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SÃO CARLOS SCHOOL OF ENGINEERING**

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**Nonlinear supersonic flutter analysis of reinforced
laminated curved panels**

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**Nonlinear supersonic flutter analysis of reinforced
laminated curved panels**

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“Then you will know the truth, and the truth will set you free.”
John 8:32.

ABSTRACT

CABRAL, M. V. **Nonlinear supersonic flutter analysis of reinforced laminated curved panels**. 2021. 109p. MSc Dissertation - São Carlos School of Engineering, University of São Paulo, São Carlos, 2021.

The self-excited aeroelastic instability of thin plates or shells in the supersonic regime is called panel flutter, which may cause severe structural failure in aircraft and spacecraft. Thus, a reliable modeling of such phenomenon is crucial for safely predicting the lifespan of aircraft skins. Indeed, aeronautical skins are typically composed of large internally reinforced panels. The presence of the stiffening components subdivides the panel into several cells, which may interact structurally. However, the ample majority of published studies concerning the aeroelastic behavior of shells and plates treats each skin panel as an isolated structure. In this context, the present research project aims to investigate the effects of the structural coupling between the multiple curved panels and to assess the inaccuracies of the single-panel model by systematically comparing its results with those from the current multibay model. Moreover, the curvature effects will also be investigated. To do that, the Mindlin shallow shell theory coupled with the nonlinear von Kármán strains was applied. For the aerodynamic model, the first-order piston theory is applied, which is suitable for high-supersonic flows. The energy equations are discretized through the Finite Element Method, and the resulting aeroelastic equations of motion are solved in the time domain by using a Newmark integration method. The final algorithm is verified and validated through comparison with numerical and analytical solutions from the literature and with the commercial finite element software, ABAQUS. Symmetric and asymmetric composite laminated double cylindrical panels with different curvature ratios were analyzed for the streamwise and cross-stream configurations concerning the supersonic flow direction. Therefore, the effect of curvature, flow direction, and lamination scheme in the pre- and post flutter behavior of curved cylindrical panels were investigated.

Keywords: Aeroelasticity, Multibay panel flutter, Shallow shells, Finite element method.

RESUMO

CABRAL, M. V. **Análise não linear do flutter supersônico de painéis curvos laminados reforçados**. 2021. 109p. Dissertação (Mestrado) - Escola de Engenharia de São Carlos, Universidade de São Paulo, São Carlos, 2021.

A instabilidade aeroelástica autoexcitada de placas ou cascas finas no regime supersônico é denominada flutter de painel e é capaz de causar falhas estruturais graves em aeronaves e veículos espaciais. Desse modo, para a previsão segura da vida em fadiga de revestimentos aeroespaciais, é fundamental que sejam realizadas análises confiáveis deste fenômeno. De fato, revestimentos aeronáuticos são tipicamente compostos por grandes painéis reforçados internamente. A presença desses elementos acaba subdividindo o painel em várias células menores capazes de interagir estruturalmente. No entanto, a grande maioria dos trabalhos publicados a respeito do comportamento aeroelástico de cascas e placas trata cada painel como uma estrutura isolada. Nesse contexto, o presente projeto de pesquisa tem como objetivo investigar os efeitos do acoplamento estrutural entre múltiplos painéis curvos e avaliar as imprecisões do modelo de painel isolado através de uma análise comparativa sistemática dos resultados de painéis isolados com o do presente modelo multicélula. Além disso, os efeitos de curvatura também serão investigados. Para tanto, a teoria de Mindlin para cascas rasas será utilizada em conjunto com as deformações não lineares de von Kármán. Para o modelo aerodinâmico, foi utilizada a teoria de primeira ordem do pistão, adequada para escoamentos supersônicos. As equações de energia são discretizadas através do método dos elementos finitos, obtendo-se, então, as equações aeroelásticas do movimento. Essas equações são resolvidas no domínio do tempo através de um método de integração de Newmark. O código final é verificado e validado por meio da comparação com soluções numéricas e analíticas da literatura e com o *software* comercial de elementos finitos, ABAQUS. Painéis cilíndricos de compósito com laminação simétrica e assimétrica com diferentes razões de curvatura foram analisados para as configurações *streamwise* e *cross-stream* referentes à direção do escoamento supersônico. Portanto, o efeito da curvatura, da direção do fluido e do esquema de laminação no comportamento anterior e posterior ao *flutter* de painéis curvos cilíndricos foram investigados.

Palavras-chave: Aeroelasticidade, *Flutter* de múltiplos painéis, Cascas rasas, Método dos elementos finitos.

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LIST OF ABBREVIATIONS AND ACRONYMS

CFD	Computational Fluid Dynamics
DOF	Degree of freedom
FEM	Finite Element Method
LCO	Limit cycle oscillations
ODE	Ordinary differential equation
PVW	Principle of Virtual Work

LIST OF SYMBOLS

$\mathbf{A}$	Plate in-plane (membrane) matrix
A_e	Elemental domain
a	Plate length
$\mathbf{B}$	Plate coupling (bending-membrane) matrix
$\mathbf{B}_m, \mathbf{B}_\theta$	Shape function derivative matrices
$\mathbf{B}_c$	Shape function matrix
$\mathbf{C}$	Plate shear (transverse) matrix
$\bar{\mathbf{C}}$	Uncorrected shear matrix
$\mathbf{c}_a$	Normalized elemental aerodynamic damping matrix
D	Reference flexural stiffness
$\mathbf{D}$	Plate bending matrix
d	Vertical distance between the center of the cylinder and a horizontal axis
$\mathbf{G}$	Global damping matrix
$\mathbf{H}$	Nonlinear stiffness matrix
H	The highest point of the curvature (midplane initial deformation) in relation to a horizontal axis
h	Panel thickness
H_i	Panel shape function related to the i -th elemental node
$\mathbf{H}_b, \mathbf{H}_u, \mathbf{H}_v$	Plate shape function vectors
$\hat{\mathbf{K}}$	Modified stiffness matrix
$\mathbf{K}_0$	Global zeroth-order stiffness matrix
$\mathbf{k}_0$	Elemental zeroth-order stiffness matrix
$\mathbf{K}_1$	Global first-order stiffness matrix

$\mathbf{k}_1$	Elemental first-order stiffness matrix
k_1, k_2	Plate shear correction factors
$\mathbf{K}_2$	Global second-order stiffness matrix
$\mathbf{k}_2$	Elemental second-order stiffness matrix
$\mathbf{k}_a$	Normalized elemental aerodynamic stiffness matrix
$\mathbf{k}_A$	Elemental in-plane stiffness matrix
$\mathbf{k}_B$	Elemental coupling stiffness matrix
$\mathbf{k}_C$	Elemental shear stiffness matrix
$\mathbf{k}_D$	Elemental bending stiffness matrix
M	Mach number
$\mathbf{m}$	Elemental mass matrix
$\mathbf{m}_x, \mathbf{m}_y, \mathbf{m}_z$	Auxiliary mass matrices
$\hat{\mathbf{P}}$	Modified force vector
p	Arbitrary external pressure (step load)
$\mathbf{P}_0$	Static aerodynamic load vector
p_0	Static aerodynamic load
q	Free-stream dynamic pressure
$\bar{Q}_p$	Plate in-plane constitutive matrix concerning the (x, y, z) system
$\bar{Q}_s$	Plate shear constitutive matrix concerning the (x, y, z) system
u	Plate displacement in the x direction
$\mathbf{u}$	Elemental DOF vector
$\mathbf{u}_b$	Elemental transverse DOF vector
$\mathbf{u}_m$	Elemental membrane DOF vector
$\mathbf{u}_r$	Elemental bending DOF vector
$\mathbf{U}$	Global DOF vector
$\hat{u}, \hat{v}, \hat{w}_1$	Midsurface displacements

v	Plate displacement in the y direction
w	Plate displacement in the z direction
$\hat{w}_0$	Midsurface initial deformation
α	Flow incidence angle with respect to the x -axis
γ_{xy}, γ_{xz}	Shear strains
δ	Variational operator
δW	Generic virtual work
δW_{int}	Internal work
δW_{ext}	External work
Δp	Pressure difference (lower-upper surface)
Δt	Time step
$\Delta x, \Delta y$	Element sizes
$\overline{\Delta x}$	Minimum element size
ε_p	Plate in-plane (membrane) strain
ε_s	Plate shear strain
ε_m	Membrane strain vector
ε_θ	Nonlinear strain vector
ε_{w_0}	Curvature strain vector
θ	Layer-wise fiber orientation angle
Θ_0	Plate slope matrix due to the initial deformation
Θ	Plate slope matrix due to the virtual deformation
κ	Bending strain vector
λ	Dimensionless dynamic pressure
λ_f	Critical dimensionless dynamic pressure
λ_{jump}	Dimensionless dynamic pressure at jump discontinuity occurrence
μ	Flow-panel mass ratio

ν	Poisson's ratio
ρ	Plate density
ρ_∞	Free-stream flow density
ϕ_x, ϕ_y	Midsurface normal rotations
ω_0	Reference frequency

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1 INTRODUCTION

Aeroelastic phenomena have been encountered since the beginning of aviation; however, after the Second World War, with the emergence of supersonic vehicles, new aeroelastic problems emerged, such as the panel flutter, which can cause structural failures in aircraft panels and aerospace launch vehicles. According to Dowel (1970), several parameters associated with this phenomenon are also important for other physical problems. For example, any variation in panel geometry or loading, which is significant for panel flutter, will also be important for noise response and sonic fatigue. This includes pressurization, buckling under thermal stress, and even the aerodynamic loading due to structural motion (DOWEL, 1970). Therefore, panel flutter is a phenomenon still intensively investigated and of considerable practical interest.

Panel flutter is a self-excited aeroelastic instability of thin plates or shells that occurs more frequently, but not exclusively, in supersonic flight (HESS; GIBSON, 1962). Its occurrence has been investigated in both flat and curved panels. In the study of curved panels, the following cases receive special attention: 1) Curvature perpendicular to the flow, namely cross-stream curvature, and 2) Curvature in the flow direction, namely streamwise curvature.

Unlike flat panels, curved panels can also possess aerodynamic static loads capable of significantly change the frequency range of the flutter occurrence. Therefore, it is necessary to include this effect in the aeroelastic system. Azzouz (2005) performed a linear and nonlinear analysis of isotropic cylindrical panels, using 2D and 3D panel models. He used the Marguerre curved plate theory to develop the curved panel element and the first-order piston theory to account for the aerodynamic loads. The final reduced nonlinear finite element multi-modal equations account for pre-flutter and flutter behavior with curvature effect. His analysis showed globally that curvature has a detrimental effect on the flutter boundaries for 2D panels, but it is beneficial for 3D panels at an optimum height-rise. He also study the influence of the yaw flow angle for 3D panels and showed that yaw flow angle are beneficial for the flutter boundaries for certain curvature values. More recently, Ghoman and Azzouz (2012) and Azzouz (2009) analyzed the supersonic flutter of composite streamwise curved single-panel using the Marguerre theory to model shallow shells and the first-order piston theory to describe the aerodynamic behavior. However, this analysis did not include the study of limit cycle oscillations. Zhou et al. (2018) also used first-order piston theory and the finite element method to analyze the cylindrical curved plate post-flutter behavior. They analyzed the nonlinear response of curved composite plates with embedded microfiber composite actuators in supersonic airflow.

Voss (1961) and Dowell (1969) were pioneers in analyzing curved panels subject to the cross-stream flow direction. They were followed by other authors, such as Matsuzaki (1973) and Bismarck-Nasr (1993). Matsuzaki (1973) used the linear theory to examine the influences of the in-plane boundary conditions and the panel geometry on natural vibration and flutter characteristics of cylindrically curved, simply supported panels. He concluded that the flutter characteristics of cross-stream panels are strongly affected by the in-plane boundary conditions and the panel geometry. Most of the available literature uses linear theory, not contemplating the post-flutter behavior of cross-stream curved panels subjected to supersonic flow.

According to Mei et al. (1999a), the early studies in nonlinear panel flutter using classical methods have been limited to two or three-dimensional rectangular plates with all four edges simply supported or clamped. Later, Olson (1967) used the Finite Element Method (FEM) to study the linear panel flutter problem. Subsequently, Yang and Sung (1977), Bismarck-Nasr (1992), and many other authors also applied the finite element method to study the supersonic limit cycle oscillations (LCO) in post-flutter regime. Among the advantages of using the finite element method is its more versatile applicability, since it includes easily and conveniently effects as aerodynamic damping, flow angularities, in-plane stress, thermal loads, and complex panel configurations and support conditions. The main disadvantage of use FEM to perform this nonlinear aeroelastic analysis is the high computational cost when compared with linearized updated mode approximations or modal decomposition techniques.

With respect to the aerodynamic model, the oldest and most popular approach is the piston theory. As the aerodynamic load risen from piston theory depends on the panel displacements, the corresponding work terms may be evaluated in the form of an aerodynamic damping matrix and as contributions to the stiffness matrix. Accordingly, piston theory configures one simple model derived from supersonic potential flow theory. For shells, there is also a load term corresponding to the aerodynamic static load, which depends on the shell curvature and the dynamic pressure. Bein et al. (1993) compares the results for the hypersonic shallow shell panel flutter obtained from an Euler solver with those attained through first-, second-, and third-order piston theories. He showed that the piston theory results are in close agreement with those from the Euler equations solution. More recently, Alder (2015) compared the quasi-steady piston theory and the linearized potential flow theory with the Euler solution and showed that the results approach each other for Mach numbers of 1.6 and larger.

Aeronautical skins are typically composed of large internally reinforced panels with multiple stringers, ribs, longerons and frames, as illustrated in Figure 1. These reinforcement elements end up dividing the panel in several cells that, during the flutter, can interact structurally and make the aeroelastic movement more complex and potentially

dangerous. Most of the works concerning the aeroelastic analysis of multiple cells are restricted to the linear flutter analysis, beginning with Dowell (1964) and Rodden (1964). In their approach, also known as multibay analysis, the stiffeners were modelled as simple supports positioned between the subpanels. At that time, limited by the restrictions of linear modelling, the general conclusion reached by authors was that the critical flutter speeds of panels with multiple cells are very similar to those of single-bay panels. The following studies concerning the multibay panel flutter remained restricted to the linear analysis (LOCK; FARKAS, 1965; MUKHERJEE; PARTHAN, 1995; SHIAU; CHANG, 1991; PANY; PARTHAN, 2003). Liao and Sun (1993) performed an investigation of the flutter instability of laminated composite flat and curved plates submitted to supersonic flow. The effects of several parameters were analyzed, such as the angle of fiber slope, the lamination scheme, and the reinforcers position in the occurrence of the flutter in reinforced and non-reinforced plates and shells. Castro et al. (2016) analyzed the linear aeroelastic behavior of curved panels using semi-analytical modeling, and modeled the base of the reinforcers as a structural element. In most of the works, the analysis focused on the effects of stiffener addition on the panel's aeroelastic behavior, making a comparison between the original panel (without reinforcers) and the same panel with one or more reinforcers. However, recently, Pacheco et al. (2017c) presented a nonlinear panel flutter analysis considering the effects of the structural coupling between the multiple cells in a panel, rather than to quantifying the additional reinforcement brought by stiffeners. Each cell of the reinforced flat panel was compared with an individual panel of the same dimensions. It was found that the flutter mechanism is substantially influenced by the nonlinear coupling between adjacent panels, leading to jumps and bifurcations in the amplitude of the limit cycle oscillation diagrams. His work was followed by Guimarães et al. (2020), who used the Rayleigh-Ritz method and gave particular attention to the aeroelastic comparison on the post-flutter behavior between conventional and tow steered laminates for multibay flat panels.

More recently, Fernandes and Tamijani (2017) devised an aeroelastic model with curvilinear stiffeners and showed that such unconventional geometric features could be explored for efficient flutter suppression. He used the first-order piston theory approximation to the linear potential flow theory to perform a linear flutter analysis of curvilinear-stiffened plates subjected to supersonic flow. Moreover, Vescovini et al. (2020) performed a novel semi-analytical strategy for the analysis of stiffened panels characterized by variable-stiffness skin and curvilinear stringers. The approach relies on the combined use of first-order theories for the plate and beam elements, along with the Ritz method for seeking an approximate solution. The formulation has been developed for several solution procedures, including static, free-vibration, buckling, and thermal-buckling analyses.

