasibility is assured and then asymptotic stability ensues.

Theorem 1: Problem 1 is recursively feasible.

Proof: The reasoning follows a similar procedure to what is done in (Korda et al., 2011), since essentially the same parameterization is employed, changing the positional to the velocity form.

The artificial reference $x_{v,tar}$ may be taken as its previous value, $x_{v,tar}(k+1) = x_{v,tar}(k)$. It is obvious that with this choice $y_{tar}(k+1) = y_{tar}(k)$. Regarding the remaining decision variables, it is possible to show that a feasible solution $\left(\tilde{D},\tilde{G}\right)$ at time k can be used to generate a feasible solution at time *k*+1 for all possible disturbances $w(k)$. More specifically, the elements corresponding to time step *k*+1 up to *k*+*m* are taken as a time shift of the solution in *k*. The last control action is obtained from the observation that the augmented state is constrained to an invariant set, therefore $\Delta u(k+m) = K_{\Psi}\Psi(k+m)$ is feasible. All control move components concerning $\Delta w(k)$ are accommodated in $\tilde{D}$, which is the free term. The remaining feedback actions remain in $\tilde{G}$. The explicit expression for $\left(\tilde{D},\tilde{G}\right)$ is given next:

$$\tilde{D} = \begin{bmatrix} d(k+1) + G(k+1,k)\Delta w(k) \\ d(k+2) + G(k+2,k)\Delta w(k) \\ \vdots \\ d(k+m-1) + G(k+m-1,k)\Delta w(k) \\ d_f \end{bmatrix}, \tilde{G} = \begin{bmatrix} 0 & 0 \\ \hat{G} & 0 \\ G_f & 0 \end{bmatrix} \tag{6-28}$$

With

$$d_f = K_s\left(A_{\Psi}^{m}\Psi(k) + \bar{B}_{m,\Psi}\bar{D} + \left[\bar{B}_{m,\Psi}\bar{G} + \bar{F}_{m,\Psi}M_w\right]_{1:nw}\Delta w(k)\right)$$

$$\hat{G} = \begin{bmatrix} G(k+2,k+1) & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ G(k+m-1,k+1) & \cdots & G(k+m-1,k+m-2) & 0 \end{bmatrix}$$

$$G_f = K_s\left(\left[\bar{B}_{m,\Psi}\bar{G} + \bar{F}_{m,\Psi}M_w\right]_{nw+1:m\cdot nw}\right) \tag{6-29}$$

The notation $\left[X\right]_{a:b}$ stands for the matrix obtained from X by extracting its columns from *a* to *b*. □

Next, we proceed to show that the feasible solution at time *k*+1 implies that the cost function is decreasing in mean as long as the initial state is sufficiently removed from the desired set-point. In contrast to nominal model predictive control, the best that can be achieved is the convergence in mean to the set-point, since the additive noise is constantly steering the outputs away from their reference points.

Theorem 2: Let P be a terminal cost as stated in Assumption 1. If noise is assumed to be zero-mean, then, considering the sequential solution of Problem 1 if there are no set-point changes, the following bound is valid :

$$\mathbb{E}_{\mathfrak{X}(k)}\left[V(k+1)\right]-V(k)\leq 2tr\left(F_v^T\left(I_{nx}-A_{CL}^T\right)C_v^T PC_vF_v\Sigma_W\right)-\left\|y(k)-y_{tar}\right\|_Q^2 \qquad (6\text{-}30)$$

Proof: Let $\left(\bar{D},\bar{G}\right)$ be the optimal solution of Problem 1 at time *k* and $\left(\tilde{D},\tilde{G}\right)$ a feasible solution obtained as in Theorem 1. Also, consider that y_{tar} is constant from time *k* to *k+1* and using the relation $A_v x_{v,tar}=x_{v,tar}$ it is possible to show that:

$$\begin{aligned}
&x_v(k+m+1)=A_v x(k+m)+B_v\Delta u(k+m)+F_v\Delta w(k+m)\\
&=A_v x_v(k+m)+B_vK_s\left(x_v(k+m)-x_{v,tar}\right)+F_v\Delta w(k+m)\\
&=A_{CL}x_v(k+m)-B_vK_sx_{v,tar}+F_v\Delta w(k+m)\\
&=A_{CL}x_v(k+m)-B_vK_sx_{v,tar}+F_v\Delta w(k+m)+\left(x_{v,tar}-A_vx_{v,tar}\right)\\
&\Rightarrow\\
&x_v(k+m+1)-x_{v,tar}=A_{CL}\left(x(k+m)-x_{v,tar}\right)+F_v\Delta w(k+m)
\end{aligned} \qquad (6\text{-}31)$$

Then, it follows that

$$
\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[V(k+1)\right]-V(k)\\
&=\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|y(k+m+1)-y_{tar}\right\|_P^2\right]\\
&+\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|y(k+m)-y_{tar}\right\|_Q^2+\left\|\Delta u(k+m)\right\|_R^2\right]\\
&-\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|y(k+m)-y_{tar}\right\|_P^2+\left\|y(k)-y_{tar}\right\|_Q^2+\left\|\Delta u(k)\right\|_R^2\right]\\
&=\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|C_vA_{CL}\left(x(k+m)-x_{v,tar}\right)+C_vF_v\Delta w(k+m)\right\|_P^2\right]\\
&+\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|C_v\left(x_v(k+m)-x_{v,tar}\right)\right\|_Q^2+\left\|K_s\left(x(k+m)-x_{v,tar}\right)\right\|_R^2\right]\\
&-\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|C_v\left(x_v(k+m)-x_{v,tar}\right)\right\|_P^2+\left\|y(k)-y_{tar}\right\|_Q^2+\left\|\Delta u(k)\right\|_R^2\right]
\end{aligned}
\tag{6-32}
$$

The first term may be decomposed as follows

$$
\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|C_vA_{CL}\left(x(k+m)-x_{v,tar}\right)+C_vF_v\Delta w(k+m)\right\|_P^2\right]=\\
&=\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|C_vA_{CL}\left(x(k+m)-x_{v,tar}\right)\right\|_P^2+\left\|C_vF_v\Delta w(k+m)\right\|_P^2\right]+\\
&+2\mathbb{E}_{\mathfrak{X}(k)}\left[\left(C_vA_{CL}\left(x(k+m)-x_{v,tar}\right)\right)^T PC_vF_v\Delta w(k+m)\right]\\
&=\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|C_vA_{CL}\left(x(k+m)-x_{v,tar}\right)\right\|_P^2\right]+2tr\left(F_v^TC_v^TPC_vF_v\Sigma_W\right)+\\
&+2tr\left(A_{CL}^TC_v^TPC_vF_v\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+m)\left(x(k+m)-x_{v,tar}\right)^T\right]\right)
\end{aligned}
\tag{6-33}
$$

Let us carefully inspect the cross-term of (6-33):

$$
\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+m)\left(x(k+m)-x_{v,tar}\right)^T\right]=\\
&\mathbb{E}_{\mathfrak{X}(k)}\left[\left(w(k+m)-w(k+m-1)\right)x(k+m)^T\right]=\\
&-\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-1)\left(A_vx(k+m-1)+B_v\Delta u(k+m-1)+F_v\Delta w(k+m-1)\right)^T\right]=\\
&-\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-1)\left(F_v\left(w(k+m-1)-w(k+m-2)\right)\right)^T\right]=\\
&-\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-1)w(k+m-1)^T\right]F_v^T=-\Sigma_WF_v^T
\end{aligned}
\tag{6-34}
$$

The first equality of (6-34) is justified since the assumption that $\{w(k)\}$ is identically distributed causes $\{\Delta w(k)\}$ to be zero-mean and therefore the term

containing $x_{v,tar}$ vanishes. The second equality follows since $x(k+m)$ and $\Delta w(k+m)$ are not correlated. The third is a consequence that $x(k+m-1)$ and $\Delta u(k+m-1)$ are not correlated to the noise in the same instant and the zero-mean assumption. Thus, the only non-zero component in (6-34) is the one relating $w(k+m-1)$ with itself.

Substituting the result back in (6-33)

$$\begin{aligned}&\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|CA_{CL}\left(x(k+m)-x_{v,tar}\right)+CF\Delta w(k+m)\right\|_P^2\right]=\\&=\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|CA_{CL}\left(x(k+m)-x_{v,tar}\right)\right\|_P^2\right]+2tr\left(F^TC^TPCF\Sigma_W\right)-2tr\left(F^TA_{CL}{}^TC^TPCF\Sigma_W\right)\end{aligned} \tag{6-35}$$

Noticing that the terminal cost is given by $A_{CL}{}^TPA_{CL}+Q+K_s{}^TRK_s=P$, it follows that (6-32) is equal to:

$$\begin{aligned}&\mathbb{E}_{\mathfrak{X}(k)}\left[V(k+1)\right]-V(k)=\\&=\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|C_vA_{CL}\left(x_v(k+m)-x_{v,tar}\right)\right\|_P^2+\left\|C_v\left(x_v(k+m)-x_{v,tar}\right)\right\|_Q^2+\left\|K_s\left(x_v(k+m)-x_{v,tar}\right)\right\|_R^2\right]\\&-\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|C_v\left(x_v(k+m)-x_{v,tar}\right)\right\|_P^2+\left\|y(k)-y_{tar}\right\|_Q^2+\left\|\Delta u(k)\right\|_R^2\right]\\&+2tr\left(F_v^T\left(I_{nx}-A_{CL}{}^T\right)C_v{}^TPC_vF_v\Sigma_W\right)\\&=2tr\left(F_v^T\left(I_{nx}-A_{CL}{}^T\right)C_v{}^TPC_vF_v\Sigma_W\right)-\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|y(k)-y_{tar}\right\|_Q^2+\left\|\Delta u(k)\right\|_R^2\right]\\&\le 2tr\left(F^T\left(I_{nx}-A_{CL}{}^T\right)C^TPCF\Sigma_W\right)-\left\|y(k)-y_{tar}\right\|_Q^2\end{aligned} \tag{6-36}$$

□

The last inequality resembles the definition of a supermartingale. Indeed, while the system is sufficiently removed from the set-point, the cost is guaranteed to decrease on average. As it approaches a region in the neighborhood of the set-point, the sign of the right hand size of (6-36)becomes positive and there are no more guarantees of decreasing. This corresponds to the situation where the state fluctuates randomly inside the terminal set.