Concerning the nonlinear aeroelastic analysis of multibay panels, recently Cabral et al. (2021a) investigated the influence of curvature in the nonlinear supersonic flutter of

Figure 1 – Typical fuselage skin of aerospace vehicles.

Source: <https://www.wingsoverquebec.com/?p=8356>
<http://www.aerospacengineering.net>

isotropic multibay panels. They analyzed a two-bay cylindrically curved panel subjected to streamwise flow, showing that the nonlinear coupling between adjacent bays under supersonic flutter leads to higher displacements and stronger sensitivity to initial conditions. However, the nonlinear aeroelastic behavior of multibay curved panels remains scarce in the literature. A better understanding of this phenomenon would be achieved through analysis of the influence of several parameters such as: different materials, flow direction, boundary conditions, and geometry.

Therefore, this research project aims to study the effects of the structural coupling between the multiple curved panels to fill an existing gap in the current literature. The influence of the flow direction, lamination scheme and curvature ratio will be accounted. The external loads correspond to the aerodynamic loads coming from the supersonic flow, which are composed by the auto excitation arisen from the nonstationary aerodynamic and the static loads due to the panel's curvature. For the structural model, the nonlinear Mindlin shallow shell formulation is applied to describe the behavior of the cylindrically curved panel coupled with the nonlinear von Kármán strain-displacement relations. Consequently, the structural model is able to account for nonlinearities due to large deflections of thin and moderately thick shallow shells. Furthermore, the present model considers either isotropic and composite laminated materials, which have become a trending due to its high stiffness and low weight. For the aerodynamic model, the first-order piston theory is applied, which is suitable for high-supersonic flows. The energy equations are discretized through the Finite Element Method, and the resulting aeroelastic equations of motion are solved in the time domain by using a Newmark integration method. The stiffener is modeled as simply supporters that allow rotation between the bays. The streamwise and the cross-stream flow directions are analyzed, which refer to the flow direction aligned and perpendicular to curvature, respectively.

1.1 Objectives

The emergence of supersonic vehicles brought new challenges to the design of aerospace structures. In this context, the panel flutter has been getting attention since aerospace vehicles' structural skins are composed of panels that can undergo large-amplitude, high-frequency aeroelastic oscillations during supersonic flight. Consequently, these structures can suffer fatigue failures and lead aircraft or spacial vehicles to collapse. The importance of a better representation of this phenomenon is increased with the development of commercial supersonic aircraft, for instance, the X-59 QueSST that Lockheed Martin and NASA are currently developing. However, the multibay analysis of panels subjected to supersonic flutter remains scarce, even though the external skins of high-speed flight aerospace structures are composed of multiple adjacent curved panels that can interact. Therefore, the present research project proposes an aeroelastic model to analyze the pre- and post-flutter behavior of multibay isotropic and composite laminated shallow shells, since most of the panels in an aerospace vehicle are actually curved. The main goal is to study how the flutter mechanism is influenced by the nonlinear coupling between adjacent panels, which is not found in the current literature in respect to curved panels. A deep comprehension of the nonlinear dynamics concerning the panel structural performances will contribute to supersonic vehicles' design. Along these lines, the present research project proposes the implementation and validation of a finite element aeroelastic model for reinforced curved panels. The intent it is to bring some improvement in the modelling by considering the structural interaction between adjacent bays. The cross-stream two-bay model is compared with the single panel arrangement to identify the similarities and discrepancies between these approaches. Moreover, the influence of the lamination scheme, curvature ratio, and flow direction in the aeroelastic behavior of cylindrical multibay curved panels will be investigated.

1.2 Dissertation structure

Besides this introduction, this work is divided into five more chapters. Chapter 2 presents the aeroelastic model, comprising the structural model, aerodynamic model, the finite element discretization, the final aeroelastic equations of motion and the solution methods. Chapter 3 presents the model validation, which were performed in multiple levels: linear analysis (vertical deflections and fundamental frequencies), structural nonlinear plate vibration, flutter boundary prediction and nonlinear aeroelastic analysis. The results were verified through comparison with the literature and with the commercial finite element software ABAQUS. Chapter 4 presents and discusses the aeroelastic analysis concerning streamwise and cross-stream isotropic multibay panels, focusing into examine the nonlinear dynamics. Chapter 5 presents a more extensive study on composite laminated panels, investigating the influence of the lamination scheme, curvature ratio and flow direction

in the aeroelastic behavior of curved panels. It also conducts a systematic comparative analysis in which the aeroelastic response of cross-stream coupled panels is confronted with the equivalent single panel arrangement. Chapter 6 draws some conclusions, summarizes the findings and highlights of this work, and also brings suggestions for future works.

2 AEROELASTIC MODEL

This Chapter is divided into four sections. First, the nonlinear Mindlin shallow shell formulation is presented to build the structural model, and the Finite Element Method formulation is applied to extract the equations of motion from the energy equations. A brief section explains how the geometry of the shallow shell is defined. The next section describes the aerodynamic model, which then is combined with the structural model through the application of the Principle of Virtual Work (PVW), resulting in the aeroelastic model. Finally, the numerical schemes used for solving the aeroelastic equations are addressed.

2.1 The nonlinear Mindlin shallow shell formulation

To analyze the behavior of shallow shells, the Mindlin formulation is coupled with the Marguerre shallow shell theory. The Mindlin kinematic theory is an evolution of the classical plate theory that considers transverse shear deformability. Unlike the Kirchhoff formulations, the Mindlin assumptions do not require that normals to the midsurface remain normal after deformation. This enables the kinematics modeling of moderately thick panels. The displacement field at a point (x, y, z) at a time t is (PICA; WOOD, 1980):

$$\begin{aligned} u(x, y, z, t) &= \hat{u}(x, y, t) - z\phi_x(x, y, t), \\ v(x, y, z, t) &= \hat{v}(x, y, t) - z\phi_y(x, y, t), \\ w(x, y, z, t) &= \hat{w}_0(x, y) + \hat{w}_1(x, y, t), \end{aligned} \tag{2.1}$$

in which u , v , w are the displacements in the x , y , z directions, respectively. They are expressed as functions of the midplane translations, $\hat{u}$, $\hat{v}$, $\hat{w}_1$; the midplane initial deformation related to the panel curvature, $\hat{w}_0$; and independent normal rotations ϕ_x and ϕ_y . The quantities ϕ_x and ϕ_y are the rotation of the normals around the y and x axes, respectively, independent of the translational displacements. The coupling between Mindlin shallow shell and the von Kármán nonlinear strain-displacement relations allows the resulting structural model to account for geometrical non-linearities arising from large-displacement motion. For Mindlin shallow shell, the relevant midplane Green's strain

vector is revised to give (PICA; WOOD, 1980):

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \end{Bmatrix} - \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial \hat{w}_0}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial \hat{w}_0}{\partial y}\right)^2 \\ \frac{\partial \hat{w}_0}{\partial x}\frac{\partial \hat{w}_0}{\partial y} \\ \frac{\partial \hat{w}_0}{\partial x} + \frac{\partial \hat{w}_0}{\partial x}\frac{\partial \hat{w}_0}{\partial z} \\ \frac{\partial \hat{w}_0}{\partial y} + \frac{\partial \hat{w}_0}{\partial y}\frac{\partial \hat{w}_0}{\partial z} \end{Bmatrix}. \quad (2.2)$$

Introducing the von Kármán assumptions, which imply that derivatives of u and v with respect to x , y , z are small, and noting that w is independent of z , allows Green's strain to be rewritten in terms of the midplane deformations of Eq. (2.1) as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \hat{u}}{\partial x} \\ \frac{\partial \hat{v}}{\partial y} \\ \frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x} \\ \frac{\partial \hat{w}_1}{\partial x} - \phi_x \\ \frac{\partial \hat{w}_1}{\partial y} - \phi_y \end{Bmatrix} + z \begin{Bmatrix} -\frac{\partial \phi_x}{\partial x} \\ -\frac{\partial \phi_y}{\partial y} \\ -\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial \hat{w}_1}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial \hat{w}_1}{\partial y}\right)^2 \\ \frac{\partial \hat{w}_1}{\partial x}\frac{\partial \hat{w}_1}{\partial y} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \frac{\partial \hat{w}_0}{\partial x}\frac{\partial \hat{w}_1}{\partial x} \\ \frac{\partial \hat{w}_0}{\partial y}\frac{\partial \hat{w}_1}{\partial y} \\ \frac{\partial \hat{w}_0}{\partial x}\frac{\partial \hat{w}_1}{\partial y} + \frac{\partial \hat{w}_1}{\partial x}\frac{\partial \hat{w}_0}{\partial y} \\ 0 \\ 0 \end{Bmatrix}. \quad (2.3)$$

The strains of Eq. (2.3) can be separated into in-plane strain, $\{\varepsilon_p\}$, and transverse shear strain, $\{\varepsilon_s\}$:

$$\{\varepsilon_p\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \{\varepsilon_m\} + z\{\kappa\} + \{\varepsilon_\theta\} + \{\varepsilon_{w_0}\}, \quad (2.4)$$

and

$$\{\varepsilon_s\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \hat{w}_1}{\partial x} - \phi_x \\ \frac{\partial \hat{w}_1}{\partial y} - \phi_y \end{Bmatrix}, \quad (2.5)$$

where $\{\varepsilon_m\}$ is the membrane strain vector, $\{\kappa\}$ is the bending strain vector, $\{\varepsilon_\theta\}$ is the nonlinear strain vector, and $\{\varepsilon_{w_0}\}$ is the curvature strain vector. The in-plane and

transverse stresses are

$$\begin{aligned}\{\sigma_p\} &= [\bar{Q}_p] \{\varepsilon_p\}, \\ \{\sigma_s\} &= [\bar{Q}_s] \{\varepsilon_s\},\end{aligned}\tag{2.6}$$

in which $[\bar{Q}_p]$ and $[\bar{Q}_s]$ are the constitutive matrices concerning the (x, y, z) system. These matrices are obtained through rotation from the local system within each ply (1,2,3) to the global (x, y, z) system. The fiber orientation angle, θ , for a generic ply is shown in Figure 2. Details about the computation of the matrices $[\bar{Q}_p]$ and $[\bar{Q}_s]$ may be found in Reddy (2003).

Figure 2 – A laminate made up of laminae with different fiber orientations.

Source: Elaborated by the author.

The stresses and strains are combined by the Principle of Virtual Work, which states an equality between the virtual work done by internal and external forces. The internal work is computed as the strain energy variation caused by the stress field acting over a virtual strain field, with the variational operator, δ , indicating virtual quantities. After substitution of the constitutive equations, the virtual strain energy becomes:

$$\delta W_{int} = \int_{V_p} \left(\delta \varepsilon_p^T \bar{Q}_p \varepsilon_p + \delta \varepsilon_s^T \bar{Q}_s \varepsilon_s \right) dV.\tag{2.7}$$

The differential, dV , can be factored into a thickness differential and a surface differential. Furthermore, the in-plane strain can be split into $\varepsilon_0 + z\kappa$, in which $\varepsilon_0 = \varepsilon_m + \varepsilon_\theta + \varepsilon_{w_0}$. Then, the internal work can be written as:

$$\delta W_{int} = \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} [(\delta\varepsilon_0 + z\delta\kappa)^T \bar{Q}_p(\varepsilon_0 + z\kappa) + \delta\varepsilon_s^T \bar{Q}_s \varepsilon_s] dz dA. \quad (2.8)$$

where h is the shell's thickness. Through the products expansion and integration along the thickness, Eq. (2.8) becomes:

$$\delta W_{int} = \int_A (\delta\varepsilon_0^T \mathbf{A}\varepsilon_0 + \delta\varepsilon_0^T \mathbf{B}\kappa + \delta\kappa^T \mathbf{B}\varepsilon_0 + \delta\kappa^T \mathbf{D}\kappa + \delta\varepsilon_s^T \mathbf{C}\varepsilon_s) dA, \quad (2.9)$$

in which

$$(\mathbf{A}, \mathbf{B}, \mathbf{D}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, z^2) \bar{Q}_p dz, \quad (2.10)$$

and $\mathbf{C}$ is a function of $\bar{\mathbf{C}}$:

$$\bar{\mathbf{C}} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_s dz. \quad (2.11)$$

The $\mathbf{C}$ matrix corresponds to the $\bar{\mathbf{C}}$ matrix corrected for nonuniform shear. This correction is necessary because the Mindlin plate model accounts for shear deformability, but introduces an unrealistic kinematic consideration by assuming shear strains to be constant along the thickness. In fact, the transverse shear strains normally present a near-parabolic distribution, being zero at the borders, $z = \pm \frac{h}{2}$, and assuming maximum value in between. Therefore, the shear correction factor is applied to compensate such discrepancy between the method assumptions and real distributions. For isotropic panels, the shear correction factor usually assumed is 5/6, while for composite panels they depend on the lamination scheme (WHITNEY, 1972). The calculation of the shear correction factor for the composite panels analyzed in this work follows the methodology addressed by Whitney (1972).

The external work has a contribution coming from an external load field and another one coming from the inertial forces due to the plate's acceleration field:

$$\delta W_{ext} = \int_A \delta w \Delta p dA - \int_V \rho \left(\delta u \frac{\partial^2 u}{\partial t^2} + \delta v \frac{\partial^2 v}{\partial t^2} + \delta w \frac{\partial^2 w}{\partial t^2} \right) dV, \quad (2.12)$$

in which ρ is the material density. For now, the pressure field, Δp , is assumed to be arbitrary independent upon the displacement field. Later, in section 2.3, the aerodynamic pressure shall be calculated as a function of the displacement field.