6.2. Case studies

The system described in Pannocchia & Rawlings (2003) is a CSTR considering an arbitrary reaction, A → B. The controlled variables are the liquid level, reactor temperature and concentration of A, which are steered using the jacket temperature and outlet flow rate. A general view of the reactor is provided in Figure 6-1. More details concerning the model may be found in the cited reference.

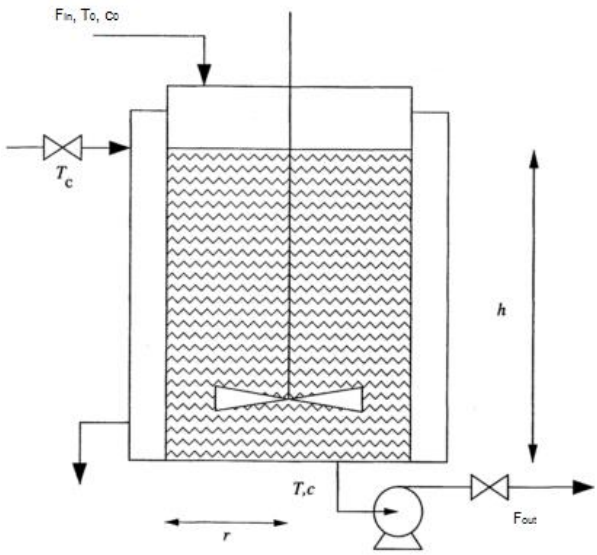


Figure 6-1: Schematic representation of the considered system - (Pannocchia & Rawlings, 2003)

It is evident that the system only has 2 degrees of freedom, so that not all controlled variables can be driven independently. Since temperature and concentration are highly coupled in this case, we have chosen to control only the first one. A complete description of the variables in this problem is given in Table 6-1:

Table 6-1: Variables of the linearized CSTR

Variable	Description
y_1	Reactant concentration (kmol/L)
y_2	Reactor temperature (K)
y_3	Reactor level (m)
u_1	Jacket temperature (K)
u_2	Output flowrate (L/min)
w_1	Process noise of y_1(kmol/L)
w_2	Process noise of y_2(K)
w_3	Process noise of y_3(m)

The state space (positional) model has been obtained through a linearization around the same steady state as described in the referred work. A sampling time of 15 seconds has been chosen to the discretization of the model. Since the order of magnitude of the outputs vary over a wide range, normalizing factors of 80 kmol/L, 8K and 0.1m were used. The definition of scaled variables is the same of Chapter 4, (4-8). The resulting discrete time state space description, already considering scaled variables, is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.7448 & -0.1961 & -0.0375 \\ 0.0563 & 0.7806 & -0.1091 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} -0.0070 & 0.0004 \\ 0.0580 & 0.0009 \\ 0 & -0.0165 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$
$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \end{bmatrix} \qquad (6\text{-}37)$$
$$\begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Truncated Gaussian noise with zero-mean and standard deviation of 20 kmol/L, 2 K and 0.025 m for concentration, temperature and level, respectively, was added in the states at all sampling times. The truncation has been performed at three standard deviations, as previously.

Tuning parameters and constraints are presented in Table 6-2 and Table 6-3, respectively. The input limits have been relaxed in comparison to the simulations of Pannocchia & Rawlings (2003) because they would saturate in the presence of the robust version of the chance constraint (6-8). Tuning matrix *Q* is given for the scaled outputs, but the results are shown in engineering units.

Table 6-2: Input and control moves bounds

Input	Minimum value $(u_{\min})$	Maximum value $(u_{\max})$	Maximum change $(\Delta u_{\max})$
u_1	200K	420 K	10 K
u_2	0L/min	400 L/min	150 L/min

Table 6-3: Tuning parameters - CSTR

Parameter	Value
Q	$\begin{bmatrix}0 & 1 & 10\end{bmatrix}$
R	$\begin{bmatrix}10^{-4} & 10^{-4}\end{bmatrix}$
Q_{λ}	$\begin{bmatrix}10^{3} & 10^{3} & 10^{3}\end{bmatrix}$
m	4

Finally, a soft constraint was included in y_3, imposing the level to satisfy the following relation:

$$\mathbb{P}\left[y_3 \leq y_{sp}(3) + 0.02\right] \geq 0.9 \tag{6-38}$$

Two cases studies are considered in this section. In the first, regulator case is presented for a fixed noise realization, in order to analyze the behavior of the objective function. The second case study is a Monte Carlo approach as done in Chapters 4 and 5.

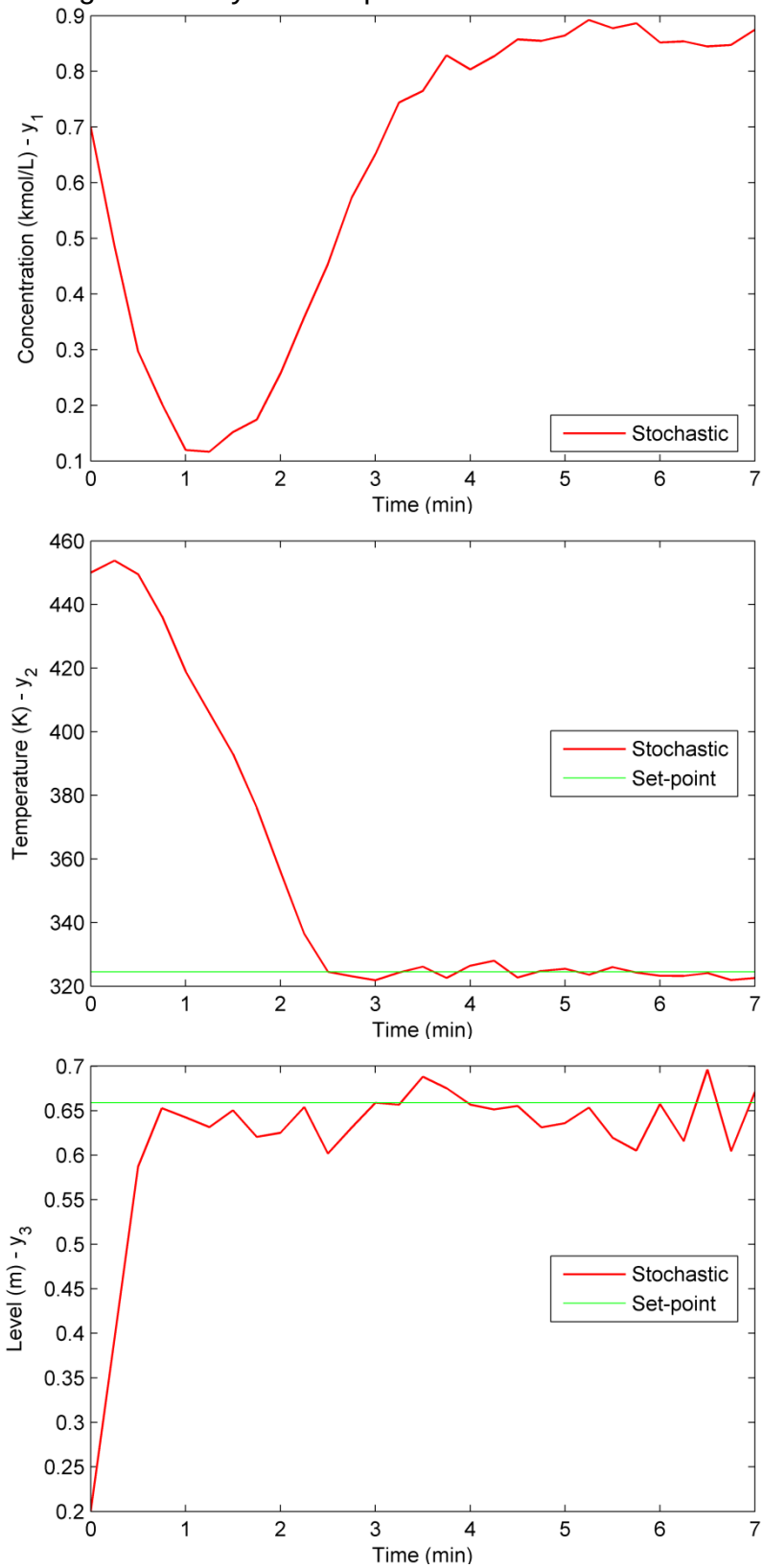
In the first case study, the system starts at a point removed from the origin and is directed towards it, as resumed in Table 6-4:

Table 6-4: Simulation operating points in the first case study

System initial conditions		Output set-point
Inputs	Outputs	
370 K	0.700kmol/L	0.877kmol/L
20 L/min	450 K	324.5 K
	0.2 m	0.659 m

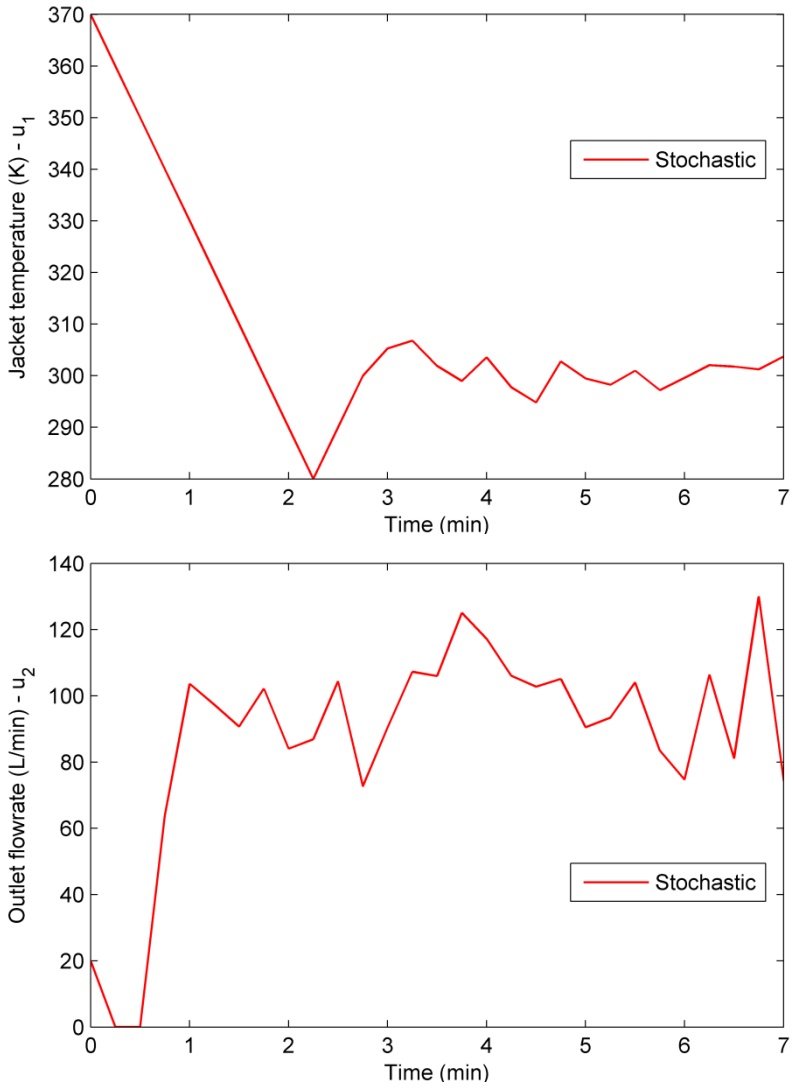
The behavior of the system in closed-loop is illustrated in Figures 6-2 and 6-3.

Figure 6-2: System outputs – one noise realization



Source: own elaboration

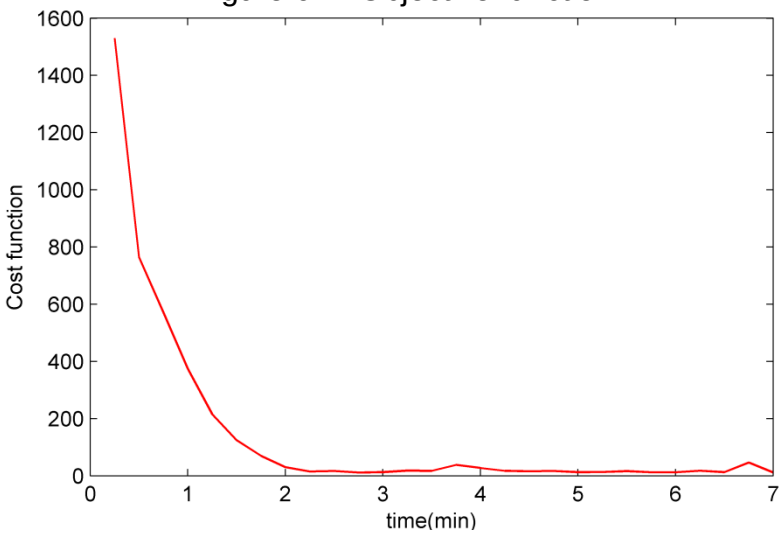
Figure 6-3: System inputs – one noise realization



Source: own elaboration

After a transient where the temperature rate constraint is active, the system oscillates around a steady state near the set-point. The expected behavior of the objective function is consistent with this simulation is composed of a phase of decrease followed by some oscillations, as depicted in Figure 6-7:

Figure 6-4: Objective function



Source: own elaboration

It is worthwhile to mention that the notion of stochastic stability and convergence of the objective function is a little different from the standard MPC theory, since the cost expected to decrease only when the initial state is sufficiently removed from the origin. Even in that situation, it is theoretically possible to observe some increases due to the probabilistic nature of the bound(6-30), which is present in Figure 6-7. The increases that happen at 3.5 and 6.5 minutes occur when the right-hand size of (6-30) becomes positive, in accordance with the result of Theorem 2.

Regarding the second case study, we consider the same framework of Chapter 4, comparing the proposed controller to two benchmarks: a standard nominal MPC and a LQR, considering the same tuning parameters. The Monte Carlo technique of previous chapters, with 1000 repetitions of simulations with different noise sequences, was reprised.

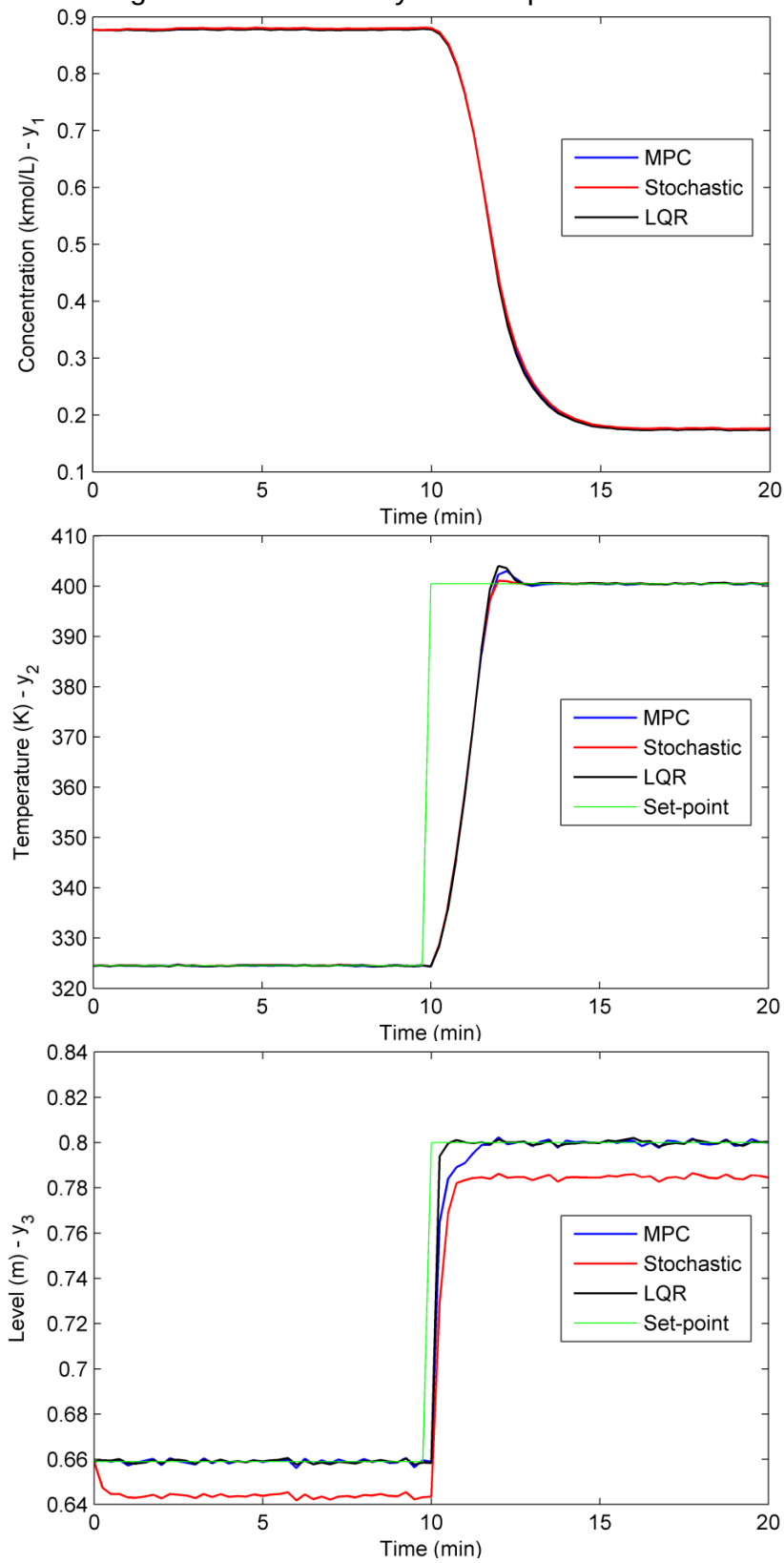
The simulation was composed of two parts: regulation at the origin in the first half and set-point tracking in the second, in a total of 20 minutes (80 sampling times). The exact conditions are given in Table 6-5:

Table 6-5: Simulation operating points in the second case study

System initial conditions		Output set-points	
Inputs	Outputs	$0 \leq k \leq 40$	$41 \leq k \leq 80$
100 L/min	0.877kmol/L	0.877kmol/L	0.1 kmol/L
300 K	324.5 K	324.5 K	400.5 K
	0.659 m	0.659 m	0.8 m

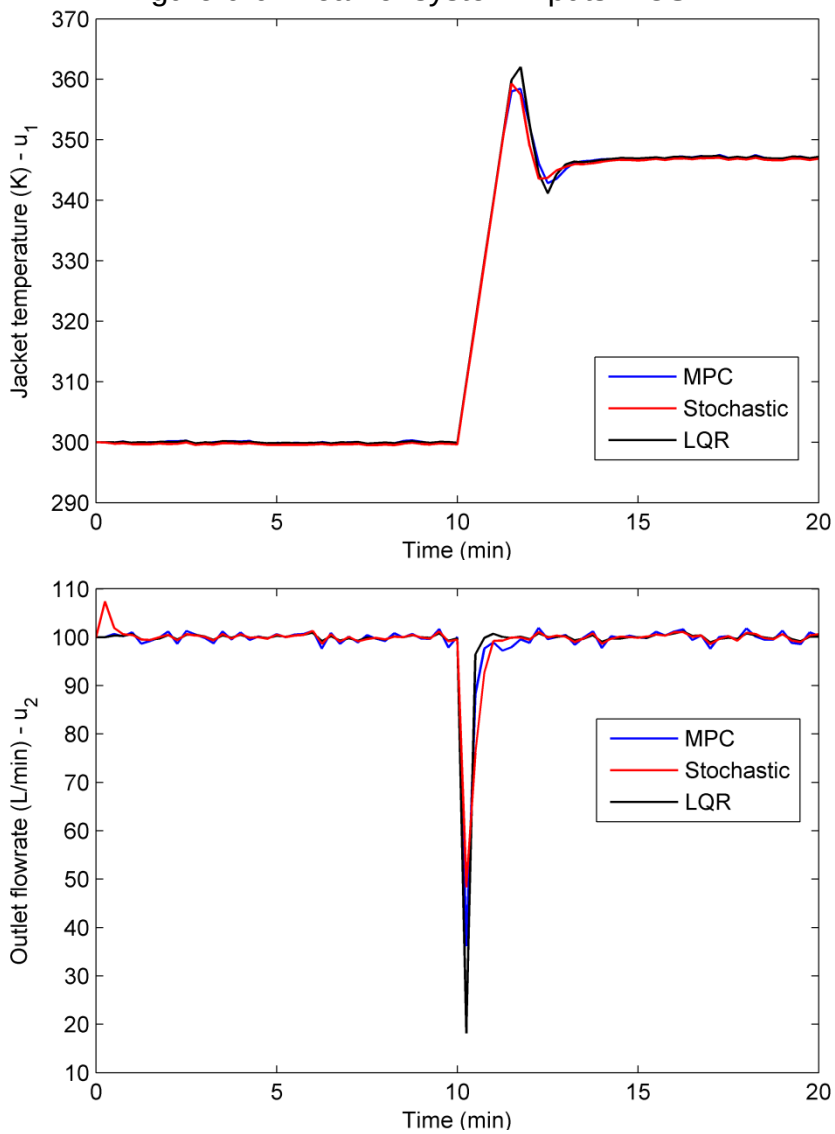
Figures 6-5 and 6-6 show the mean inputs and outputs.