Upon substitution of Eq. (2.1) into Eq. (2.12), and knowing that $\delta w = \delta \hat{w}_1 + \delta \hat{w}_0 \Rightarrow \delta w = \delta \hat{w}_1$, since $\hat{w}_0$ is the initial displacement that characterizes the shallow shell curvature, the expression of external work, after integration through the thickness, become:

$$\delta W_{ext} = \int_A \delta \hat{w}_1 \Delta p dA - \int_A \rho \left[h \left(\delta \hat{u} \frac{\partial^2 \hat{u}}{\partial t^2} + \delta \hat{v} \frac{\partial^2 \hat{v}}{\partial t^2} + \delta \hat{w}_1 \frac{\partial^2 \hat{w}_1}{\partial t^2} \right) + \frac{h^3}{12} \left(\delta \phi_x \frac{\partial^2 \phi_x}{\partial t^2} + \delta \phi_y \frac{\partial^2 \phi_y}{\partial t^2} \right) \right] dA. \quad (2.13)$$

To find the complete aeroelastic equations of motion from the energy equations, the PVW, which states: $\delta W_{int} = \delta W_{ext}$, is applied in association with the Finite Element Method. The element chosen to compose the mesh was the four-node lagrangian quadrilateral element, which has five degrees of freedom (DOF) per node: $\hat{u}$, $\hat{v}$, $\hat{w}_1$, ϕ_x , ϕ_y . They are grouped in membrane DOFs: $\hat{u}$, $\hat{v}$; transverse DOF: $\hat{w}_1$; and rotation DOFs: ϕ_x , ϕ_y . Therefore, for convenience, the elemental DOF vector becomes:

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_m \\ \mathbf{u}_b \\ \mathbf{u}_r \end{Bmatrix}, \quad (2.14)$$

with

$$\mathbf{u}_m = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}, \mathbf{u}_b = \begin{Bmatrix} w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{Bmatrix}, \mathbf{u}_r = \begin{Bmatrix} \phi_{x1} \\ \phi_{y1} \\ \phi_{x2} \\ \phi_{y2} \\ \phi_{x3} \\ \phi_{y3} \\ \phi_{x4} \\ \phi_{y4} \end{Bmatrix}, \quad (2.15)$$

in which the indices 1 to 4 refer to the elemental nodes. The node numbering adopted for the finite element is the counterclockwise direction, with node 1 brought to the origin, which is a standard procedure to simplify explicit shape function calculation. Each node has an associated shape function, whose properties are:

$$H_i(x_j, y_j) = \begin{cases} 1, & \text{if } j = i; \\ 0, & \text{if } j \neq i. \end{cases}$$

The expression above provides four equations for each shape function where $j = 1, 2, 3, 4$. The bilinear functions are generated through substitution of this expression above

in the general bilinear polynomial: $H_i(x, y) = a_i x + b_i y + c_i xy + d_i$, which gives:

$$H_1(x, y) = 1 - \frac{x}{\Delta x} - \frac{y}{\Delta y} + \frac{xy}{\Delta x \Delta y}, \quad (2.16)$$

$$H_2(x, y) = \frac{x}{\Delta x} - \frac{xy}{\Delta x \Delta y}, \quad (2.17)$$

$$H_3(x, y) = \frac{xy}{\Delta x \Delta y}, \quad (2.18)$$

$$H_4(x, y) = \frac{y}{\Delta y} - \frac{xy}{\Delta x \Delta y}, \quad (2.19)$$

in which Δx and Δy are the element sizes in x and y axes directions, respectively. It is possible to approximate any displacement in terms of shape functions and the respective nodal values. For instance, the transverse displacement can be interpolated as

$$\hat{w}_1(x, y, t) = \sum_{n=1}^4 H_i(x, y) \hat{w}_{1i}(t). \quad (2.20)$$

Similarly, the midplane initial deformation, $\hat{w}_0$, is interpolated as:

$$\hat{w}_0(x, y) = \sum_{n=1}^4 H_i(x, y) \hat{w}_{0i}. \quad (2.21)$$

It is worth mentioning that the initial deformation does not vary with time. Hence, Eq. (2.21) can be rearranged as:

$$\hat{w}_0 = \mathbf{H}_b^T \mathbf{u}_0, \quad (2.22)$$

where $\mathbf{H}_b = \{H_1 \ H_2 \ H_3 \ H_4\}^T$, and $\mathbf{u}_0$ is the vector of initial transverse vertical deformation due to the initial panel curvature, which is written as $\mathbf{u}_0 = \{w_{01} \ w_{02} \ w_{03} \ w_{04}\}^T$ for an element. Similarly, Eq. (2.20) can be written as:

$$\hat{w}_1 = \mathbf{H}_b^T \mathbf{u}_b. \quad (2.23)$$

The remaining DOFs may be written as

$$\hat{u} = \mathbf{H}_u^T \mathbf{u}_m, \quad \hat{v} = \mathbf{H}_v^T \mathbf{u}_m, \quad \phi_x = \mathbf{H}_u^T \mathbf{u}_r, \quad \phi_y = \mathbf{H}_v^T \mathbf{u}_r, \quad (2.24)$$

in which $\mathbf{H}_u = \{H_1 \ 0 \ H_2 \ 0 \ H_3 \ 0 \ H_4 \ 0\}^T$ and $\mathbf{H}_v = \{0 \ H_1 \ 0 \ H_2 \ 0 \ H_3 \ 0 \ H_4\}^T$. Accordingly, each of the strain components can be written in terms of nodal values and shape functions, and introduces the resulting discretized strains into the PVW.

The membrane strain, $\{\varepsilon_m\}$, can be written in the discretized form as

$$\{\varepsilon_m\} = \begin{Bmatrix} \frac{\partial \hat{u}}{\partial x} \\ \frac{\partial \hat{v}}{\partial y} \\ \frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial(\mathbf{H}_u^T \mathbf{u}_m)}{\partial x} \\ \frac{\partial(\mathbf{H}_v^T \mathbf{u}_m)}{\partial y} \\ \frac{\partial(\mathbf{H}_u^T \mathbf{u}_m)}{\partial y} + \frac{\partial(\mathbf{H}_v^T \mathbf{u}_m)}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \mathbf{H}_u^T}{\partial x} \\ \frac{\partial \mathbf{H}_v^T}{\partial y} \\ \frac{\partial \mathbf{H}_u^T}{\partial y} + \frac{\partial \mathbf{H}_v^T}{\partial x} \end{Bmatrix} \mathbf{u}_m = \mathbf{B}_m \mathbf{u}_m, \quad (2.25)$$

in which

$$\mathbf{B}_m = \begin{bmatrix} \frac{\partial H_1}{\partial x} & 0 & \frac{\partial H_2}{\partial x} & 0 & \frac{\partial H_3}{\partial x} & 0 & \frac{\partial H_4}{\partial x} & 0 \\ 0 & \frac{\partial H_1}{\partial y} & 0 & \frac{\partial H_2}{\partial y} & 0 & \frac{\partial H_3}{\partial y} & 0 & \frac{\partial H_4}{\partial y} \\ \frac{\partial H_1}{\partial y} & \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial y} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial y} & \frac{\partial H_3}{\partial x} & \frac{\partial H_4}{\partial y} & \frac{\partial H_4}{\partial x} \end{bmatrix}. \quad (2.26)$$

Similarly, the bending vector, $\{\kappa\}$, becomes

$$\{\kappa\} = -\mathbf{B}_m \mathbf{u}_r. \quad (2.27)$$

The nonlinear strain, $\{\varepsilon_\theta\}$, can be obtained through the procedure:

$$\{\varepsilon_\theta\} = \begin{Bmatrix} \frac{1}{2} \left(\frac{\partial \hat{w}_1}{\partial x} \right)^2 \\ \frac{1}{2} \left(\frac{\partial \hat{w}_1}{\partial y} \right)^2 \\ \frac{\partial \hat{w}_1}{\partial x} \frac{\partial \hat{w}_1}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{\partial \hat{w}_1}{\partial x} & 0 \\ 0 & \frac{\partial \hat{w}_1}{\partial y} \\ \frac{\partial \hat{w}_1}{\partial y} & \frac{\partial \hat{w}_1}{\partial x} \end{bmatrix} \begin{Bmatrix} \frac{\partial \hat{w}_1}{\partial x} \\ \frac{\partial \hat{w}_1}{\partial y} \end{Bmatrix} = \frac{1}{2} \boldsymbol{\Theta}_1 \begin{Bmatrix} \frac{\partial(\mathbf{H}_b^T \mathbf{u}_b)}{\partial x} \\ \frac{\partial(\mathbf{H}_b^T \mathbf{u}_b)}{\partial y} \end{Bmatrix} = \frac{1}{2} \boldsymbol{\Theta}_1 \mathbf{B}_\theta \mathbf{u}_b, \quad (2.28)$$

in which

$$\mathbf{B}_\theta = \begin{bmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_2}{\partial x} & \frac{\partial H_3}{\partial x} & \frac{\partial H_4}{\partial x} \\ \frac{\partial H_1}{\partial y} & \frac{\partial H_2}{\partial y} & \frac{\partial H_3}{\partial y} & \frac{\partial H_4}{\partial y} \end{bmatrix}, \quad (2.29)$$

and

$$\boldsymbol{\Theta}_1 = \begin{bmatrix} \frac{\partial \hat{w}_1}{\partial x} & 0 \\ 0 & \frac{\partial \hat{w}_1}{\partial y} \\ \frac{\partial \hat{w}_1}{\partial y} & \frac{\partial \hat{w}_1}{\partial x} \end{bmatrix}. \quad (2.30)$$

Similarly, the curvature strain vector, $\{\varepsilon_{w_0}\}$, can be written as

$$\{\varepsilon_{w_0}\} = \begin{Bmatrix} \frac{\partial \hat{w}_0}{\partial x} \frac{\partial \hat{w}_1}{\partial x} \\ \frac{\partial \hat{w}_0}{\partial y} \frac{\partial \hat{w}_1}{\partial y} \\ \frac{\partial \hat{w}_0}{\partial x} \frac{\partial \hat{w}_1}{\partial y} + \frac{\partial \hat{w}_0}{\partial y} \frac{\partial \hat{w}_1}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \hat{w}_0}{\partial x} & 0 \\ 0 & \frac{\partial \hat{w}_0}{\partial y} \\ \frac{\partial \hat{w}_0}{\partial y} & \frac{\partial \hat{w}_0}{\partial x} \end{bmatrix} \begin{Bmatrix} \frac{\partial \hat{w}_1}{\partial x} \\ \frac{\partial \hat{w}_1}{\partial y} \end{Bmatrix} = \boldsymbol{\Theta}_0 \begin{Bmatrix} \frac{\partial(\mathbf{H}_b^T \mathbf{u}_b)}{\partial x} \\ \frac{\partial(\mathbf{H}_b^T \mathbf{u}_b)}{\partial y} \end{Bmatrix} = \boldsymbol{\Theta}_0 \mathbf{B}_\theta \mathbf{u}_b, \quad (2.31)$$

in which

$$\boldsymbol{\Theta}_0 = \begin{bmatrix} \frac{\partial \hat{w}_0}{\partial x} & 0 \\ 0 & \frac{\partial \hat{w}_0}{\partial y} \\ \frac{\partial \hat{w}_0}{\partial y} & \frac{\partial \hat{w}_0}{\partial x} \end{bmatrix}. \quad (2.32)$$

The transverse shear strain, $\{\varepsilon_s\}$, can be written as

$$\{\varepsilon_s\} = \begin{Bmatrix} \frac{\partial \hat{w}_1}{\partial x} \\ \frac{\partial \hat{w}_1}{\partial y} \end{Bmatrix} - \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} = \mathbf{B}_\theta \mathbf{u}_b - \mathbf{B}_c \mathbf{u}_r, \quad (2.33)$$

in which

$$\mathbf{B}_c = \begin{bmatrix} H_1 & 0 & H_2 & 0 & H_3 & 0 & H_4 & 0 \\ 0 & H_1 & 0 & H_2 & 0 & H_3 & 0 & H_4 \end{bmatrix}. \quad (2.34)$$

The virtual strains must also be discretized. For the linear terms, the results are:

$$\delta \varepsilon_m = \delta(\mathbf{B}_m \mathbf{u}_m) = \mathbf{B}_m \delta \mathbf{u}_m, \quad (2.35)$$

$$\delta \kappa = -\delta(\mathbf{B}_m \mathbf{u}_r) = -\mathbf{B}_m \delta \mathbf{u}_r, \quad (2.36)$$

$$\delta \varepsilon_{w_0} = \delta(\mathbf{\Theta}_0 \mathbf{B}_\theta \mathbf{u}_b) = \mathbf{\Theta}_0 \mathbf{B}_\theta \delta \mathbf{u}_b, \quad (2.37)$$

$$\delta \varepsilon_s = \delta(\mathbf{B}_\theta \mathbf{u}_b - \mathbf{B}_c \mathbf{u}_r) = \mathbf{B}_\theta \delta \mathbf{u}_b - \mathbf{B}_c \delta \mathbf{u}_r \quad (2.38)$$

It is important to note that the initial curvature, $\hat{w}_0$, does not vary with time, consequently the matrix $\mathbf{\Theta}_0$ (Eq. (2.32)) is constant during the time variation. The nonlinear component yields:

$$\delta \{\varepsilon_\theta\} = \begin{Bmatrix} \delta(\frac{1}{2}(\frac{\partial \hat{w}_1}{\partial x})^2) \\ \delta(\frac{1}{2}(\frac{\partial \hat{w}_1}{\partial y})^2) \\ \delta(\frac{\partial \hat{w}_1}{\partial x} \frac{\partial \hat{w}_1}{\partial y}) \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \hat{w}_1}{\partial x} \delta(\frac{\partial \hat{w}_1}{\partial x}) \\ \frac{\partial \hat{w}_1}{\partial y} \delta(\frac{\partial \hat{w}_1}{\partial y}) \\ \frac{\partial \hat{w}_1}{\partial y} \delta(\frac{\partial \hat{w}_1}{\partial x}) \frac{\partial \hat{w}_1}{\partial x} \delta(\frac{\partial \hat{w}_1}{\partial y}) \end{Bmatrix} = \begin{bmatrix} \frac{\partial \hat{w}_1}{\partial x} & 0 \\ 0 & \frac{\partial \hat{w}_1}{\partial y} \\ \frac{\partial \hat{w}_1}{\partial y} & \frac{\partial \hat{w}_1}{\partial x} \end{bmatrix} \delta \begin{Bmatrix} \frac{\partial \hat{w}_1}{\partial x} \\ \frac{\partial \hat{w}_1}{\partial y} \end{Bmatrix} = \mathbf{\Theta}_1 \mathbf{B}_\theta \delta \mathbf{u}_r. \quad (2.39)$$

Since all strain components and their virtual quantities have been written in discrete form, the next step is to determine the stiffness matrices. Accordingly, Eq. (2.9) must be discretized and written in terms of nodal degrees of freedom and their time derivatives. Unlike isotropic and symmetric laminates, the coupling stiffness, $\mathbf{B}$, is not zero for general asymmetric laminate. Each term of Eq. (2.9) will be treated separately and in the element domain. The first term comes from the midsurface strain and includes the contribution due to initial curvature:

$$\int_{A_e} \delta \varepsilon_0^T \mathbf{A} \varepsilon_0 dA, \quad (2.40)$$

with A_e being the elemental domain. The element matrices are calculated individually and are ultimately assembled in the form of global matrices and vectors.

After substitution of Eq. (2.25), (2.28), (2.31), (2.35), (2.37) and (2.39), Eq. (2.40) becomes:

$$\int_{A_e} \delta \varepsilon_0^T \mathbf{A} \varepsilon_0 \, dA = \int_{A_e} (\mathbf{B}_m \delta \mathbf{u}_m + \boldsymbol{\Theta}_1 \mathbf{B}_\theta \delta \mathbf{u}_b + \boldsymbol{\Theta}_0 \mathbf{B}_\theta \delta \mathbf{u}_b)^T \mathbf{A} (\mathbf{B}_m \mathbf{u}_m + \frac{1}{2} \boldsymbol{\Theta}_1 \mathbf{B}_\theta \mathbf{u}_b + \boldsymbol{\Theta}_0 \mathbf{B}_\theta \mathbf{u}_b) \, dA. \quad (2.41)$$

Multiplication of the terms of Eq. (2.41) yields:

$$\begin{aligned} \int_{A_e} \delta \varepsilon_0^T \mathbf{A} \varepsilon_0 \, dA = \int_{A_e} \left[\delta \mathbf{u}_m^T (\mathbf{B}_m^T \mathbf{A} \mathbf{B}_m) \mathbf{u}_m + \delta \mathbf{u}_m^T \left(\frac{1}{2} \mathbf{B}_m^T \mathbf{A} \boldsymbol{\Theta}_1 \mathbf{B}_\theta \right) \mathbf{u}_b + \right. \\ \delta \mathbf{u}_m^T (\mathbf{B}_m^T \mathbf{A} \boldsymbol{\Theta}_0 \mathbf{B}_\theta) \mathbf{u}_b + \delta \mathbf{u}_b^T (\mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{A} \mathbf{B}_m) \mathbf{u}_m + \delta \mathbf{u}_b^T \left(\frac{1}{2} \mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{A} \boldsymbol{\Theta}_1 \mathbf{B}_\theta \right) \mathbf{u}_b + \\ \delta \mathbf{u}_b^T (\mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{A} \boldsymbol{\Theta}_0 \mathbf{B}_\theta) \mathbf{u}_b + \delta \mathbf{u}_b^T (\mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{A} \mathbf{B}_m) \mathbf{u}_m + \delta \mathbf{u}_b^T \left(\frac{1}{2} \mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{A} \boldsymbol{\Theta}_1 \mathbf{B}_\theta \right) \mathbf{u}_b + \\ \left. \delta \mathbf{u}_b^T (\mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{A} \boldsymbol{\Theta}_0 \mathbf{B}_\theta) \mathbf{u}_b \right] \, dA, \end{aligned} \quad (2.42)$$

which can be conveniently rewritten as:

$$\left\{ \begin{matrix} \delta \mathbf{u}_m^T & \delta \mathbf{u}_b^T & \delta \mathbf{u}_\phi^T \end{matrix} \right\} \mathbf{k}_A \left\{ \begin{matrix} \mathbf{u}_m \\ \mathbf{u}_b \\ \mathbf{u}_\phi \end{matrix} \right\} = \delta \mathbf{u}^T \mathbf{k}_A \mathbf{u}, \quad (2.43)$$

where

$$\mathbf{k}_A = \int_{A_e} \left(\begin{bmatrix} \mathbf{B}_m^T \mathbf{A} \mathbf{B}_m & \mathbf{B}_m^T \mathbf{A} \boldsymbol{\Theta} \mathbf{B}_\theta & 0 \\ \mathbf{B}_\theta^T (\boldsymbol{\Theta}_1^T + \boldsymbol{\Theta}_0^T) \mathbf{A} \mathbf{B}_m & \mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{A} \boldsymbol{\Theta} \mathbf{B}_\theta + \mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{A} \boldsymbol{\Theta} \mathbf{B}_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) dA, \quad (2.44)$$

in which $\boldsymbol{\Theta} = \frac{1}{2} \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_0$.

The second and the third terms of Eq. (2.9) come from the coupling between bending and axial deformations. The expansion of the second term yields:

$$\int_{A_e} \delta \varepsilon_0^T \mathbf{B} \kappa = \int_{A_e} (\mathbf{B}_m \delta \mathbf{u}_m + \boldsymbol{\Theta}_1 \mathbf{B}_\theta \delta \mathbf{u}_b + \boldsymbol{\Theta}_0 \mathbf{B}_\theta \delta \mathbf{u}_b)^T \mathbf{B} (-\mathbf{B}_m \mathbf{u}_r) \, dA, \quad (2.45)$$

which leads to

$$\begin{aligned} \int_{A_e} \delta \varepsilon_0^T \mathbf{B} \kappa = \int_{A_e} \left[\delta \mathbf{u}_m^T (-\mathbf{B}_m^T \mathbf{B} \mathbf{B}_m) + \delta \mathbf{u}_b^T (-\mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{B} \mathbf{B}_m) + \right. \\ \left. \delta \mathbf{u}_b^T (-\mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{B} \mathbf{B}_m) \right] \mathbf{u}_r \, dA. \end{aligned} \quad (2.46)$$

The third term, on its turn, becomes

$$\int_{A_e} \delta \kappa^T \mathbf{B} \varepsilon_0 \, dA = \int_{A_e} (-\mathbf{B}_m \delta \mathbf{u}_r)^T \mathbf{B} (\mathbf{B}_m \mathbf{u}_m + \frac{1}{2} \mathbf{\Theta}_1 \mathbf{B}_\theta \mathbf{u}_b + \mathbf{\Theta}_0 \mathbf{B}_\theta \mathbf{u}_b) \, dA, \quad (2.47)$$

which results in

$$\int_{A_e} \delta \kappa^T \mathbf{B} \varepsilon_0 \, dA = \int_{A_e} \delta \mathbf{u}_r^T \left[(-\mathbf{B}_m^T \mathbf{B} \mathbf{B}_m) \mathbf{u}_m + (-\mathbf{B}_m^T \mathbf{B} \frac{1}{2} \mathbf{\Theta}_1 \mathbf{B}_\theta) \mathbf{u}_b + (-\mathbf{B}_m^T \mathbf{B} \mathbf{\Theta}_0 \mathbf{B}_\theta) \mathbf{u}_b \right] \, dA. \quad (2.48)$$

Finally, the sum of the second and third term, Eq. (2.46) and (2.48), after discretization, becomes:

$$\int_{A_e} \delta \varepsilon_0^T \mathbf{B} \kappa + \int_{A_e} \delta \kappa^T \mathbf{B} \varepsilon_0 \, dA = \delta \mathbf{u}^T \mathbf{k}_B \mathbf{u}, \quad (2.49)$$

in which

$$\mathbf{k}_B = - \int_{A_e} \begin{bmatrix} 0 & 0 & \mathbf{B}_m^T \mathbf{B} \mathbf{B}_m \\ 0 & 0 & \mathbf{B}_\theta^T (\mathbf{\Theta}_1^T + \mathbf{\Theta}_0^T) \mathbf{B} \mathbf{B}_m \\ \mathbf{B}_m^T \mathbf{B} \mathbf{B}_m & \mathbf{B}_m^T \mathbf{B} \mathbf{\Theta} \mathbf{B}_\theta & 0 \end{bmatrix} \, dA. \quad (2.50)$$

The fifth contribution to the virtual strain energy is the bending energy. Substitution of Eq.(2.27) and Eq.(2.36) into the fifth term of Eq.(2.9) results in

$$\int_{A_e} \delta \kappa^T \mathbf{D} \kappa \, dA = \int_{A_e} (-\mathbf{B}_m \delta \mathbf{u}_r)^T \mathbf{D} (-\mathbf{B}_m \mathbf{u}_r) \, dA = \delta \mathbf{u}_r^T \left(\int_{A_e} \mathbf{B}_m^T \mathbf{D} \mathbf{B}_m \, dA \right) \mathbf{u}_r, \quad (2.51)$$

which can be written as

$$\int_{A_e} \delta \kappa^T \mathbf{D} \kappa \, dA = \delta \mathbf{u}^T \mathbf{k}_D \mathbf{u}, \quad (2.52)$$

where

$$\mathbf{k}_D = \int_{A_e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{B}_m^T \mathbf{D} \mathbf{B}_m \end{bmatrix} \, dA. \quad (2.53)$$

Lastly, the energy due to transverse shear, which is the last term of Eq. (2.9), after discretization, becomes:

$$\begin{aligned} \int_{A_e} \delta \varepsilon_s^T \mathbf{C} \varepsilon_s \, dA &= \int_{A_e} (\mathbf{B}_\theta \delta \mathbf{u}_b - \mathbf{B}_c \delta \mathbf{u}_r)^T \mathbf{C} (\mathbf{B}_\theta \mathbf{u}_b - \mathbf{B}_c \mathbf{u}_r) \, dA = \\ \int_{A_e} \left[\delta \mathbf{u}_b^T (\mathbf{B}_\theta^T \mathbf{C} \mathbf{B}_\theta) \mathbf{u}_b - \delta \mathbf{u}_b^T (\mathbf{B}_\theta^T \mathbf{C} \mathbf{B}_c) \mathbf{u}_r - \delta \mathbf{u}_r^T (\mathbf{B}_c^T \mathbf{C} \mathbf{B}_\theta) \mathbf{u}_b + \right. \\ \left. \delta \mathbf{u}_r^T (\mathbf{B}_c^T \mathbf{C} \mathbf{B}_c) \mathbf{u}_r \right] \, dA &= \delta \mathbf{u}^T \mathbf{k}_C \mathbf{u}, \end{aligned} \quad (2.54)$$

in which

$$\mathbf{k}_C = \int_{A_e} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{B}_\theta^T \mathbf{C} \mathbf{B}_\theta & -\mathbf{B}_\theta^T \mathbf{C} \mathbf{B}_c \\ 0 & -\mathbf{B}_c^T \mathbf{C} \mathbf{B}_\theta & \mathbf{B}_c^T \mathbf{C} \mathbf{B}_c \end{pmatrix} dA. \quad (2.55)$$

Finally, the virtual internal work within an element e is

$$\delta W_{int}^e = \delta \mathbf{u}^T (\mathbf{k}_A + \mathbf{k}_B + \mathbf{k}_C + \mathbf{K}_D) \mathbf{u}. \quad (2.56)$$

If the obtained stiffness matrices are rearranged according to the order of the terms as zeroth-, first- and second-order, the virtual strain energy of Eq. (2.56) within an element e becomes:

$$\delta W_{int}^e = \delta \mathbf{u}^T (\mathbf{k}_0 + \mathbf{k}_1 + \mathbf{k}_2) \mathbf{u}, \quad (2.57)$$

where

$$\mathbf{k}_0 = \int_{A_e} \begin{pmatrix} \mathbf{B}_m^T \mathbf{A} \mathbf{B}_m & \mathbf{B}_m^T \mathbf{A} \boldsymbol{\Theta}_0 \mathbf{B}_\theta & -\mathbf{B}_m^T \mathbf{B} \mathbf{B}_m \\ \mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{A} \mathbf{B}_m & \mathbf{B}_\theta^T \mathbf{C} \mathbf{B}_\theta + \mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{A} \boldsymbol{\Theta}_0 \mathbf{B}_\theta & -\mathbf{B}_\theta^T \mathbf{C} \mathbf{B}_c - \mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{B} \mathbf{B}_m \\ -\mathbf{B}_m^T \mathbf{B} \mathbf{B}_m & -\mathbf{B}_c^T \mathbf{C} \mathbf{B}_\theta - \mathbf{B}_m^T \mathbf{B} \boldsymbol{\Theta}_0 \mathbf{B}_\theta & \mathbf{B}_m^T \mathbf{D} \mathbf{B}_m + \mathbf{B}_c^T \mathbf{C} \mathbf{B}_c \end{pmatrix} dA, \quad (2.58)$$

$$\mathbf{k}_1 = \int_{A_e} \begin{pmatrix} 0 & \frac{1}{2} \mathbf{B}_m^T \mathbf{A} \boldsymbol{\Theta}_1 \mathbf{B}_\theta & 0 \\ \mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{A} \mathbf{B}_m & \mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{A} \boldsymbol{\Theta}_0 \mathbf{B}_\theta + \frac{1}{2} \mathbf{B}_\theta^T \boldsymbol{\Theta}_0^T \mathbf{A} \boldsymbol{\Theta}_1 \mathbf{B}_\theta & -\mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{B} \mathbf{B}_m \\ 0 & -\frac{1}{2} \mathbf{B}_m^T \mathbf{B} \boldsymbol{\Theta}_1 \mathbf{B}_\theta & 0 \end{pmatrix} dA, \quad (2.59)$$

$$\mathbf{k}_2 = \int_{A_e} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} \mathbf{B}_\theta^T \boldsymbol{\Theta}_1^T \mathbf{A} \boldsymbol{\Theta}_1 \mathbf{B}_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} dA. \quad (2.60)$$

The detailed numerical integration procedure within the element can be found in Szabó and Babuška (1991). Moreover, as the panel thickness-to-length ratio becomes smaller, the discretized strain energy due to transverse shear becomes progressively higher than the other strain energy terms, which is physically inconsistent. This issue is called

shear locking, and might be overcome through selective integration (FERREIRA, 2008). This technique consists in using fewer Gauss points to evaluate the transverse-shear-related stiffness matrices than for the other matrices. Nonetheless, this reduced integration technique is only required when the length-to-thickness ratio becomes large, i.e., $L/h > 40$. In the analysis performed in this work, four points were used for the main stiffness and mass terms, whilst $\mathbf{k}_C$ is evaluated with only one integration point.

The classical linear Mindlin plate theory is recovered by setting the first- and second-order matrices equals to zero. The zeroth-order stiffness matrix, $\mathbf{k}_0$, is constant, whereas $\mathbf{k}_1$ and $\mathbf{k}_2$ depend upon the displacements, $\mathbf{u}$, linearly and quadratically, respectively. After the finite element discretization and assembling all the elemental contributions, the internal work can be written as:

$$\delta W_{int} = \delta \mathbf{U}^T (\mathbf{K}_0 + \mathbf{K}_1 + \mathbf{K}_2) \mathbf{U}, \quad (2.61)$$

where $\mathbf{U}$ is the total DOF vector, which means that this vector contains all the mesh degrees of freedom. In Eq. (2.61), the stiffness matrices are also written in capital letters, meaning that they account for global quantities, which are obtained after assembling all elemental contributions.

As expressed in Eq. (2.13), the external work has contributions due to aerodynamic pressure, which will be detailed in Section 2.3, and inertial forces. The discretization of the external work due to inertial forces within an element becomes simply

$$-\delta \mathbf{u}^T \mathbf{m} \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad (2.62)$$

with $\mathbf{m}$ being the elemental mass matrix, given by

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_m & 0 & 0 \\ 0 & \mathbf{m}_b & 0 \\ 0 & 0 & \mathbf{m}_r \end{bmatrix} = \begin{bmatrix} h(\mathbf{m}_x + \mathbf{m}_y) & 0 & 0 \\ 0 & h\mathbf{m}_z & 0 \\ 0 & 0 & \frac{h^3}{12}(\mathbf{m}_x + \mathbf{m}_y) \end{bmatrix}, \quad (2.63)$$

in which

$$\mathbf{m}_x = \int_{A_e} \rho \mathbf{H}_u \mathbf{H}_u^T dA, \quad (2.64)$$

$$\mathbf{m}_y = \int_{A_e} \rho \mathbf{H}_v \mathbf{H}_v^T dA, \quad (2.65)$$

$$\mathbf{m}_z = \int_{A_e} \rho \mathbf{H}_b \mathbf{H}_b^T dA. \quad (2.66)$$

$$(2.67)$$

In section 2.4, the Principle of Virtual Work will be applied and the aeroelastic equations of motion will be determined.

2.2 Geometry definition

The midplane initial deformation, $\hat{w}_0$, presented in Equation (2.1) defines the geometry of the shallow shell. It can be any kind of geometry as long as the curvature is shallow. The shallowness of a shell is defined by the relation between the rise of the shell relatively to its other dimensions. Khazaeinejad and Usmani (2016) illustrates the ranges of applicability of various shell theories for modeling cylindrical shells. A shallow shell typically has a rise of less than one-fifth of the smallest dimension of its planform, which could be written as $H/a \leq 1/5$. Other ways to characterize the shallowness of a shell is through the relation between the rise and the thickness, H/h , and also through the relation between the radius of curvature (cylinder radius) and the panel length, R/a .

The present work comprises cylindrical shallow shells with the streamwise and cross-stream configurations, with respect to the flow direction. The reinforcers are modeled as fixed supports, allowing only rotation between the panels, and are transversely placed in relation to the flow direction. The choice for this position of the reinforcers was based in Pacheco et al. (2017a), who showed that reinforcers aligned with the flow direction affects mildly the limit cycle oscillation amplitude of flat composite panels, while the tandem direction with respect to the flow produces high amplitudes for the downstream bay and jumps in the oscillation amplitude. Figure 3 illustrates the streamwise curvature, in which the flow is aligned with the curvature direction, $\alpha = 0$ (α is the flow incidence angle with respect to the x -axis).

Figure 3 – Streamwise two-bay panel model.

Source: Elaborated by the author.

The geometry of the shallow shell with constant streamwise curvature is obtained from the cylinder equation considering the cylinder radius, R , that allows the deflection

H for the given panel length, b , expressed as:

$$\begin{aligned}\hat{w}_0(x, y) &= \sqrt{R^2 - \left(x - \frac{b}{2}\right)^2} - d, \\ d &= R - H,\end{aligned}\tag{2.68}$$

where d is the vertical distance between the center of the cylinder and the x -axis, H is the highest point of the curvature (midplane initial deformation) in relation to the x -axis, and h is the panel thickness.

Figure 4 – Cross-stream two-bay panel model.

Source: Elaborated by the author.

Analogously, the geometry of the cross-stream curvature, as illustrated in Figure 4, is given by:

$$\hat{w}_0(x, y) = \sqrt{R^2 - \left(y - \frac{a}{2}\right)^2} - d,\tag{2.69}$$

where d is the distance between the cylinder center and the y -axis and a is the length of each bay.

2.3 Aerodynamic model

The most employed approach concerning the unsteady aerodynamic model in panel flutter problems is the piston theory (MEI et al., 1999b), which is derived from the supersonic potential flow theory. More recently, Alder (2015) compared the quasi-steady piston theory and the linearized potential flow theory with the Euler solution and showed that the results approach each other for Mach numbers of 1.6 and larger. Dowell (1974) discussed the applicability of the piston theory for cylindrical shells geometry, and also applied the first-order piston theory to the nonlinear analysis of curved panels. Amirzadegan and Dowell (2019) made a correlation of experimental and computational

results of streamwise curved panels, comparing their findings with the results of an early, classical experiment of Anderson (1962). They found adequate qualitative agreement between theory and experiment by using the Galerkin method and the first-order piston theory. Indeed, a more widely applicable aerodynamic model, as the full potential flow theory, Euler or Navier-Stokes fluid models would provide better quantitative results. However, the simplicity and the lower computational cost of the piston theory justify its use, since it also provides good qualitative results and serves as a guide to more selective studies of the flutter phenomena and its dynamic behavior. The first-order piston theory was also applied to analyze the flutter phenomena of cylindrical panels by many authors, such as: Zhou et al. (2018), Ghoman and Azzouz (2012), and Azzouz (2009).