Figure 6-5: Mean of system outputs – CSTR



Source: own elaboration

Figure 6-6: Mean of system inputs – CSTR

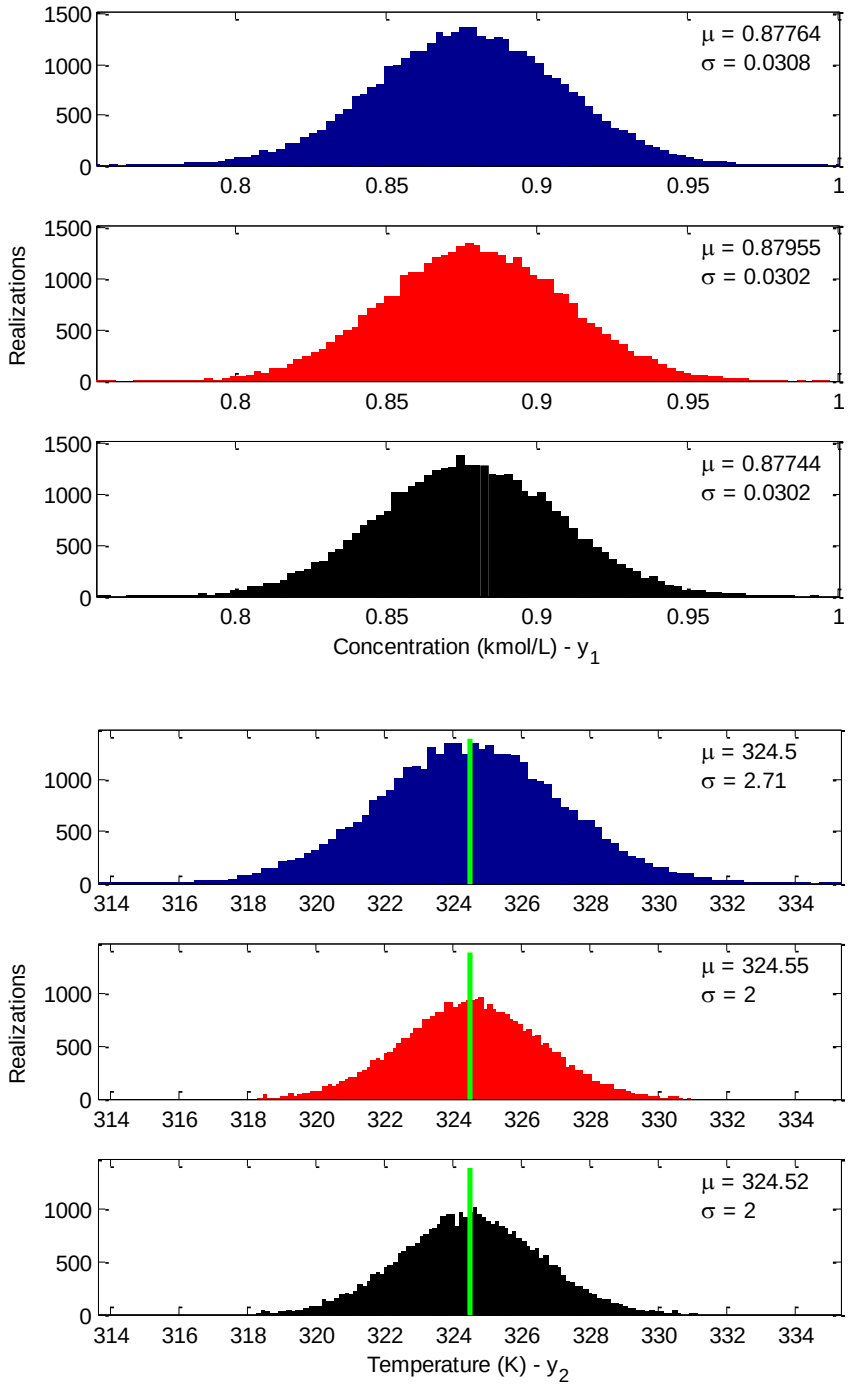


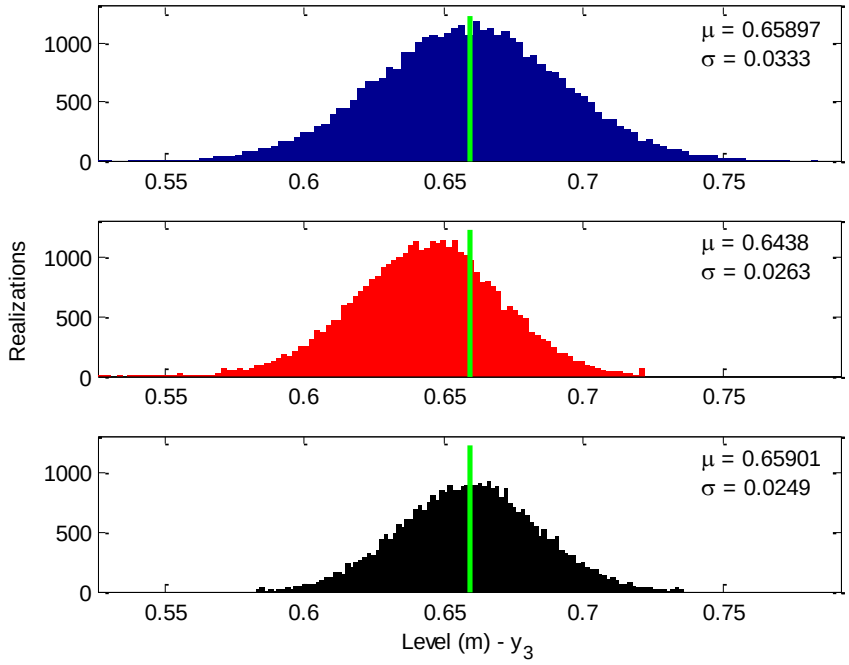
Source: own elaboration

As previously, the soft constraint induces a back off over one of the outputs. Without this constraint, the controller is able to perform the set-point tracking, which is illustrated by the mean trajectory of y_2. The main difference in comparison to the formulation of Chapter 3 is not explicit in the previous figures: if the input constraints are tightened, the soft constraint is respected in an overly conservative fashion, causing for instance the system to stabilize in a position where no constraint violations are reported for most of the simulations.

The comparison of the distribution of system outputs is presented in Figures 6-7 and 6-8:

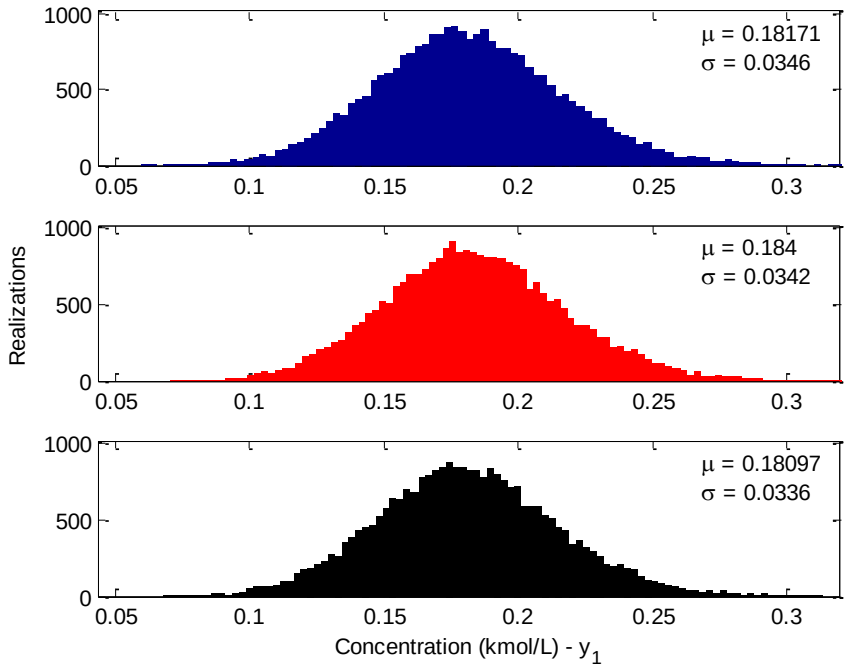
Figure 6-7: Distribution of system outputs between 1 and 10 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green