The aerodynamic loading modeled by the piston theory depends on the panel displacements, and the corresponding work terms may be evaluated in the form of an aerodynamic damping matrix and contributions to the stiffness matrix. The first-order piston theory provides an unsteady pressure term as a function of the panel transverse motion for a flow field parallel to the panel surface with a particular flow incidence angle α with respect to x -axis. Regarding curved panels, there is also a static pressure due to their initial curvature, which contributes to total pressure term (Ghoman; Azzouz, 2012):

$$\Delta p(\hat{w}(x, y, t)) = -\frac{2q}{\beta} \times \left(\frac{\partial \hat{w}_1}{\partial x} \cos \alpha + \frac{\partial \hat{w}_1}{\partial y} \sin \alpha + \frac{M^2 - 2}{M^2 - 1} \frac{1}{V} \times \frac{\partial \hat{w}_1}{\partial t} \right) - \frac{2q}{\beta} \times \left(\frac{\partial \hat{w}_0}{\partial x} \cos \alpha + \frac{\partial \hat{w}_0}{\partial y} \sin \alpha \right), \quad (2.70)$$

where $\beta = \sqrt{M^2 - 1}$, q , V and M are the free-stream dynamic pressure, velocity and Mach number, respectively. The first part of the equation provides the pressure due to deformation of the panel during the flutter occurrence, and the second part provides the static aerodynamic load due to panel curvature. The simplicity of this theory comes from the fact that the aerodynamic pressure is linearly related to the panel's instantaneous shape and velocity, and does not require fluid meshes. Hence, the resulting aeroelastic system becomes considerably simpler and computationally cheaper than CFD-based solvers.

The effect of cross-stream and streamwise flow direction will be analyzed. The cross-stream case refers to the flow perpendicular to the curvature (Fig. 4), and the streamwise case refers to the flow in the same direction as the curvature (Fig. 3). Note that $\frac{\partial \hat{w}_0}{\partial x} = 0$ and $\sin \alpha = 0$ for the cross-stream case, so the last term of Equation (2.70) becomes null. Therefore, only the streamwise case allows static aerodynamic loads.

To make parametric studies simpler, Eq. (2.70) can be conveniently rewritten as:

$$\Delta p = -\lambda \left(\frac{D}{a^3} \right) \left[\frac{\partial \hat{w}_1}{\partial x} \cos \alpha + \frac{\partial \hat{w}_1}{\partial y} \sin \alpha \right] - g_a(\rho h \omega_0) \frac{\partial \hat{w}_1}{\partial t} - \lambda \left(\frac{D}{a^3} \right) \left[\frac{\partial \hat{w}_0}{\partial x} \cos \alpha + \frac{\partial \hat{w}_0}{\partial y} \sin \alpha \right], \quad (2.71)$$

where a is the panel length, D (Eq. (2.72)) is a reference flexural stiffness, and ω_0 (Eq. (2.73)) is a reference frequency. Furthermore, the analysis is usually parameterized with respect to two dimensionless coefficients: the dynamic pressure, λ (Eq. (2.74)), and the damping factor, g_a (Eq. (2.75)):

$$D = \frac{h^3 E_1}{12(1 - \nu_{12}\nu_{21})}, \quad (2.72)$$

$$\omega_0 = \sqrt{\frac{D}{\rho h a^4}}, \quad (2.73)$$

$$\lambda = \frac{2qa^3}{D\sqrt{M^2 - 1}}, \quad (2.74)$$

$$g_a = \sqrt{\frac{\lambda\mu}{\sqrt{M^2 - 1}} \left(\frac{M^2 - 2}{M^2 - 1} \right)} \approx \sqrt{\frac{\lambda\mu}{M}}, \quad (2.75)$$

in which $\mu = \frac{a\rho_\infty}{\rho}$ is the mass ratio. Hence, the work done by the pressure field over a virtual displacement field is:

$$\delta W_{ext}^{flow} = \int_A (\delta \hat{w}_1 \Delta p) dA = \sum_{n=1}^{n_e} \int_{A_e} (\delta \hat{w}_1 \Delta p) dA, \quad (2.76)$$

in which n_e is the number of elements. Substitution of Eq. (2.71) into Eq. (2.76) results in three terms. One related to the aerodynamic stiffness, another related to the aerodynamic damping and the last one related to the static aerodynamic load due to initial curvature. These calculations will be shown for one element.

Upon substitution of Eq. (2.23) and Eq. (2.71) into Eq. (2.76) and knowing that $\hat{w}_1 = \mathbf{H}_b^T \mathbf{u}_b = \mathbf{u}_b^T \mathbf{H}_b$, the three terms of the external work due the aerodynamic pressure are:

$$\begin{aligned} \delta W_{ext1}^{flow} &= \int_{A_e} \delta \hat{w}_1 \left[-\lambda \left(\frac{D}{a^3} \right) \left(\frac{\partial \hat{w}_1}{\partial x} \cos \alpha + \frac{\partial \hat{w}_1}{\partial y} \sin \alpha \right) \right] dA = \\ &\delta \mathbf{u}_b^T \int_{A_e} \mathbf{H}_b \left[-\lambda \left(\frac{D}{a^3} \right) \left(\frac{\partial \mathbf{H}_b^T}{\partial x} \cos \alpha + \frac{\partial \mathbf{H}_b^T}{\partial y} \sin \alpha \right) \right] dA \mathbf{u}_b, \end{aligned} \quad (2.77)$$

$$\delta W_{ext2}^{flow} = \int_{A_e} -\delta \hat{w}_1 g_a (\rho h \omega_0) \frac{\partial \hat{w}_1}{\partial t} = -g_a \rho h \omega_0 \delta \mathbf{u}_b \int_{A_e} \mathbf{H}_b [\mathbf{H}_b^T dA] \frac{\partial \mathbf{u}_b}{\partial t} dA, \quad (2.78)$$

$$\begin{aligned} \delta W_{ext3}^{flow} &= \int_{A_e} \delta \hat{w}_1 \left[-\lambda \left(\frac{D}{a^3} \right) \left(\frac{\partial \hat{w}_0}{\partial x} \cos \alpha + \frac{\partial \hat{w}_0}{\partial y} \sin \alpha \right) \right] dA = \\ &\delta \mathbf{u}_b^T \int_{A_e} \mathbf{H}_b \left[-\lambda \left(\frac{D}{a^3} \right) \left(\frac{\partial \mathbf{H}_b^T}{\partial x} \cos \alpha + \frac{\partial \mathbf{H}_b^T}{\partial y} \sin \alpha \right) \right] dA \mathbf{u}_0. \end{aligned} \quad (2.79)$$

The elemental stiffness and damping matrices obtained from Eq. (2.77) and Eq. (2.78), respectively, are:

$$\mathbf{k}_a = \frac{D}{a^3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \int_{A_e} \mathbf{H}_b \left(\frac{\partial \mathbf{H}_b^T}{\partial x} \cos \alpha + \frac{\partial \mathbf{H}_b^T}{\partial y} \sin \alpha \right) dA & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.80)$$

and

$$\mathbf{c}_a = \rho h \omega_0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \int_{A_e} \mathbf{H}_b \mathbf{H}_b^T dA & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.81)$$

The static aerodynamic pressure at one element is obtained from Eq. (2.79) is:

$$p_0 = \frac{D}{a^3} \left[\int_{A_e} \mathbf{H}_b \left(\frac{\partial \mathbf{H}_b^T}{\partial x} \cos \alpha + \frac{\partial \mathbf{H}_b^T}{\partial y} \sin \alpha \right) dA \right] \mathbf{u}_0. \quad (2.82)$$

After assembling all the elemental contributions, the external work due to the flow becomes:

$$\delta W_{ext}^{flow} = -\delta \mathbf{U}^T (\lambda \mathbf{K}_a \mathbf{U} + g_a \mathbf{C}_a \dot{\mathbf{U}} + \lambda \mathbf{P}_0). \quad (2.83)$$

Therefore, the aerodynamic load is translated into two matrices that can be introduced into the pre-existing structural model, and a load vector that can be introduced as an external force acting upon the panel surface.

2.4 Aeroelastic equation of motion and solution methods

The complete aeroelastic equation of motion is obtained by applying the equality stated by the Principle of Virtual Work. Equation (2.61) determines the internal work, while Eq. (2.83) refers to the external work due to aerodynamic loads. Assembling these equations along with the work due to the panel's acceleration field, Eq. (2.62), through the equality stated by the PVW, $\delta W_{int} = \delta W_{ext}$, results:

$$\delta \mathbf{U}^T (\mathbf{K}_0 + \mathbf{K}_1 + \mathbf{K}_2) \mathbf{U} = -\delta \mathbf{U}^T (\lambda \mathbf{K}_a \mathbf{U} + g_a \mathbf{C}_a \dot{\mathbf{U}} + \lambda \mathbf{P}_0) - \delta \mathbf{U}^T (\mathbf{M} \ddot{\mathbf{U}}), \quad (2.84)$$

where $\mathbf{M}$ is the total mass matrix, which is obtained by assembling all elemental contributions, $\mathbf{m}$, Eq. (2.63). The right side of the equation refers to the external work, while the left side accounts for the internal work. Since the PVW states that the internal and external virtual works must be equal for any compatible virtual displacement field $\delta \mathbf{U}$, Eq. (2.84) becomes:

$$\mathbf{M} \ddot{\mathbf{U}} + g_a \mathbf{C}_a \dot{\mathbf{U}} + (\lambda \mathbf{K}_a + \mathbf{K}_0 + \mathbf{K}_1 + \mathbf{K}_2) \mathbf{U} = -\lambda \mathbf{P}_0, \quad (2.85)$$

which is the system of ordinary differential equations (ODEs) that dictates the motion of the discretized shallow shell system. Moreover, it is a nonlinear ODE system, as the values of matrices $\mathbf{K}_1$ and $\mathbf{K}_2$ depend on the solution, $\mathbf{U}$, at each corresponding time that is being evaluated. In order to solve Eq. (2.85), numerical time integration marching is required. To facilitate this, Eq. (2.85) can be rewritten as

$$\mathbf{M}\ddot{\mathbf{U}} + \left[\mathbf{G} \left(\lambda, \frac{\mu}{M} \right) \right] \dot{\mathbf{U}} + [\mathbf{K}(\lambda) + \mathbf{H}(\mathbf{U})]\mathbf{U} = \mathbf{F}(\lambda), \quad (2.86)$$

in which

$$\mathbf{K}(\lambda) = \lambda \mathbf{K}_a + \mathbf{K}_0, \quad (2.87)$$

$$\mathbf{F}(\lambda) = -\lambda \mathbf{P}_0 \quad (2.88)$$

$$\mathbf{G} \left(\lambda, \frac{\mu}{M} \right) = \sqrt{\frac{\lambda \mu}{M}} \mathbf{C}_a, \quad (2.89)$$

$$\mathbf{H}(\mathbf{U}) = \mathbf{K}_1(\mathbf{U}) + \mathbf{K}_2(\mathbf{U}). \quad (2.90)$$

Since the geometry and material properties were fixed, the aeroelastic system relies on λ and μ/M , which essentially represent the flight condition. However, to solve the aeroelastic equations, it is necessary to apply the boundary conditions. For the analysis showed in Chapters 4 and 5, the set boundary condition was all edges simply supported. Nonetheless, symmetry boundary condition was applied, which consists of restricting perpendicular displacements and parallel rotations to the edge that is cut by the symmetry plane. To determine the symmetry plane, the geometry, material properties, and loads should be considered in general. Li and Reid (1992) provide an extensive study about symmetry conditions for composite laminates. In the panel flutter problem, the existence of mathematical symmetry requires symmetry of material properties, geometry, flow direction, boundary conditions and initial conditions. Therefore, for all cases analyzed in the present work, which consist of isotropic and cross-ply laminate cylindrical panels, the symmetry plane is always aligned with the flow direction and placed in the half of panel width.

With the boundary conditions properly set, the aeroelastic analysis may begin. The pre- and post-flutter behavior will be analyzed and the flutter boundary will be determined.

2.4.1 Linear aeroelastic analysis

To study the aeroelastic behavior, it is important to obtain the stability boundary, which corresponds to the critical speed at the flutter onset. The flutter boundary can be determined by increasing the dynamic pressure to which the panel is subjected.

Before the flutter onset, the panel presents damped motion. Flat and cross-stream panels vibrate around $\hat{w}_1 = 0$ until the aerodynamic damping eventually causes the motion to cease. Streamwise panels presents steady-state deflections due to the presence of static aerodynamic loads. At the flutter onset, the panel experiences oscillations of the order of the thickness (AZZOUZ, 2005).

To predict the flutter boundary of cross-stream curved panels, which are not subjected to static aerodynamic loads as the streamwise panels, the linear part of Eq. (2.85) is solved for a range of λ until the system becomes unstable, indicated by the occurrence of negative damping or mode coalescence. This linear approach is suitable for cross-stream panels because Eq. (2.85) becomes homogeneous, since $\mathbf{P}_0$ is zero when the flow is perpendicular to the curvature direction. Therefore, the eigenvalue problem:

$$\det(\omega^2 \mathbf{M} + \omega \mathbf{C}_a + \mathbf{K}_0 + \lambda \mathbf{K}_a) = 0, \quad (2.91)$$

is obtained, where ω is the eigenvalue. The smallest dynamic pressure for which Eq. (2.91) reaches aeroelastic instability, λ_f , characterizes the flutter condition.

For $\lambda > \lambda_f$, the equations of motion including all the nonlinear terms, Eq. (2.85), are solved directly in the time domain through the Newmark integration scheme presented by Akay (1980). The flutter boundary of streamwise panels were found through this same procedure.