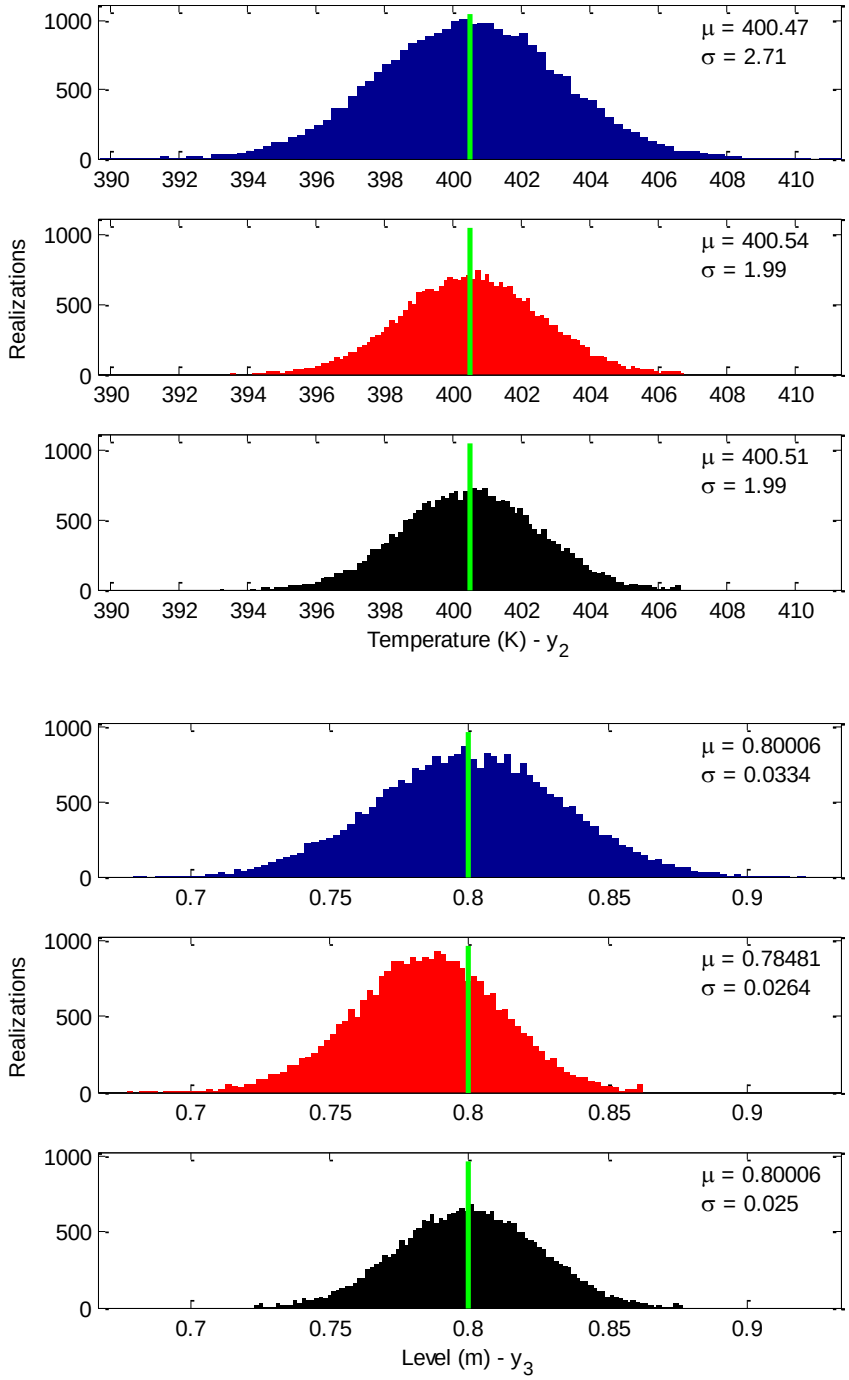




Source: own elaboration

Figure 6-8: Distribution of system outputs between 13.5 and 20 min for MPC (blue), Stochastic controller (red) and Truncated LQR (black). Set-point in green





Source: own elaboration

From the previous figures, we notice that the relaxed input constraints allowed the LQR controller to achieve optimal performance, which corresponds to the lowest variances for all variables, at both steady states. At some cases, the proposed controller is able to match the optimal variance, indicating that it may be possible to

guarantee local optimality of a SMPC. The accumulated costs are summarized in Table 6-6:

Table 6-6: Average weighted integral square error - CSTR

Standard MPC J_{MPC}	Stochastic controller J_{Sto}	Truncated LQR J_{LQR}
451.5	436.9	405.7

Besides the observation of the last paragraph, it must be kept in mind that the benchmark controllers ignore the probabilistic constraint, which has a direct impact over the cost due to the shift of the mean of y_3. Since the cost of the stochastic controller with stability guarantee is slightly greater than the obtained in a nearly-optimal behavior (increase of 7.7%), it follows that the proposed algorithm is an interesting candidate for practical applications.

Regarding the soft constraint, the effect of mean displacement is clear from Figures 6-4 and 6-5. Constraint violation occurred with frequencies given in Table 6-7, showing that the proposal was more conservative in this case in comparison to the controller of Chapter 3. However, the observed violation frequency is not far away from the prescribed limit of 10%, showing that conservativeness is not exaggerated in this formulation.

Table 6-7: Percentage of violation of the soft constraint– CSTR

First steady-state ($4 \leq k \leq 40$)		Second steady-state ($54 \leq k \leq 80$)	
Controller	Violations	Controller	Violations
MPC	26.76%	MPC	27.34%
Stochastic	8.59%	Stochastic	8.73%
LQR	20.85%	LQR	21.35%

In summary, the modified stochastic controller trades off performance and stability, as usual. In situations where the noise level is small in comparison to the effect that allowed control moves may produce, its application is favored. This condition is common in practice, because there is no control algorithm that performs well in the presence of excessive noise and limited control action.

7. Conclusions

7.1. Main contributions

The problem of controlling a linear system subject to bounded additive noise has been tackled in this work. More specifically, we focused on the set-point tracking problem under these circumstances. The algorithm proposed in Chapter 3 meets most of the requirements initially defined: consideration of hard and soft constraint, adequate computational time and attractive performance in comparison to alternatives from the literature.

From a more formal point of view, the modified algorithm of Chapter 6 is more significant since it guarantees recursive feasibility and asymptotical stability in the presence of persistent disturbances. The hypothesis of finite support is crucial in establishing those results, which are very similar in spirit to the robust model predictive control approach. Important features of the proposed controller in contrast to robust formulations are the probabilistic nature of state constraints and the objective value expressed as an expected value. Less conservativeness and better performance may be achieved over the worst-case typical approach.

The case studies illustrated the set-point tracking ability of the proposed controller, an innovation when compared to the usual literature dedicated to the regulator case. The integral action was obtained from the incremental description of the system, inspired from works on nominal MPC. An additional feature of the case studies is the consideration of truncated skew normal noise, which is more challenging than the simple Gaussian noise since its mean is not zero.

Another major innovation of this work is the proposal of a zone control strategy in Chapter 5, which, to the best of our knowledge, has not been done in the SMPC literature. A significant reduction of the control action has been achieved with such strategy, since the algorithm does not try to over control the random perturbations. Therefore, in a practical application it could achieve a compromise between output tracking and conservation of the actuators.

Finally, Chapter 6 provides a stochastic controller with stability guarantee, the first result in the literature for the tracking problem. The additional constraints necessary to prove recursive feasibility lead to a more conservative formulation that

may not achieve good performance in the presence of stringent input constraints. However, the conditions where the controller could be successfully employed are not uncommon; hence the result is valuable from both theoretical and practical viewpoints.

7.2. Future work directions

All control algorithms within this thesis are based on the assumption of bounded noise support. This hypothesis is justifiable in an industrial framework, but its suppression would lead to interesting theoretical contributions. Recursive feasibility would no longer be obtainable through worst-case considerations, and therefore it is not trivial to handle the problem.

Another research line may be derived from the suppression of the full state measurement assumption. Output feedback is not a novelty in stochastic predictive control, since (Hokayem et al., 2012) and (Cannon et al., 2012) have already worked on this issue, but not considering the set-point tracking problem. It would be necessary to study how the inclusion of different observers changes the proposed algorithm and its stability results. An interesting candidate is the Moving Horizon Estimator (MHE) due to its bounds on the estimation error that could be employed by the controller.

Noise distribution is assumed to be known and time invariant. However, it is possible in reality that a plant is subject to stronger or weaker disturbances as time evolves, which should be considered by the controller. Similarly, all noise is supposed to enter the system at every sampling time. Therefore, the case of disturbances with different time rates has not been addressed, even though it may be used to model an important situation in practice where there is a fast process noise coupled with a slower noise that changes the operating point.

The zone control problem could be tackled through an approach more heavily based on invariant sets, since these entities are natural extensions of equilibrium points. Ideally, the controller would perform no control move if the system is predicted to continue in a given set, acting only when necessary. That approach would be even more attractive than the one in Chapter 5 in terms of equipment protection, but its stability analysis could be a lot more challenging.

REFERENCES

Åstrom, K. J. (2006). *Introduction to Stochastic Control Theory*. New York: Dover Publications.

Azzalini, A. (1985). A Class of Distributions Which Includes the Normal Ones. *Scandinavian Journal of Statistics*, *12*(2), 171–178.

Bernardini, D., & Bemporad, A. (2009). Scenario-based model predictive control of stochastic constrained linear systems. In *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference* (pp. 6333–6338). Shanghai.

Blanchini, F. (1999). Set invariance in control. *Automatica*, *35*, 1747–1767.

Cannon, M., Cheng, Q., Kouvaritakis, B., & Raković, S. V. (2012). Stochastic tube MPC with state estimation. *Automatica*, *48*(3), 536–541.

Cannon, M., Kouvaritakis, B., & Wu, X. (2009). Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints. *Automatica*, *45*(1), 167–172.

Couchman, P. D., Cannon, M., & Kouvaritakis, B. (2006). Stochastic MPC with inequality stability constraints. *Automatica*, *42*(12), 2169–2174.

Ferramosca, A., Limon, D., González, A. H., Alvarado, I., & Camacho, E. F. (2012). Robust MPC for tracking zone regions based on nominal predictions. *Journal of Process Control*, *22*(10), 1966–1974.

Ferramosca, A., Limon, D., González, A. H., Odloak, D., & Camacho, E. F. (2009). MPC for tracking target sets. In *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference* (pp. 8020–8025). Shanghai.

González, A. H., & Odloak, D. (2009). A stable MPC with zone control. *Journal of Process Control*, *19*(1), 110–122.

Goulart, P. J., Kerrigan, E. C., & Maciejowski, J. M. (2006). Optimization over state feedback policies for robust control with constraints. *Automatica*, *42*(4), 523–533.