2.4.2 Nonlinear aeroelastic analysis

To solve the aeroelastic equations of motion accounting for all the linear and nonlinear terms, Eq. (2.85), the modified Newmark method proposed by Akay (1980) was applied. By selecting a time step Δt , the solutions $\mathbf{U}$ and $\dot{\mathbf{U}}$ are expressed in parametric Taylor series form as follows (AKAY, 1980):

$$\dot{\mathbf{U}}_{t+\Delta t} = \dot{\mathbf{U}}_t + \left[(1 - \alpha) \ddot{\mathbf{U}}_t + \alpha \ddot{\mathbf{U}}_{t+\Delta t} \right] \Delta t, \quad (2.92)$$

$$\mathbf{U}_{t+\Delta t} = \mathbf{U}_t + \dot{\mathbf{U}}_t \Delta t + \left[\left(\frac{1}{2} - \beta \right) \ddot{\mathbf{U}}_t + \beta \ddot{\mathbf{U}}_{t+\Delta t} \right] \Delta t^2, \quad (2.93)$$

where α and β are parameters selected to control accuracy and stability of the scheme. The values $\alpha = 1/2$ and $\beta = 1/4$ were adopted, which leads to the constant-average acceleration method:

$$\dot{\mathbf{U}}_{t+\Delta t} = \dot{\mathbf{U}}_t + \frac{\Delta t}{2} (\ddot{\mathbf{U}}_{t+\Delta t} + \ddot{\mathbf{U}}_t) \quad (2.94)$$

$$\mathbf{U}_{t+\Delta t} = \mathbf{U}_t + \Delta t \dot{\mathbf{U}}_t + \left(\frac{\Delta t}{2} \right)^2 (\ddot{\mathbf{U}}_{t+\Delta t} + \ddot{\mathbf{U}}_t), \quad (2.95)$$

Rearranging Equations (2.94) and (2.95), we obtain:

$$\ddot{\mathbf{U}}_{t+\Delta t} = -\ddot{\mathbf{U}}_t + \left(\frac{2}{\Delta t} \right)^2 (\mathbf{U}_{t+\Delta t} - \mathbf{U}_t - \Delta t \dot{\mathbf{U}}_t), \quad (2.96)$$

and

$$\dot{\mathbf{U}}_{t+\Delta t} = -\dot{\mathbf{U}}_t + \frac{2}{\Delta t}(\mathbf{U}_{t+\Delta t} - \mathbf{U}_t). \quad (2.97)$$

Substitution of Equations (2.96) and (2.97) into Eq. (2.86) yields:

$$\left(\hat{\mathbf{K}} + \frac{\Delta t^2}{4} \mathbf{H}_{t+\Delta t} \right) \mathbf{U}_{t+\Delta t} = \frac{\Delta t^2}{4} \mathbf{F}_{t+\Delta t} + \hat{\mathbf{P}}_t, \quad (2.98)$$

where

$$\hat{\mathbf{K}} = \mathbf{M} + \frac{\Delta t}{2} \mathbf{G} + \frac{\Delta t^2}{4} \mathbf{K}, \quad (2.99)$$

$$\hat{\mathbf{P}}_t = \mathbf{M} \left[\mathbf{U}_t + \Delta t \dot{\mathbf{U}}_t + \frac{\Delta t^2}{4} \ddot{\mathbf{U}}_t \right] + \mathbf{G} \left[\frac{\Delta t}{2} \mathbf{U}_t + \frac{\Delta t^2}{4} \dot{\mathbf{U}}_t \right]. \quad (2.100)$$

The efficiency of this approach relies on the fact that the coefficient matrix, $\hat{\mathbf{K}}$, is constant for all iterations and time steps, so it needs to be inverted only once, before the time integration begins. The load term $\mathbf{F}_{t+\Delta t}$ is either assumed zero (for flutter boundary prediction and for the nonlinear aeroelastic analysis of cross-stream shallow shells or flat panels) or $-\lambda \mathbf{P}_0$, which corresponds to the static aerodynamic load acting upon the streamwise panel model. Therefore, this load term does not vary in time: $\mathbf{F}_{t+\Delta t} = \mathbf{F}$.

Since Equation (2.98) is nonlinear, as $\mathbf{H}_{t+\Delta t}$ is a function of $\mathbf{U}_{t+\Delta t}$, it has to be solved iteratively until equilibrium is established at each time step. The iterative approach proposed by Akay (1980) consists into transferring the nonlinear term $\frac{\Delta t^2}{4} \mathbf{H}_{t+\Delta t} \mathbf{U}_{t+\Delta t}$ to the right hand side of Eq. (2.98). Therefore, the iterative equations become:

$$\hat{\mathbf{K}} \mathbf{U}_{t+\Delta t}^{i+1} = \hat{\mathbf{F}}_t - \frac{\Delta t^2}{4} \mathbf{H}_{t+\Delta t}^i \mathbf{U}_{t+\Delta t}^i, \quad (2.101)$$

in which

$$\hat{\mathbf{F}}_t = \frac{\Delta t^2}{4} \mathbf{F} + \mathbf{M} \left[\mathbf{U}_t + \Delta t \dot{\mathbf{U}}_t + \frac{\Delta t^2}{4} \ddot{\mathbf{U}}_t \right] + \mathbf{G} \left[\frac{\Delta t}{2} \mathbf{U}_t + \frac{\Delta t^2}{4} \dot{\mathbf{U}}_t \right]. \quad (2.102)$$

The iterations should continue until a sufficiently accurate solution at each time step is achieved. Then, the new solution, $i + 1$, is calculated from the previous iteration as:

$$\mathbf{U}_{t+\Delta t}^{i+1} = \left(-\frac{\Delta t^2}{4} \hat{\mathbf{K}}^{-1} \mathbf{H}_{t+\Delta t}^i \right) \mathbf{U}_{t+\Delta t}^i + \hat{\mathbf{K}}^{-1} \hat{\mathbf{F}}. \quad (2.103)$$

The computational time needed for each time step is drastically reduced due to the fact that no inversion of the nonlinear stiffness matrix is required.

The converged values of a previous time step are used for the first iteration of the next time step, that is, $\mathbf{U}_{t+\Delta t}^0 = \mathbf{U}_t$. After convergence, the values of $\ddot{\mathbf{U}}_t$ and $\dot{\mathbf{U}}_t$ are updated through Eq. (2.96) and (2.97).

This variation of the Newmark method is very efficient, but comes as a trade-off for unconditional stability (AKAY, 1980). Thus, the aforementioned Newmark scheme is conditionally stable, so a minimum time step is required for convergence. The time steps used in the analysis were based on the estimate of the critical time step proposed by LEECH (1965):

$$\Delta t \leq \frac{1}{4} \left(\frac{\rho h}{D} \right)^{1/2} \overline{\Delta x}^2, \quad (2.104)$$

which can be rewritten as:

$$\Delta t \leq \frac{1}{\omega_0} \left(\frac{\overline{\Delta x}}{2a} \right)^2, \quad (2.105)$$

where $\overline{\Delta x}$ corresponds to the minimum element size in the mesh.

3 MODEL VALIDATION

The presented aeroelastic model was validated through comparison with the available literature and the finite element commercial software ABAQUS. The validations comprise: linear analysis of cylindrical and spherical composite shallow shells (natural frequencies and transverse displacements) for different curvature ratios; flutter boundary of cross-stream cross-ply and angle-ply composite shallow shells; nonlinear structural vibration of cylindrical panels compared with ABAQUS results; nonlinear aeroelastic analysis of flat panels ($H/h = 0$) and flutter boundary of streamwise curved panels through comparison with theoretical and experimental results.

3.1 Linear validation

The linear validation was made through comparison with Reddy (1984). The present model was validated against the free vibration of laminated composite cylindrical shells and vertical deflections of spherical shells under uniformly distributed load.

The nondimensional natural frequencies, $\bar{\omega} = \omega a^2 \sqrt{\rho/E_2}/h$ of the present model was compared with the analytical solutions of Reddy (1984). The cylindrical thin ($a/h = 100$) and thick ($a/h = 10$) shells are simply supported 2-layered, and 3-layered cross-ply. The simply supported boundary condition of the single square panel analyzed are: $\hat{v}(0, y, t) = \hat{v}(a, y, t) = 0$; $\hat{u}(x, 0, t) = \hat{u}(x, a, t) = 0$; $\hat{w}_1(0, y, t) = \hat{w}_1(a, y, t) = \hat{w}_1(x, a, t) = \hat{w}_1(x, 0, t) = 0$; $\phi_x(x = 0, y, t) = \phi_x(x = a, y, t) = 0$; $\phi_y(x, y = 0, t) = \phi_y(x, y = a, t) = 0$.

Tables 1 and 2 show the comparison of the nondimensionalized fundamental frequencies for the following laminates: $[0, 90]$, $[0, 90, 0]$, respectively. The curvature ratio is shown as R/a and H/h . R/a is the relation between the radius of the cylinder and the panel length, and H/h is the relation used in the present work, which is the highest point of the curvature (midplane initial deformation) in relation to the horizontal axis divided by the plate thickness, as previously defined. As shown in the Tables 1 and 2, the present model demonstrates a very good agreement with the analytical solutions of Reddy (1984).

Moreover, the nondimensional center deflection of spherical shells with different curvature ratio and thickness was analyzed. It is important to mention that the present model is able to analyze different types of curvatures, as long as the final geometry corresponds to a shallow shell. Tables 3 and 4 show the comparison of the nondimensional center deflections for thin and thick spherical shells, respectively. Note that, once again, the results agree very well, which shows that the present linear model is able to analyze and provide satisfactory results for different kinds of shallow shells.

The flutter boundary of shallow shells was also validated. Table 5 shows the

Table 1 – Comparison of the nondimensionalized fundamental frequencies, $\bar{\omega} = \omega a^2 \sqrt{\rho/E_2}/h$, versus length-to-thickness and radius-to-thickness ratio, of a simply supported 2-layered $[0, 90]$ cylindrical panel.

$[0, 90]$						
R/a	H/h	$a/h = 100$		H/h	$a/h = 10$	
		Present	Reddy		Present	Reddy
10^3	$\simeq 0$	9.699	9.687	$\simeq 0$	8.9091	8.8998
10	1.2508	11.863	11.831	0.1251	8.9421	8.8879
5	2.5063	16.767	16.668	0.2506	9.0231	8.9092
4	3.1373	19.668	19.509	0.3137	9.0815	8.9301

Table 2 – Comparison of the nondimensionalized fundamental frequencies, $\bar{\omega} = \omega a^2 \sqrt{\rho/E_2}/h$, versus length-to-thickness and radius-to-thickness ratio, of a simply supported 3-layered $[0, 90, 0]$ cylindrical panel.

$[0, 90, 0]$						
R/a	H/h	$a/h = 100$		H/h	$a/h = 10$	
		Present	Reddy		Present	Reddy
10^3	$\simeq 0$	15.201	15.183	$\simeq 0$	12.172	12.162
10	1.2508	16.654	16.625	0.1251	12.189	12.173
5	2.5063	20.409	20.332	0.2506	12.240	12.207
4	3.1373	22.832	22.709	0.3137	12.278	12.233

comparison of the critical dimensionless aerodynamic pressure of a cross-ply and a angle-ply 5-layered cylindrical shell exposed to cross-stream flow with the following boundary conditions: all edges clamped and all edges simply supported, as specified by Singha and Mandal (2007). The results of the present model are in good agreement with Singha and Mandal (2007), who neglected the aerodynamic damping, used a 16-node degenerated isoparametric shell element and solved the eigenvalue problem through the QR method. Here the aerodynamic damping were considered to be very small, with $\mu/M = 1/1000$.

Once the model is validated, it is important to investigate the mesh convergence for the multibay curved panel. The isotropic multibay curved panel analyzed here is subjected to cross-stream flow, Fig. 4, and has the same properties of the one that will be further analyzed in this work: $E_1 = 155 \text{ GPa}$, $E_2 = 8.07 \text{ GPa}$, $G_{12} = 4.55 \text{ GPa}$, $\rho = 1550 \text{ kg/m}^3$,

Table 3 – Comparison of the nondimensional center deflection versus radius-to-thickness ratio of spherical shells under uniformly distributed load for $a/h=10$.

$a/h = 10$							
R/a	H/h	[0, 90]		[0, 90, 0]		[0, 90, 90, 0]	
		Present	Reddy	Present	Reddy	Present	Reddy
10^3	0.0025	19.4628	19.469	10.2224	10.220	10.2522	10.251
10	0.25031	19.0185	19.065	10.0921	10.110	10.1223	10.141

$\bar{W} = (w_1 h^3 E_2 / q_0 a^4) * 10^3$, q_0 =intensity of transverse load.
 Note: $a/b = 1.0$, material: $E_1/E_2 = 25$; $G_{23} = 0.2E_2$; $G_{13} = G_{12} = 0.5E_2$; $\nu_{12}=0.25$.

Table 4 – Comparison of the nondimensional center deflection versus radius-to-thickness ratio of spherical shells under uniformly distributed load for $a/h=100$.

$a/h = 100$							
R/a	H/h	[0, 90]		[0, 90, 0]		[0, 90, 90, 0]	
		Present	Reddy	Present	Reddy	Present	Reddy
10^3	0.025	16.9665	16.980	6.694	6.6970	6.8293	6.8331
10	2.5031	5.5269	5.5428	3.6349	3.6445	3.711	3.7208

Table 5 – Comparison of the critical dimensionless aerodynamic pressure of a 5-layered cylindrical shell exposed to cross-stream flow with different boundary conditions with the Singha and Mandal (2007)'s results.

H/h	[0, 90, 0, 90, 0]				[45, -45, 45, -45, 45]			
	Simply Supported		Clamped		Simply Supported		Clamped	
	Present	Ref.	Present	Ref.	Present	Ref.	Present	Ref.
0	284.8	278.7	509.9	499.7	542.3	542.4	845.7	835.2
2	290.6	286.7	524.6	512.7	283.9	276.6	640.4	624.2

$\nu = 0.22$. Consider $b = 2a$, $h = a/100$, $\mu/M = 0.05$, and $H/h = 1$. All edges are simply supported: $\hat{u}(x, y, t) = \hat{v}(x, y, t) = \hat{w}_1(x, y, t) = 0$ for all edges; $\phi_y(x, y = 0, t) = \phi_y(x, y = a, t) = 0$; and $\phi_x(x = 0, y, t) = \phi_x(x = a, y, t) = \phi_x(x = 2a, y, t) = 0$. Symmetry condition

is used to analyze the curved plate, thus, the plate dimensions are $a/2 \times 2a$ (corresponding to $a/2 \times b$), once $b = 2a$, as previously stated.

Thereby, the flutter dynamic pressure is plotted against the mesh size for the length a , $\frac{a}{\Delta x}$. From Fig. 5, it is possible to verify that the solution converges. The value of λ_{cr} for $\frac{a}{\Delta x} = 20$ is 353.8, which is 1.11% higher than the result obtained for $\frac{a}{\Delta x} = 40$. For $\frac{a}{\Delta x} = 24$, $\lambda_{cr} = 352.8$, the difference reduces to 0.71%. This shows that $n = 24$ provides sufficient accuracy to analyze the flutter behavior of cross-stream shallow shells.

Figure 5 – Convergence study: flutter dynamic pressure for the $[0, 90, 0]$ two-bay cylindrical panel at cross-stream flow.

Source: Elaborated by the author.

3.2 Nonlinear validation

To validate the nonlinear matrices of the present model, the dynamic behavior of the cylindrical shells with no aerodynamic effect was analyzed. Time integration was done using the Newmark method and the responses were generated for arbitrary curvature ratios: $H/h = 3.14$ and $H/h = 2.51$ for single isotropic square panel, with the following parameters: $a = 1m$, $b = a$, $h = a/100$, $E = 69 MPa$, $\nu = 0.25$, $\rho = 2500 kg/m^3$. The curved panel has all edges simply supported and is subjected to a uniformly distributed vertical pressure, namely:

$$\Delta p(x, y, t) = \begin{cases} 0, & t < 0; \\ p_0, & t \geq 0; \end{cases}$$

where $p_0 = -1N/m^2$. The initial conditions are all displacements, velocities and accelerations null. The present results were compared with those from the finite element commercial software ABAQUS, as showed in Figure 6. The selected element in ABAQUS was the S4R, which is a 4-node, quadrilateral, stress/displacement shell element with reduced integration and a large-strain formulation. The Hilber-Hughes-Taylor time integrator is used by ABAQUS. Despite the different formulations, the time responses of the present model have an adequate agreement with the ABAQUS ones. The curvature ratios used for this model verification can be considered wide enough to ensure that the present model embraces appropriate capabilities to predict the nonlinear structure responses. With the wind-off condition presenting acceptable performance, the next step regards aeroelastic response performance.

Figure 6 – Comparison of the nonlinear dynamic response of a cylindrical shell with $a/b = 1$, $a/h = 100$.

Source: Elaborated by the author.

The nonlinear aeroelastic analyses were validated through comparison with the results obtained by Pacheco et al. (2017b) and Dixon and Mei (1993), which use the Kirchhoff theory for thin plates. This case study comprises a flat, rectangular single panel with the properties already presented in this section. The boundary condition is all edges simple supported, and the LCO amplitudes are taken at the $\frac{3}{4}$ position with respect to the panel center line. According to the results in Figure 7, the present model shows satisfactory agreement with those by Pacheco et al. (2017b) and Dixon and Mei (1993).

Figure 7 – Limit cycle amplitudes for different values of dynamic pressure of a single panel.

Source: Elaborated by the author.

Figure 8 – Comparison between numerical and experimental results of the flutter dynamic pressure.

Source: Elaborated by the author.