Gurobi Optimization, Inc. (2013). Gurobi Optimizer Reference Manual. Retrieved from http://www.gurobi.com (accessed Oct. 2013)

Hokayem, P., Chatterjee, D., & Lygeros, J. (2009). On stochastic receding horizon control with bounded control inputs. In *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference* (pp. 6359–6364). Shanghai.

Hokayem, P., Cinquemani, E., Chatterjee, D., Ramponi, F., & Lygeros, J. (2012). Stochastic receding horizon control with output feedback and bounded controls. *Automatica*, *48*(1), 77–88.

Kerrigan, E. C. (2005). Matlab Invariant Set Toolbox. Retrieved from http://www-control.eng.cam.ac.uk/eck21/matlab/invsetbox/index.html (accessed Oct. 2013)

Korda, M., & Cigler, J. (2012). Nonquadratic stochastic model predictive control: A tractable approach. *Automatica*, *48*(9), 2352–2358.

Korda, M., Gondhalekar, R., Cigler, J., & Oldewurtel, F. (2011). Strongly feasible stochastic model predictive control. In *IEEE Conference on Decision and Control and European Control Conference* (pp. 1245–1251). Orlando.

Kouvaritakis, B., Cannon, M., Raković, S. V., & Cheng, Q. (2010). Explicit use of probabilistic distributions in linear predictive control. *Automatica*, *46*(10), 1719–1724.

Löfberg, J. (2004). YALMIP : A toolbox for modeling and optimization in MATLAB. In *IEEE International Symposium on Computer Aided Control Systems Design* (pp. 284–289). Taipei.

Maeder, U., Borrelli, F., & Morari, M. (2009). Linear offset-free Model Predictive Control. *Automatica*, *45*(10), 2214–2222.

Mayne, D. Q., Rawlings, J. B., Rao, C. V, & Scokaert, P. O. M. (2000). Constrained model predictive control : Stability and optimality. *Automatica*, *36*, 789–814.

Ogunnaike, B. A., Lemaire, J. P., Morari, M., & Ray, W. H. (1983). Advanced Multivariable Control of a Pilot-Plant Distillation Column. *AIChE*, *29*(4), 632–640.

Pannocchia, G., & Rawlings, J. B. (2003). Disturbance models for offset-free model-predictive control. *AIChE Journal*, *49*(2), 426–437.

Primbs, J. A., & Sung, C. H. (2009). Stochastic Receding Horizon Control of Constrained Linear Systems With State and Control Multiplicative Noise. *IEEE Transactions on Automatic Control*, *54*(2), 221–230.

Ramponi, F., Chatterjee, D., Milias-Argeitis, A., Hokayem, P., & Lygeros, J. (2010). Attaining Mean Square Boundedness of a Marginally Stable Stochastic Linear System With a Bounded Control Input. *IEEE Transactions on Automatic Control*, *55*(10), 2414–2418.

Scokaert, P. O. M., & Mayne, D. Q. (1998). Min-Max Feedback Model Predictive Control for Constrained Linear Systems. *IEEE Transactions on Automatic Control*, *43*(8), 1136–1142.

Van Hessem, D. H., Scherer, C. W., & Bosgra, O. H. (2001). LMI-based closed-loop economic optimization of stochastic process operation under state and input constraints. In *Proceedings of the 40th IEEE conference on Decision and Control* (pp. 4228–4233). Orlando.

Yang, Y., Sontag, E. D., & Sussmann, H. J. (1997). Global stabilization of linear discrete-time systems with bounded feedback. *Systems & Control Letters*, *30*, 273–281.

Appendix A - Calculation of the objective function for systems subject to bounded noise

- T_1

Substituting the dynamics (3-13)in the cost (3-16), we have

$$\begin{aligned}
&T_1 = \mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{X}(k)\right\|^2_{\bar{C}^T\bar{Q}\bar{C}}\right] = \mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{A}x(k)+\bar{B}\Delta U(k)+\bar{F}\Delta\bar{W}(k)\right\|^2_{\bar{C}^T\bar{Q}\bar{C}}\right] = \\
&\mathbb{E}_{\mathfrak{X}(k)}\big[x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{A}x(k)+\Delta\bar{U}(k)^T\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{B}\Delta\bar{U}(k)+ \\
&\Delta\bar{W}(k)^T\bar{F}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\Delta\bar{W}(k)+2x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{B}\Delta\bar{U}(k)+ \\
&2x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\Delta\bar{W}(k)+2\Delta\bar{U}(k)^T\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\Delta\bar{W}(k)\big]
\end{aligned} \tag{A-1}$$

This expression can be simplified by noticing that the quadratic forms involving only x(k) and $\bar{W}(k)$ are independent of the decision variables and therefore are constants in the optimization problem.

$$\begin{aligned}
&T_1 = \underbrace{\left\|x(k)\right\|^2_{\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{A}} + tr\left(\bar{F}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\Delta\bar{W}^T(k)\right]\right)}_{cte} + \\
&2x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\left(\bar{B}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{U}(k)\right]+\bar{F}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\right]\right) \\
&+2tr\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\Delta\bar{U}^T(k)\right]\right)+tr\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{B}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{U}(k)\Delta\bar{U}^T(k)\right]\right)
\end{aligned} \tag{A-2}$$

where it was used that $tr\left(\mathbb{E}[\xi]\right)=\mathbb{E}\left[tr\left(\xi\right)\right]$ and the cyclic property of the trace, $tr\left(AB\right)=tr\left(BA\right)$.

Thus, it is necessary to calculate the mean value of control moves and noise realization, in addition to the covariance of the control moves and the cross-covariance between these inputs and the process noise.

- Mean of noise realization

Even if the original noise is zero-mean, this term does not vanish because it is an expected value conditional to the information known up to time k, which includes $w(k-1)$:

$$\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\right]=\begin{bmatrix}\mathbb{E}_{\mathfrak{X}(k)}\left[w(k)-w(k-1)\right]\\ \mathbb{E}_{\mathfrak{X}(k)}\left[w(k+1)-w(k)\right]\\ \vdots\\ \mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-1)-w(k+m-2)\right]\end{bmatrix}=\begin{bmatrix}\mathbb{E}_{\mathfrak{X}(k)}\left[w(k)\right]-w(k-1)\\ 0\\ \vdots\\ 0\end{bmatrix}= \tag{A-3}$$
$$=\mu^{\Delta w}(k)$$

Since the noise is identically distributed, only the first component may be different from zero. It is important to notice that $\mu^{\Delta w}(k)$ is time-dependent even if the distribution of $w(k)$ is stationary. Remember that $\mathbb{E}_{\mathfrak{X}(k)}\left[w(k)\right]=\mu_w$ is known from the noise distribution. Previous noise realization ($w(k-1)$) must also be taken into account at each sampling time.

- Mean of control moves

$$\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{U}(k)\right]=\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{G}\Delta\bar{W}(k)+\bar{D}\right]=$$
$$\bar{G}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\right]+\bar{D}=\bar{G}\mu^{\Delta w}(k)+\bar{D} \tag{A-4}$$

- Covariance

$$\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{U}(k)\Delta\bar{U}^T(k)\right]=\mathbb{E}_{\mathfrak{X}(k)}\left[\left(\bar{G}\Delta\bar{W}(k)+\bar{D}\right)\left(\bar{G}\Delta\bar{W}(k)+\bar{D}\right)^T\right]=$$
$$\bar{G}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\Delta\bar{W}(k)^T\right]\bar{G}^T+\bar{G}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\right]\bar{D}^T+\bar{D}\left(\bar{G}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\right]\right)^T$$
$$+\bar{D}\bar{D}^T=$$
$$G\mu^{\Delta w\Delta w}(k)G^T+\bar{G}\mu^{\Delta w}(k)\bar{D}^T+\bar{D}\mu^{\Delta w}(k)^T\bar{G}^T+\bar{D}\bar{D}^T \tag{A-5}$$

where

$$\mu^{\Delta w \Delta w}(k) = \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta \bar{W}(k)\Delta \bar{W}(k)^T\right] \tag{A-6}$$

This matrix is tridiagonal because, following the independence assumption on $\{w(k)\}$, there are only two classes of non-zero elements: those of the diagonal (involving the same noise realization) and those of the super and subdiagonal (noise realizations of consecutive time steps). Also, each of these classes is composed of two possibilities, whereas they are a function of the known noise $w(k-1)$ or not. Altogether, there are 4 possibilities that have to be analyzed individually.

1. Diagonal involving $w(k-1)$

$$\begin{aligned}
&\mu_{00}^{\Delta w \Delta w}(k) = \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k)\Delta w(k)^T\right] = \mathbb{E}_{\mathfrak{X}(k)}\left[\left(w(k)-w(k-1)\right)\left(w(k)-w(k-1)\right)^T\right] = \\
&= \mathbb{E}_{\mathfrak{X}(k)}\left[w(k)w(k)^T\right] - w(k-1)\mathbb{E}_{\mathfrak{X}(k)}\left[w(k)^T\right] - \mathbb{E}_{\mathfrak{X}(k)}\left[w(k)\right]w(k-1)^T \\
&+ w(k-1)w(k-1)^T \\
&= \Sigma_w - w(k-1)\mu_w{}^T - \mu_w w(k-1)^T + w(k-1)w(k-1)^T
\end{aligned} \tag{A-7}$$

2. Nondiagonal involving $w(k-1)$

$$\begin{aligned}
&\mu_{01}^{\Delta w \Delta w}(k) = \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k)\Delta w(k+1)^T\right] = \\
&= \mathbb{E}_{\mathfrak{X}(k)}\left[\left(w(k)-w(k-1)\right)\left(w(k+1)-w(k)\right)^T\right] = \\
&= 0 - \Sigma_w - \mu_w w(k-1)^T + \mu_w w(k-1)^T = -\Sigma_w
\end{aligned} \tag{A-8}$$

3. Diagonal not involving $w(k-1)$

$$\begin{aligned}
&\mu_{11}^{\Delta w \Delta w}(k) = \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+1)\Delta w(k+1)^T\right] = \\
&= \mathbb{E}_{\mathfrak{X}(k)}\left[\left(w(k+1)-w(k)\right)\left(w(k+1)-w(k)\right)^T\right] = 2\Sigma_w
\end{aligned} \tag{A-9}$$

4. Nondiagonal not involving $w(k-1)$

$$\begin{aligned}
&\mu_{12}^{\Delta w \Delta w}(k) = \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+1)\Delta w(k+2)^T\right] = \\
&= \mathbb{E}_{\mathfrak{X}(k)}\left[\left(w(k+1)-w(k)\right)\left(w(k+2)-w(k+1)\right)^T\right] = -\Sigma_w
\end{aligned} \tag{A-10}$$

The complete expression is

$$\mu^{\Delta w \Delta w}(k) = \begin{bmatrix} \mu_{00}^{\Delta w \Delta w}(k) & -\Sigma_w & 0 & \cdots & 0 \\ -\Sigma_w & 2\Sigma_w & -\Sigma_w & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & -\Sigma_w & 2\Sigma_w \end{bmatrix} \tag{A-11}$$

- Cross-covariance

$$\begin{aligned} &\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\Delta\bar{U}^T(k)\right] = \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\left(\bar{G}\Delta\bar{W}(k)+\bar{D}\right)^T\right] = \\ &\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\Delta\bar{W}(k)^T\right]\bar{G}^T + \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{W}(k)\right]\bar{D}^T = \\ &\mu^{\Delta w \Delta w}(k)\bar{G}^T + \mu^{\Delta w}(k)\bar{D}^T \end{aligned} \tag{A-12}$$

Substituting (A-3)-(A-5) and (A-12) onto (A-2), we finally have

$$\begin{aligned} &T_1 = \|x(k)\|^2_{\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{A}} + tr\left(\bar{F}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\mu^{\Delta w \Delta w}(k)\right) + \\ &2x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\left(\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right) \\ &+2tr\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\left(\mu^{\Delta w \Delta w}(k)\bar{G}^T + \mu^{\Delta w}(k)\bar{D}^T\right)\right) + \\ &tr\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{B}\left(\bar{G}\mu^{\Delta w \Delta w}(k)\bar{G}^T + \bar{G}\mu^{\Delta w}(k)\bar{D}^T + \bar{D}\mu^{\Delta w}(k)^T\bar{G}^T + \bar{D}\bar{D}^T\right)\right) \end{aligned} \tag{A-13}$$

- T_2

Considering the previously established expressions for the mean of noise (A-3)and control moves (A-4), it follows that

$$\begin{aligned} &\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{X}(k)\right] = \mathbb{E}_{\mathfrak{X}(k)}\left[\bar{A}x(k)+\bar{B}\Delta\bar{U}(k)+\bar{F}\Delta\bar{W}(k)\right] = \\ &\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k) \end{aligned} \tag{A-14}$$

- T_3

The evaluation of T_3 is also trivial once we have established the covariance of $\Delta U(k)$.

$$
\begin{aligned}
T_3 &= \mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\Delta\bar{U}(k)\right\|^2_{\bar{R}}\right] = \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{U}^T(k)\bar{R}\Delta\bar{U}(k)\right] = \\
&= tr\left(\bar{R}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{U}(k)\Delta\bar{U}^T(k)\right]\right) = \\
&= tr\left(\bar{R}\left(\bar{G}\mu^{\Delta w\Delta w}(k)\bar{G}^T + \bar{G}\mu^{\Delta w}(k)\bar{D}^T + \bar{D}\mu^{\Delta w}(k)^T\bar{G}^T + \bar{D}\bar{D}^T\right)\right)
\end{aligned}
\tag{A-15}
$$

Considering (3-16),(A-13)-(A-15), an expression of the forced component of the cost that does not require the on-line calculation of any expected values is given by

$$
\begin{aligned}
&V_{for}(k) = \mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{X}(k)\right\|^2_{\bar{C}^T\bar{Q}\bar{C}}\right] - 2\bar{Y}_{sp}^T\bar{Q}\bar{C}\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{X}(k)\right] + \left\|\bar{Y}_{sp}\right\|^2_{\bar{Q}} + \mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\Delta\bar{U}(k)\right\|^2_{\bar{R}}\right] = \\
&\left\|x(k)\right\|^2_{\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\bar{A}} + tr\left(\bar{F}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\mu^{\Delta w\Delta w}(k)\right) \\
&+2x(k)^T\bar{A}^T\bar{C}^T\bar{Q}\bar{C}\left(\bar{B}\left(\bar{G}\mu^{\Delta w}(k) + \bar{D}\right) + \bar{F}\mu^{\Delta w}(k)\right) \\
&+2tr\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{F}\left(\mu^{\Delta w\Delta w}(k)\bar{G}^T + \mu^{\Delta w}(k)\bar{D}^T\right)\right) \\
&+tr\left(\left(\bar{B}^T\bar{C}^T\bar{Q}\bar{C}\bar{B} + \bar{R}\right)\left(\bar{G}\mu^{\Delta w\Delta w}(k)\bar{G}^T + \bar{G}\mu^{\Delta w}(k)\bar{D}^T + \bar{D}\mu^{\Delta w}(k)^T\bar{G}^T + \bar{D}\bar{D}^T\right)\right) \\
&-2\bar{Y}_{sp}^T\bar{Q}\bar{C}\left(\bar{A}x_v(k) + \bar{B}\left(\bar{G}\mu^{\Delta w}(k) + \bar{D}\right) + \bar{F}\mu^{\Delta w}(k)\right) + \left\|\bar{Y}_{sp}\right\|^2_{\bar{Q}}
\end{aligned}
\tag{A-16}
$$

- T_4

Substituting the dynamics (3-17)in the cost (3-20),

$$
\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{\bar{X}}(k)\right\|^2_{\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}}\right] = \\
&\mathbb{E}_{\mathfrak{X}(k)}\Big[x(k+m)^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{A}}x(k+m) + \Delta\bar{\bar{U}}(k)^T\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Delta\bar{\bar{U}}(k) \\
&+\Delta\bar{\bar{W}}(k)^T\bar{\bar{F}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Delta\bar{\bar{W}}(k) + 2x_v(k+m)^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Delta\bar{\bar{U}}(k) \\
&+2x_v(k+m)^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Delta\bar{\bar{W}}(k) + 2\Delta\bar{\bar{U}}(k)^T\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Delta\bar{\bar{W}}(k)\Big]
\end{aligned}
\tag{A-17}
$$

Each of these terms should be individually analyzed because different considerations about noise correlatedness take place between state, input and noise itself.

a) Let T_m be a matrix to select only the last component of $\bar{X}(k)$, i. e., $T_m\bar{X}(k) = x_v(k+m)$. Then,

$$\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}[x_v(k+m)^T \bar{\bar{A}}^T \bar{\bar{C}}^T \bar{\bar{Q}} \bar{\bar{C}} \bar{\bar{A}} x_v(k+m)] = \\
&= tr\left(\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{\bar{A}}^T \bar{\bar{C}}^T \bar{\bar{Q}} \bar{\bar{C}} \bar{\bar{A}} \left(T_m \bar{X}(k)\right)\left(T_m \bar{X}(k)\right)^T\right]\right) \\
&= tr\left(T_m{}^T \bar{\bar{A}}^T \bar{\bar{C}}^T \bar{\bar{Q}} \bar{\bar{C}} \bar{\bar{A}} T_m \mathbb{E}_{\mathfrak{X}(k)}\left[\left(\bar{X}(k)\right)\left(\bar{X}(k)\right)^T\right]\right)
\end{aligned} \tag{A-18}$$

Using equation (3-13), we have

$$\begin{aligned}
&\Sigma_{\bar{X}} = \mathbb{E}_{\mathfrak{X}(k)}\left[\bar{X}(k)\bar{X}(k)^T\right] = \\
&= \mathbb{E}_{\mathfrak{X}(k)}\left[\left(\bar{A}x_v(k) + \bar{B}\Delta\bar{U}(k) + \bar{F}\Delta\bar{W}(k)\right)\left(\bar{A}x_v(k) + \bar{B}\Delta\bar{U}(k) + \bar{F}\Delta\bar{W}(k)\right)^T\right] \\
&= \bar{A}x_v(k)x_v(k)^T\bar{A}^T + \bar{F}\mu^{\Delta w\Delta w}(k)\bar{F}^T \\
&+\bar{A}x_v(k)\mu^{\Delta wT}(k)\bar{F}^T + \bar{F}\mu^{\Delta w}(k)x_v(k)^T\bar{A}^T \\
&+\bar{B}\left(G\mu^{\Delta w\Delta w}(k)G^T + \bar{G}\mu^{\Delta w}(k)\bar{D}^T + \bar{D}\mu^{\Delta w}(k)^T\bar{G}^T + \bar{D}\bar{D}^T\right)\bar{B}^T \\
&+\bar{A}x_v(k)\left(\bar{G}\mu^{\Delta w}(k) + \bar{D}\right)^T\bar{B} + \bar{B}^T\left(\bar{G}\mu^{\Delta w}(k) + \bar{D}\right)x_v(k)^T\bar{A}^T \\
&+\bar{F}\left(\mu^{\Delta w\Delta w}(k)\bar{G}^T + \mu^{\Delta w}(k)\bar{D}^T\right)\bar{B}^T + \bar{B}\left(\mu^{\Delta w\Delta w}(k)\bar{G}^T + \mu^{\Delta w}(k)\bar{D}^T\right)^T\bar{F}^T
\end{aligned} \tag{A-19}$$

Note that the cross terms between $x_v(k)$ and $\Delta\bar{W}(k)$ are only an additive contribution to the cost and do not interfere in the optimization. All calculations follow from (A-4)-(A-12),(A-13).

b) First, we use the cyclic property to rearrange this term

$$\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{U}}(k)^T \bar{\bar{B}}^T \bar{\bar{C}}^T \bar{\bar{Q}} \bar{\bar{C}} \bar{\bar{B}} \Delta\bar{\bar{U}}(k)\right] = \\
&= tr\left(\bar{\bar{B}}^T \bar{\bar{C}}^T \bar{\bar{Q}} \bar{\bar{C}} \bar{\bar{B}} \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{U}}(k)\Delta\bar{\bar{U}}(k)^T\right]\right) \\
&= tr\left(\bar{\bar{B}}^T \bar{\bar{C}}^T \bar{\bar{Q}} \bar{\bar{C}} \bar{\bar{B}} \Sigma_{UU}\right)
\end{aligned} \tag{A-20}$$