The last validation, presented in Figure 8, concerns the critical dynamic pressure analysis of streamwise shallow shells with different curvature ratios. Due to presence of aerodynamic static loads, the equation of motion is non-homogeneous and the critical dynamic pressure is found through the nonlinear dynamic analysis of the complete aeroelastic equations of motion. Thus, successively values of λ are analyzed to find the condition, λ_f , when the system changes from steady-state equilibrium to LCO motion. Figure 8 presents a comparison between numerical (for $H/h= 0, 1, 2, 3, 4$, and 5) and experimental results (for $H/h= 0$ and 7.9) of the flutter pressure boundary. The study case

corresponds to the experimental data due to Anderson (1962) for square panels clamped on the four edges with streamwise curvature. Later, Dowell (1969) analyzed the same plate with a 2D model, and recently Amirzadegan and Dowell (2019) performed a comparison between the experimental results and a 3D model. The results from the present model are compared with Amirzadegan and Dowell (2019) and Anderson (1962) results. The square clamped panel has the following properties: $a = 9$ inches, $a/b = 1$, $h = 0.008$ inches, $E = 15 \times 10^6$ psi, $\nu = 0.3$, and the relation μ/M is 0.1, which is the same adopted by Amirzadegan and Dowell (2019). In Anderson (1962) experiments, the values of μ/M varied with H/h , however the numerical analysis of Amirzadegan and Dowell (2019) found out that differences are very small between various values of M and μ (less than 1%).

Amirzadegan and Dowell (2019) discussed the physical sources of the quantitative discrepancies between theory and experiment, namely, the manufacturing imperfections and thermal stresses, since the experimental plate was extremely thin ($h = 0.008$ inches), flexibility in the in-plane edge supports, and the effect of a shallow cavity beneath the panels. Experiments in panel flutter are very scarce in the literature, hence to do another experiment considering the lessons learned over the last decades would be of great value. Comparison between the results from the present model with Amirzadegan and Dowell (2019) computations shows a good agreement, besides the different approaches adopted to perform the analysis. Amirzadegan and Dowell (2019) use modal analysis with 12 modes shapes, combining 6 streamwise modes and 2 symmetric spanwise modes. The present model uses a time domain solution and a 24×12 mesh (the symmetry condition was applied) to solve the differential equations of motion obtained through application of the finite element method.

4 AEROELASTIC ANALYSIS OF MULTIBAY ISOTROPIC CURVED PANELS

The multibay analysis of panels subjected to supersonic flutter remains scarce, even though the external skins of high-speed flight aerospace structures are composed of multiple adjacent panels that can interact. Although curved panels possess a more complex behavior, especially in the post-flutter regime, the investigation of adjacent curved panels interaction remains restrained to linear analysis, comprising only the pre-flutter regime. It is well known that flight vehicles' structural skins often possess some curvature, reassuring the relevance of taking such geometric features into account. Along these lines, a study of reinforced curved panels with respect to various aspects of their aeroelastic behavior is proposed.

The cross-stream and streamwise models with $H/h = 1.25$ are analyzed in this chapter to provide a deep understanding of the nonlinear aeroelastic dynamic behavior of curved panels. The aeroelastic behavior of each of the two square cells is compared to that of a single, isolated square panel concerning the cross-stream model. Cabral et al. (2021b) reports a numerical investigation of shallow shells' linear and nonlinear dynamic behavior exposed to cross-stream supersonic flutter for various curvature ratios. Cabral et al. (2021a) investigated the influence of streamwise curvature on the nonlinear supersonic flutter of isotropic multibay panels.

The multibay curved panel is modeled with all edges simply supported, including that between the adjacent bays (as illustrated in Figure 9), that is:

$$\begin{aligned} \hat{u}(x, y, t) = \hat{v}(x, y, t) = \hat{w}_1(x, y, t) &= 0 \quad , \quad \text{for all edges,} \\ \phi_y(x, y = 0, t) = \phi_y(x, y = a, t) &= 0 \quad , \quad \text{and} \\ \phi_x(x = 0, y, t) = \phi_x(x = a, y, t) = \phi_x(x = 2a, y, t) &= 0 \quad . \end{aligned}$$

The isotropic material properties are the same as in Cabral et al. (2021a), where $b = 2a$, $h = a/100$, $\nu = 0.25$, $\mu/M = 0.01$, and H/h determines the panel's curvature ratio, here assumed 1.25. The shallow shell hypothesis implies an initial panel curvature, $\hat{w}_0$. Hence, the initial and net deformations in the z -direction can be summed to determine the vertical displacements, that is: $w(x, y, z, t) = \hat{w}_0(x, y) + \hat{w}_1(x, y, t)$ (*cf.* Equation (2.1)). A simplified notation is adopted to describe the vertical net displacements: $W = \hat{w}_1(x, y, t)$. Figure 9 presents the reference points P1 and P2, at positions $(x/2a = 0.375, y/a = 0.5)$ and $(x/2a = 0.875, y/a = 0.5)$, respectively. At P1 and P2 points, the study of the static and dynamic behavior of the panel is performed. For a single panel model, the reference point is located at the $\frac{3}{4}$ position along the x -direction.

Figure 9 – Top view of the multibay panel exhibiting the points P1 and P2 used as reference.

Source: Elaborated by the author.

4.1 Cross-stream isotropic panel

Figure 4 illustrates the cross-stream two-bay panel model to be analyzed in this section. The panels are connected by simply supporters that allow rotation between them. To have a better understanding on how the interaction between adjacent bays affects its aeroelastic behavior, the results will be compared with the equivalent single panel arrangement.

Figure 10 depicts the amplitude envelope for the single and the two-bay panel for the curvature ratio of $H/h = 1.25$. The initial condition for the time integration adopts a very small distributed displacement imposed to the panels, corresponding to a uniform pressure $p = 0.010(E(h/a)^4)$ to its surface. From now on, this initial condition is referred as *ST* initial condition. After the flutter onset, limit cycle oscillations are observed in the panel. The flutter onset is the same for both single and multibay panel arrangements, $\lambda_f = 524.2$. It is possible to see two jump discontinuities in the two-bay panel post-flutter oscillation amplitudes. The single panel behavior resembles the downstream bay (at the point P2) near to the flutter boundary. In contrast, the first bay moves with small amplitudes. As the dynamic pressure increases, the upstream bay starts to behave very similarly to the single panel, and this similarity becomes even more pronounced as the dynamic pressure increases.

The reasons for the sudden jumps in post-flutter oscillation amplitudes are examined. The first jump discontinuity occurs at $\lambda = 660$. Figure 11a shows the fluttering shape at the panel midsection ($y = a/2$) for $\lambda = 657$ and 660. The panel's bays seem to maintain the same motion pattern before and after the jump phenomenon, vibrating close to in-phase motion. The time responses depicted in Figures 11c and 11e confirm such oscillation feature. However, at the second jump discontinuity (*cf.* Figure 11b), there is a shift in the motion pattern, from in-phase motion to out-of-phase motion, which confirms a change in

Figure 10 – Bifurcation diagram for a cross-stream two-bay panel with $H/h=1.25$ curvature and ST initial condition.

Source: Elaborated by the author.

the flutter mechanism. The time responses in Figures 11d and 11f reassure the change in the motion patterns when the dynamic pressure increases from 687 to 688. To confirm the changes in motion pattern, Figure 12 depicts the phase angle for the λ range presented in the amplitude envelope of Figure 10.

The sign of the phase angle depends on the reference point P1 or P2. For all phase angle plots, the point P2 at the second bay was adopted as the reference. The first shift in the phase angle reported in Figure 12, from 0° to $345.48^\circ (-14.52^\circ)$, is detailed in the fluttering shapes and time response of Figure 11, which clarifies that the motion pattern maintains a roughly in-phase motion after this first jump discontinuity. Therefore, this first jump discontinuity does not represent a change in the flutter mechanism and will be further investigated. Concerning the second jump discontinuity at $\lambda = 688$, there is an evident change in the phase angle. Before this jump occurs, the phase angle is either near 0° or 360° , which is roughly an in-phase motion. At $\lambda = 688$, there is a shift in the phase angle to values between 128° and 136° , which demonstrates an out-of-phase motion pattern. Indeed, the second jump discontinuity is related to a change in the panel motion during flutter.

The amplitude jump discontinuities and the change in the flutter mechanism suggest different LCO branches for this two-bay panel. Other initial conditions were applied to investigate the coexistence of bifurcation branches, resulting in the amplitude envelope presented in Figure 13. The initial conditions used to analyze the $H/h = 1.25$ curvature ratio are:

ST – Static deflection due to a **uniform pressure** applied to the panel's surface: $p =$

Figure 11 – Fluttering shapes and time reponses for pre- and post-jump conditions for $H/h = 1.25$ with the application of ST initial condition.

$$0.01(E(h/a)^4).$$

NST_B – The amplitudes of the LCO arising from a post-jump condition are taken as initial condition for a subsequent analysis with a slightly decreased λ . Following, the

Figure 12 – Phase difference angle between the motions of points P1 and P2 achieved through application of ST , NSD_F , and NSD_B initial condition for the $H/h = 1.25$ curvature ratio.

Source: Elaborated by the author.

latest results are employed for another simulation, considering an even smaller λ . This operations are sequentially performed for a range of dynamic pressures.

NST_F – The LCO amplitudes of a pre-jump condition are used as initial condition for a subsequent analysis with a small increment in λ . Following, the latest results are employed for another simulation, considering an even greater λ . This operations are sequentially performed for a range of λ .

NSD_B – It is the same procedure of NST_B , except that the jump is characterized by the discontinuity found at $\lambda = 660$. Hence, the motion conditions for $\lambda = 660$ are used as initial conditions to simulate $\lambda = 655$.

NSD_F – It is the same procedure of NST_F , except that the jump here is characterized by the discontinuity found at $\lambda = 660$.

The NSF_F and NSF_B initial conditions are the same as Cabral et al. (2021a) used to examine the mutual existence of different LCO branches for two-bays streamwise panels. Concerning the first jump discontinuity, the NSD_B and NSD_F initial conditions were applied. The NSD_B condition was applied for $\lambda < 660$. The latest results are employed for a new simulation with a even more decreased λ , repeating the calculations until the amplitude branches merge into each other at $\lambda = 645$. For $\lambda > 660$, the NSD_F condition was applied. Through the phase angle of Figure 12, it is seen that for all ST , NSD_B , and NSD_F initial conditions, the motion mechanism of flutter resembles in-phase motion for $\lambda < 688$ (until the second jump discontinuity). The fluttering shapes were plotted in

Figure 13 – Bifurcation diagram for a cross-stream two-bay panel with $H/h=1.25$ curvature.

Source: Elaborated by the author.

Figure 14b and 14c, at the beginning and end of the bifurcation branch. In both cases, it is seen that the bays oscillates close to in-phase motion.

With respect to the second amplitude jump discontinuity at $\lambda = 688$, the NST_B and NST_F initial conditions were applied. The NST_B is analogous to the NSD_B , and the NST_F is analogous to NSD_F . The only difference is the value of λ from which the LCO amplitudes were used as the initial condition, corresponding to the second jump discontinuity around $\lambda = 688$. For instance, to achieve the high-amplitude LCO for $\lambda = 687$, the maximum limit cycle amplitude of 688 was used, which corresponds to the NST_B condition. The new LCO achieved for 687 was used as initial condition for $\lambda = 685$, and so on.

The panel motion shape inspection reveals the flutter mechanism. Figure 14 presents the fluttering shapes for various λ in the range where different LCO branches co-exist. It is possible to see that the highest LCO branch maintains its motion pattern, with the bays vibrating close to out-of-phase motion until $\lambda = \lambda_{jump}$. In its turn, the LCOs found through the application of ST initial condition presented in Figure 10 also maintains an in-plane motion pattern. However, for $\lambda \geq \lambda_{jump}$, the responses obtained through the application of ST shifts their motion pattern to out-of-phase, while the results obtained through NST_F initial condition shifts from out-of-phase to in-phase motion. Such a phenomenon can be confirmed by the phase angle plot in Figure 15, which clearly shows a change in how the bays oscillate at $\lambda = \lambda_{jump}$.

Figures 14b and 14c present the fluttering shapes of the three branches. Despite the changes in amplitudes due to ST , NSD_F , and NSD_B initial conditions, the motion

pattern is held (especially for the downwind bay, since the change in amplitude of the upwind bay is mild). In conclusion, this LCO branch's existence is due to sensitivity to the initial conditions, not due to a change in the flutter mechanism. Therefore, this cross-stream two-bay curved panel with $H/h = 1.25$ showed a complex behavior and a higher sensitivity concerning initial conditions.

Figure 14 – Fluttering shapes ($y = a/2$) for $H/h = 1.25$ cross-stream curvature.

Source: Elaborated by the author.

Figure 15 – Motion phase angle at points P1 and P2 achieved through application of NST_B and NST_F initial condition for the $H/h = 1.25$ curvature ratio.

Source: Elaborated by the author.

4.2 Streamwise isotropic panel

A cylindrical shallow shell with $H/h = 1.25$ curvature ratio and streamwise flow direction is analyzed here, as illustrated in Figure 3. The model parameters are the same as in the previous analyses.

The bifurcation diagram in Figure 16 shows steady-state deflections occurring for $0 \leq \lambda \leq 751$, which are related to damping oscillations. The pre-flutter equilibrium point of the first bay (P1) is in the opposite direction of the curvature, while the second bay (P2) deflects in the same direction of the curvature. Differently from the cross-stream model of section 4.1, the pre-flutter region of streamwise panels is characterized by nonzero static deflections due to the presence of aerodynamic static loads. For each dynamic pressure, the steady case is computed firstly, and the deflections are applied to the panel at the initial time of the solution procedure. This initial condition will be referred as *STP* initial condition from now on. The application of *STP* promotes the flutter onset near $\lambda = 751$ outlined by a violent jump to large LCO amplitudes. By looking at the post-flutter peak-to-peak amplitudes of the limit cycle oscillations, it is revealed that the positive amplitudes of the second bay are larger than the negatives until a certain λ value near 800. Then, the negatives amplitudes become larger. In comparison, the first bay has greater deflections in the opposite direction of the panel curvature for the evaluated range of dynamic pressure.

Upstream and downstream bays present different post-flutter behavior. Right after

Figure 16 – Bifurcation diagram for a streamwise two-bay panel of $H/h = 1.25$ curvature ratio with *STP* initial condition.

Source: Elaborated by the author.

Figure 17 – Bifurcation diagram for a streamwise two-bay panel of $H/h = 1.25$ curvature ratio with *STD2* initial condition.

Source: Elaborated by the author.

the flutter onset, the downstream bay possesses limit cycle oscillations with two peak amplitudes, as observed by the peak-to-peak amplitudes at the point P2 in Figure 16. As λ increases, the oscillations progressively become into single amplitude. With respect to the

upstream bay, the LCO amplitudes at the point P1 show a more conventional oscillatory motion, mainly in opposition to the panel curvature.

The coexistence of limit cycle oscillations branches for the same range of dynamic pressure was verified for lower curved multibay panels by Cabral et al. (2021a). To investigate the coexistence of LCO branches for this $H/h = 1.25$ curvature, an additional initial condition were applied: *STD2* — half of the static deflection due to static aerodynamic loading.

The results for the *STP* initial condition are confronted to the results of simulations using other initial condition denoted as *STD2*. Figure 17 presents the bifurcation diagram obtained through application of *STD2* initial condition. The higher LCO amplitudes are connected by solid lines, while isolated points indicate intermediate oscillatory peaks. The bifurcation jumps are represented by dashed lines. It can be observed that the behavior obtained with the *STD2* initial condition is far more complex than with employment of the *STP* one. The steady-state equilibrium regions for *STP* and *STD2* initial conditions are the same until $\lambda = 750$. However, the sudden jump characterizing the flutter onset occurs near $\lambda = 760$ for the *STD2*, which indicates a sub-critical bifurcation. The post-flutter LCO amplitudes for the points P1 and P2 in the cases of *STP* and *STD2* are basically the same. By increasing λ near to 795, the LCO branches shift to lower amplitude branches. Then, a sudden jump occurs near to $\lambda = 805$, leading the system again to steady-state equilibrium. For the range of λ between 805 and 810, the panel remains in steady-state equilibrium, followed by another jump to similar LCO branches as found prior to the steady-state equilibrium region. With a dynamic pressure increment, a last jump discontinuity is observed. The LCO amplitudes increase moderately after this transition until the dynamic pressure of 900. Figures 18a and 18b show details of the time responses in the transition from the low-amplitude LCO branch to the high one ($\lambda=827$ and 828). Besides the increase in the amplitude of both bays, the downstream bay also changes from single to multiple peaks LCO. For this higher LCO branch, both bays achieve single amplitude LCO, as can be seen in Figure 17.