We are considering a fixed feedback law for the inputs after the control horizon, therefore control moves are correlated only if they correspond to the same time step or consecutive time steps,

$$\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta u(k+i)\Delta u(k+j)^T\right]=\begin{cases} K\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+i-1)\Delta w(k+i-1)^T\right]K^T, \text{ if } i=j \\ K\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+i-1)\Delta w(k+i)^T\right]K^T, \text{ if } i=j+1 \\ 0, \text{ if } |i-j|>1 \end{cases} \tag{A-21}$$

By analogy with the previously defined matrix $\mu^{\Delta w\Delta w}$, Σ_{UU} is also a tridiagonal matrix given by:

$$\Sigma_{UU}=\begin{bmatrix} 2K\Sigma_w K^T & -K\Sigma_w K^T & 0 & \cdots & 0 \\ -K\Sigma_w K^T & 2K\Sigma_w K^T & -K\Sigma_w K^T & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & -K\Sigma_w K^T & 2K\Sigma_w K^T \end{bmatrix} \tag{A-22}$$

c) This term is another constant and may be dropped from the optimization,

$$\begin{aligned} &\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{W}}(k)^T\bar{\bar{F}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Delta\bar{\bar{W}}(k)\right]= \\ &=tr\left(\bar{\bar{F}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{W}}(k)\Delta\bar{\bar{W}}(k)^T\right]\right) \\ &=tr\left(\bar{\bar{F}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\bar{\bar{\mu}}^{\Delta w\Delta w}\right) \end{aligned} \tag{A-23}$$

The covariance matrix is also similar to $\mu^{\Delta w\Delta w}$, as follows:

$$\bar{\bar{\mu}}^{\Delta w\Delta w}=\begin{bmatrix} 2\Sigma_w & -\Sigma_w & 0 & \cdots & 0 \\ -\Sigma_w & 2\Sigma_w & -\Sigma_w & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & -\Sigma_w & 2\Sigma_w \end{bmatrix} \tag{A-24}$$

$$\bar{\bar{\mu}}^{\Delta w\Delta w}\in\mathbb{R}^{(p-m)nw\times(p-m)nw}$$

d) The fourth component may be rearranged as

$$\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[2x(k+m)^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Delta\bar{\bar{U}}(k)\right]= \\
&=2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{U}}(k)x_v(k+m)^T\right]\right) \\
&=2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UX}\right)
\end{aligned} \tag{A-25}$$

The only control move that is correlated to $x_v(k+m)$ is $\Delta u(k+m)$, because both depend on $w(k+m-1)$. Thus, only the first element of Σ_{UX} is nonzero.

$$\begin{aligned}
&\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta u(k+m)x_v(k+m)^T\right]= \\
&=\mathbb{E}_{\mathfrak{X}(k)}\left[K\Delta w(k+m-1)\left(\bar{A}x_v(k)+\bar{B}\Delta\bar{U}(k)+\bar{F}_v\Delta\bar{W}(k)\right)^T T_m{}^T\right] \\
&=\mathbb{E}_{\mathfrak{X}(k)}\left[K\left(w(k+m-1)-w(k+m-2)\right)\left(\bar{A}x_v(k)+\bar{B}\Delta\bar{U}(k)+\bar{F}_v\Delta\bar{W}(k)\right)^T T_m{}^T\right] \\
&=K\mu^w\left(\bar{A}x_v(k)\right)^T T_m{}^T-K\mu^w\left(\bar{A}x_v(k)\right)^T T_m{}^T \\
&+K\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-1)\left(\bar{B}\bar{D}+\left(\bar{B}\bar{G}+\bar{F}_v\right)\Delta\bar{W}\right)^T\right]T_m{}^T \\
&-K\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-2)\left(\bar{B}\bar{D}+\left(\bar{B}\bar{G}+\bar{F}_v\right)\Delta\bar{W}\right)^T\right]T_m{}^T \\
&=K\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-1)\Delta\bar{W}^T\right]\left(\bar{B}\bar{G}+\bar{F}_v\right)^T T_m{}^T \\
&-K\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-2)\Delta\bar{W}^T\right]\left(\bar{B}\bar{G}+\bar{F}_v\right)^T T_m{}^T
\end{aligned} \tag{A-26}$$

Since

$$\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-1)\Delta\bar{W}^T\right]=\left[-\mu^w w(k-1)^T \quad \underbrace{0_{nw} \quad \cdots \quad 0_{nw}}_{m-2} \quad \Sigma_w\right] \tag{A-27}$$

And

$$\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-2)\Delta\bar{W}^T\right]=\left[-\mu^w w(k-1)^T \quad \underbrace{0_{nw} \quad \cdots \quad 0_{nw}}_{m-3} \quad \Sigma_w \quad -\Sigma_w\right] \tag{A-28}$$

Therefore, the expression for Σ_{UX} is

$$\Sigma_{UX} = \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{U}}(k)x(k+m)^T\right] =$$
$$\begin{bmatrix} K\begin{bmatrix} 0_{nw} & \underbrace{0_{nw} \quad \cdots \quad 0_{nw}}_{m-3} & -\Sigma_w & 2\Sigma_w \end{bmatrix}\left(\bar{B}\bar{G}+\bar{F}_v\right)^T T_m{}^T \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{A-29}$$

e) The component relating noise and state $x_v(k+m)$ is similar to the previous,

$$\begin{aligned} &\mathbb{E}_{\mathfrak{X}(k)}\left[2x(k+m)^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Delta\bar{\bar{W}}(k)\right] = \\ &= 2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{W}}(k)x(k+m)^T\right]\right) \\ &= 2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WX}\right) \end{aligned} \tag{A-30}$$

The only non-zero component of Σ_{WX} is the first, which is equal to

$$\begin{aligned} &\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+m)x_v(k+m)^T\right] = \\ &= \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+m)\left(T_m\bar{X}(k)\right)^T\right] \\ &= \mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+m)\left(\bar{A}x(k)+\bar{B}\Delta\bar{U}(k)+\bar{F}\Delta\bar{W}(k)\right)^T T_m{}^T\right] \\ &= \mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m)\Delta\bar{W}^T\right]\left(\bar{B}\bar{G}+\bar{F}_v\right)^T T_m{}^T \\ &-\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m-1)\Delta\bar{W}^T\right]\left(\bar{B}\bar{G}+\bar{F}_v\right)^T T_m{}^T \end{aligned} \tag{A-31}$$

It is necessary to evaluate one new term,

$$\mathbb{E}_{\mathfrak{X}(k)}\left[w(k+m)\Delta\bar{W}^T\right] = \begin{bmatrix} -\mu^w w(k-1)^T & \underbrace{0_{nw} \quad \cdots \quad 0_{nw}}_{m-1} \end{bmatrix} \tag{A-32}$$

Therefore,

$$\Sigma_{WX} = \begin{bmatrix} \begin{bmatrix} 0 & \underbrace{0_{nw} \quad \cdots \quad 0_{nw}}_{m-2} & -\Sigma_w \end{bmatrix} \left(\bar{B}\bar{G} + \bar{F}_v\right)^T T_m{}^T \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{A-33}$$

f) The last term of $\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{\bar{X}}(k)\right\|^2_{\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}}\right]$ is

$$\begin{aligned} &\mathbb{E}_{\mathfrak{X}(k)}\left[2\Delta\bar{\bar{U}}(k)^T\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Delta\bar{\bar{W}}(k)\right] \\ &= 2tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{W}}(k)\Delta\bar{\bar{U}}(k)^T\right]\right) \\ &= 2tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WU}\right) \end{aligned} \tag{A-34}$$

This term is also a constant since it is a function only of future disturbances. Each control move $\Delta u(k+j)$ (with $j>m$) depends on $\Delta w(k+j-1)$, thus it is correlated only with both $\Delta w(k+j)$ and $\Delta w(k+j-1)$. It follows that the nonzero elements of Σ_{WU} are the principal diagonal and the superdiagonal:

$$\Sigma_{WU} = \begin{bmatrix} -\Sigma_w K^T & 2K\Sigma_w K^T & 0 & \cdots & 0 \\ 0 & -\Sigma_w K^T & 2\Sigma_w K^T & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & -\Sigma_w K^T & 2\Sigma_w K^T \end{bmatrix} \tag{A-35}$$

Putting together (A-18), (A-20), (A-23), (A-25), (A-30) and(A-34), the expression for T_4 is:

$$\mathbb{E}_{\mathfrak{X}(k)}\left[\left\|\bar{\bar{X}}(k)\right\|^2_{\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}}\right]=$$
$$tr\left(T_m{}^T\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{A}}T_m\Sigma_{\bar{X}}\right)+tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UU}\right)+tr\left(\bar{\bar{F}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\bar{\bar{\mu}}^{\Delta w\Delta w}\right) \tag{A-36}$$
$$+2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{B}}\Sigma_{UX}\right)+2tr\left(\bar{\bar{A}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WX}\right)+2tr\left(\bar{\bar{B}}^T\bar{\bar{C}}^T\bar{\bar{Q}}\bar{\bar{C}}\bar{\bar{F}}\Sigma_{WU}\right)$$

- T_5

This term is simpler than the preceding,

$$\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{\bar{X}}(k)\right]=\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{\bar{A}}x(k+m)+\bar{\bar{B}}\Delta\bar{\bar{U}}(k)+\bar{\bar{F}}\bar{\bar{W}}(k)\right]$$
$$=\bar{\bar{A}}T_m\mathbb{E}_{\mathfrak{X}(k)}\left[\bar{X}(k)\right]+\bar{\bar{B}}\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta\bar{\bar{U}}(k)\right]+0$$
$$=\bar{\bar{A}}T_m\left(\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)+\bar{\bar{B}}\begin{bmatrix}K\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+m-1)\right]\\ \vdots\\ K\mathbb{E}_{\mathfrak{X}(k)}\left[\Delta w(k+p-1)\right]\end{bmatrix} \tag{A-37}$$
$$=\bar{\bar{A}}T_m\left(\bar{A}x_v(k)+\bar{B}\left(\bar{G}\mu^{\Delta w}(k)+\bar{D}\right)+\bar{F}\mu^{\Delta w}(k)\right)$$