A comparison between the LCO behavior achieved with the application of the *STP* and *STD2* initial conditions was performed and depicted in Figures 19 and 20. Figures 19a and 19b show the fluttering shapes for $\lambda = 830$ and $\lambda = 900$, respectively. The vertical displacements presented in Figure 19 correspond to the sum of the initial and net deformations in the z -direction: $w(x, y, z, t) = \hat{w}_0(x, y) + \hat{w}_1(x, y, t)$ (*cf.* in Equation (2.1)), and they were computed considering the *STP* and *STD2* initial conditions. It is worth noticing that points P1 and P2 do not correspond to the maximum deflection of the respective bays. Indeed, the LCO branch at point P2 obtained through *STD2* initial condition presents lower peak-to-peak LCO amplitudes than the other LCO branch for $\lambda = 830$ with the *STP* initial condition (*cf.* in Figure 19a). Despite being a convenient

Figure 18 – Time window LCO at $\lambda=827$ and 828 concerning the application of *STD2* initial condition.

Source: Elaborated by the author.

point for a general purpose analysis, the three-quarter position might not be the best choice for detailed multibay cylindrical shells flutter investigation. Although the motion shapes present certain differences, in both cases the bays seem to vibrate roughly in out-of-phase motion. These behaviors are confirmed through the time responses of Figures 20a and 20b that refer to the application of *STP* and *STD2*, respectively, for $\lambda = 830$. It is possible to see that the oscillatory motion obtained through *STD2* initial condition are more complex and presents multiple peaks amplitudes.

Figure 19 – Fluttering shape (at $y = a/2$) for $H/h=1.25$ curvature ratio.

Source: Elaborated by the author.

For the dimensionless dynamic pressure $\lambda = 900$, Figures 21a and 21b show the time responses for the application of the *STP* and *STD2* initial conditions. In both cases, the LCO amplitudes possess only one peak and the upstream and downstream bays move

Figure 20 – Time window LCO at $\lambda = 830$ for $H/h = 1.25$ concerning the application of *STP* and *STD2* initial conditions.

with out-of-phase motions. It is verified that when the *STP* initial condition is used for both $\lambda = 830$ and $\lambda = 900$, points P1 and P2 present the same motion pattern with only a slight change in the peak values (*cf.* in Figures 19a and 19b). However, when the *STD2* initial condition is applied, the time responses for $\lambda = 830$ and $\lambda = 900$ present a pronounced change in the motion patterns of the upstream and downstream bays. These motion patterns can be observed in Figures 20b and 21b, which also reveals that the upstream bay experiences greater changes in terms of oscillation amplitudes.

Figure 21 – Time window LCO at $\lambda = 900$ for $H/h = 1.25$ concerning the application of *STP* and *STD2* initial conditions.

The LCO branches found through the application of the *STD2* initial condition for this streamwise curvature do not possess a characteristic pattern that remains over

the range of λ , in which the coexistence of different limit cycles were found. Through the application of the *STP* initial condition, which is the ordinary initial condition used to perform panel flutter analysis of curved panels, the bifurcation jumps were only found in the flutter onset point. However, when the initial disturbance is halved by applying *STD2*, the system presents various jump discontinuities and changes in stability, revealing strong sensitivity to initial conditions and complex behaviors associated with the aeroelastic model nonlinearities. In order to fully understand the system dynamics, it is necessary to investigate the response to other initial conditions. The aim of this work was restricted to point out some aspects regarding the complexity and sensitivity to initial conditions to encourage the investigation of shallow shells interaction under supersonic flow. A full investigation of the complex nonlinear behavior of curved multibay panels in supersonic flows requires different techniques, for instance, those from numerical continuation.

5 AEROELASTIC ANALYSIS OF MULTIBAY COMPOSITE CURVED PANELS

This chapter reports and discusses the results regarding the nonlinear dynamic analysis of two-bay curved composite panels, as shown in Figures 3 and 4, under streamwise and cross-stream supersonic airflow. The multibay curved panel is modeled with all edges simply supported, including that between the adjacent bays (as illustrated in Figure 9), that is:

$$\begin{aligned}\hat{u}(x, y, t) = \hat{v}(x, y, t) = \hat{w}_1(x, y, t) &= 0 \quad , \quad \text{for all edges,} \\ \phi_y(x, y = 0, t) = \phi_y(x, y = a, t) &= 0 \quad , \quad \text{and} \\ \phi_x(x = 0, y, t) = \phi_x(x = a, y, t) = \phi_x(x = 2a, y, t) &= 0 \quad .\end{aligned}$$

The material properties are the same as in Ibrahim et al. (2009a) and Ibrahim et al. (2009b): $E_1 = 155 \text{ GPa}$, $E_2 = 8.07 \text{ GPa}$, $G_{12} = 4.55 \text{ GPa}$, $\rho = 1550 \text{ kg/m}^3$, $\nu = 0.22$. Consider $b = 2a$, $h = a/100$, $\mu/M = 0.05$, and H/h determines the panel's curvature ratio, here assumed 0.025, 1, and 2. A symmetric and asymmetric cross-ply laminate were analyzed: $[0, 90, 0]$ and $[0, 90]_3$.

The shallow shell hypothesis implies an initial panel curvature, $\hat{w}_0$. Hence, the initial and net deformations in the z -direction can be summed to determine the vertical displacements, that is: $w(x, y, z, t) = \hat{w}_0(x, y) + \hat{w}_1(x, y, t)$ (*cf.* Equation (2.1)). Here, a simplified notation is adopted to describe the vertical net displacements: $W = \hat{w}_1(x, y, t)$. Figure 9 presents the reference points P1 and P2, at positions $(x/2a = 0.375, y/a = 0.5)$ and $(x/2a = 0.875, y/a = 0.5)$, respectively. The study of the static and dynamic behavior of the panel is performed at points P1 and P2. For a single panel model, the reference point is located at the $\frac{3}{4}$ position along the x -direction.

The following sections discuss the analyses of multibay and single-bay cross-stream shallow shell panels with respect to a set of pre-established curvature values. The streamwise two-bay panels arrangements are also analyzed for the same range of curvature ratios.

5.1 Cross-stream curved panels

The analysis of the cylindrical cross-stream shallow shell represented in Figure 4 is performed by using symmetry, and described by a mesh with 48×12 elements. First, the flutter boundary analysis were performed for both symmetric and asymmetric laminates for all the curvature ratios: $H/h = 0.025, 1$, and 2 . Then, the nonlinear aeroelastic analysis was done. The initial disturbance adopted is a very small distributed displacement imposed to the panels, corresponding to the application of a uniform pressure to its surface, $p = E_2(h/a)^4$. This procedure is the same adopted by many authors concerning flat panels,

and it is also necessary for curved panels with cross-stream flow direction due to the absence of aerodynamic static loads. The Newmark method described in section 2.4.2 was applied with the time step $\Delta t = \frac{10^{-1}}{\omega_0} \left(\frac{\overline{\Delta x}}{2a} \right)^2$, in which $\overline{\Delta x}$ is the minimum element size in the mesh.

5.1.1 Flutter boundary

The multibay panel flutter boundary was determined by solving the eigenvalue problem of Equation (2.91). From Table 6, it is possible to see that the effect of the number of bays in the flutter boundary of curved panels with lower curvature ratio is mild, accordant with the findings of Dowell (1964) and Pacheco et al. (2017b) for flat panels.

Table 6 – Flutter dynamic pressure per curvature ratio.

H/h	$[0, 90, 0]$		$[0, 90]_3$	
	Two-bay	Single-bay	Two-bay	Single-bay
0.025	349.4	349.4	181.9	181.9
1	353.0	353.5	185.9	185.9
2	342.4	346.0	199.8	199.8

From this linear flutter analysis, it is noticed that the flutter boundary for curved panels strongly depend on the lamination scheme and it is also affected by the curvature ratio. Moreover, as discussed by Dowell (1969) and Matsuzaki (1973), the cross-stream curvature is very sensitive to in-plane boundary condition.

5.1.2 $H/h = 0.025$ cross-stream curvature

Figure 22 depicts the bifurcation diagram for the single and the two-bay panel for the curvature ratio of $H/h = 0.025$. The bifurcation diagram comprises the peak-to-peak LCO amplitude values computed at points P1 and P2 versus the dimensionless dynamic pressure. The critical panel flutter dynamic pressure also denotes the so-called Hopf bifurcation point in the bifurcation diagram. The adopted initial condition for the time integration corresponds to a minimal distributed displacement imposed on the panels, corresponding to a uniform pressure $p = E_2(h/a)^4$ to its surface. After the flutter onset, limit cycle oscillations are observed in the panel. For this curvature, the single-bay and the multibay arrangements have the same critical dynamic pressure, λ_f . Similarly to the flat two-bay panel analyzed by Pacheco et al. (2017a), there is a discontinuity in the LCO branch at $\lambda_{jump} = 205$ and 412 for the asymmetric and the symmetric cross-ply laminates, respectively. This jump discontinuities correspond to a sudden, large increase in amplitude. Before the jump, the single bay oscillates with amplitudes similar to the downstream bay.

However, after the jump, the downstream bay's amplitude escalates to nearly three times the value that a single panel would present, while the upstream bay starts to behave like a single panel. Indeed, the upstream bay oscillates in the same way in which a single panel in the same flow condition would.

The response of the $H/h = 0.025$ curvature ratio is similar to the flat two-bay panel analyzed by Pacheco et al. (2017a), which is expected since the panel curvature is very low, leading the system to behave more likely a plate than a shell.

Figure 22 – Bifurcation diagram for a cross-stream two-bay panel with $H/h=0.025$ curvature.

Source: Elaborated by the author.

To analyze the reasons for the sudden jump in amplitude, the flutter motion of both composite laminate panels will be closely examined. Figure 23b shows the deformed panel at midsection ($y = a/2$) for $\lambda < \lambda_{jump}$ and $\lambda = \lambda_{jump}$ for the asymmetric laminate, $[0, 90]_3$, oscillating during flutter. The shapes display the deformed surface of the panel at the positive LCO peak at the reference point P2. Previous to the amplitude jump occurrence, the bays (more precisely, at points P1 and P2) demonstrate in-phase motion, whilst, subsequent to the jump, the bays start to oscillate in almost out-of-phase pattern. Figure 23a confirms this behavior by showing the phase difference angle for all the dynamic pressures presented in the bifurcation diagram of Figure 22. The phase angle corresponds to the difference between the motions of points P1 and P2 of the adjacent bays. The jump phenomenon is a noticeably change in the fluttering mechanism. Prior to the amplitude jump occurrence, the bays appear to vibrate in-phase, while at $\lambda = \lambda_{jump}$ they start to oscillate closer to an out-of-phase motion pattern.

The phase portrait are drawn for the pre- and post-flutter conditions, in Figures 23c and 23d, for the first and the second bay, respectively (represented by points P1 and P2).

It is possible to notice that besides the increase in the oscillations' amplitude, the velocity also increases, which represents a rise in the kinetic energy of the panels.

Figure 23 – Jump phenomenon of the multibay panel with $H/h = 0.025$ and $[0, 90]_3$.

A similar behavior is encountered for the symmetric laminate, $[0, 90, 0]$. The fluttering shapes of Figure 24b and the phase difference angle of Figure 24a demonstrates that the panel's motion pattern changes from in-phase motion (prior to the jump occurrence) to out-of-phase motion. The phase portraits of Figure 24c and 24d concerning the points P1 and P2, respectively, reassures the change in the motion pattern for the pre- and post-jump conditions.

After the understanding of the multibay panel behavior, a closer comparison with the single panel approach is drawn. Figure 25 shows the fluttering shapes and phase portrait of the multibay and single-bay arrangements for the $[0, 90]_3$ laminate. The pre-jump condition, $\lambda = 200$, shows that the single-panel behaves similarly to the downstream bay. After the jump, the downstream bay's amplitudes escalates and the upstream bay, that had lower amplitudes before, starts to behave likely the single-panel. Since the single-panel

does not present any jump discontinuity, it maintains the same motion pattern over the dynamic pressures analyzed. These same conclusions can be drawn for the symmetric laminated, $[0, 90, 0]$, through Figure 26.

Therefore, the single-bay approach has showed to be unsafe to predict the post-flutter behavior, whilst the structural coupling between panels has proven to play an important role in the nonlinear aeroelastic response of curved panels.

Figure 24 – Jump phenomenon of the multibay panel with $H/h = 0.025$ and $[0, 90, 0]$.

(a) Phase difference angle between the motions of points P1 and P2.

(b) Fluttering shapes ($y = a/2$).

(c) Phase portrait at point P1.

(d) Phase portrait at point P2.

Source: Elaborated by the author.

Figure 25 – Fluttering shapes and phase portrait of the multibay and single-bay for $[0, 90]_3$, $H/h = 0.025$, pre- and post-jump conditions.

Source: Elaborated by the author.

Figure 26 – Fluttering shapes of the multibay and single-bay for $[0, 90, 0]$, $H/h = 0.025$, pre- and post-jump conditions.

Source: Elaborated by the author.

5.1.3 $H/h = 1$ cross-stream curvature

The curvature ratio of $H/h = 1$ for both symmetric and asymmetric cross-ply laminate panels are examined here. Similar to the previous curvature ratio, the single panel behavior resembles the second bay in the flutter boundary's vicinity. After the jump discontinuity, the upstream bay starts to behave progressively similar to the single panel, while the amplitudes of the downstream bay escalate. Note that the jump discontinuity presented in Figure 27 is less pronounced than the previous case with $H/h = 0.025$. The reasons for the sudden jump in post-flutter oscillation amplitudes will be examined for both laminates, $[0, 90]_3$ and $[0, 90, 0]$. The first jump discontinuity occurs at $\lambda_{jump} = 250$ for the asymmetric laminate and at $\lambda_{jump} = 448$ for the symmetric laminate.

Figure 27 – Bifurcation diagram for a cross-stream two-bay panel with $H/h=1$ curvature.

Source: Elaborated by the author.

Figure 28 shows the panel's behavior near to the jump discontinuity for the asymmetric laminate, $[0, 90]_3$. Though Figure 28b is possible to see that the panel changes its motion pattern from in-phase to out-of-phase motion after the jump. To confirm that, a phase difference angle plot was drawn in Figure 28a, which shows that the panel shifts from in-phase to roughly out-of-phase motion at $\lambda_{jump} = 250$. Moreover, the phase portraits of Figure 28c and 28d reassure the change in the motion pattern after the jump occurrence. Therefore, this jump discontinuity is related to a change in the panel motion during flutter.

The same type of analysis is performed to investigate the jump discontinuity of the symmetric laminated, $[0, 90, 0]$. It is possible to see that the jump occurs due to a change in the flutter mechanism, which changes from in-phase to out-of-phase motion, according to Fig. 29b and Fig. 29a. The phase portraits of Figures 29c and 29d show a high increase in velocity and vertical displacement for the post-flutter condition. Note that the increase in velocity and vertical displacement are greater to this symmetric laminate compared

Figure 28 – Jump phenomenon of the multibay panel with $H/h = 1$ and $[0, 90]_3$.

(a) Phase difference angle between the motions of points P1 and P2.

(b) Fluttering shapes ($y = a/2$).

(c) Phase portrait at point P1.

(d) Phase portrait at point P2.

Source: Elaborated by the author.

with the asymmetric one, which is a reflection of the amount of energy involved in the jump occurrence.

To have a better understanding of the similarities and discrepancies between the single-bay and multibay approach, the phase portrait and fluttering shapes were drawn for a pre- and post-flutter condition. Figure 30 shows the fluttering shapes and phase portrait of the multibay and single-bay arrangements for the $[0, 90]_3$ laminate. Similarly to the previous curvature ratio of $H/h = 0.025$, the pre-jump condition, $\lambda = 200$, shows that the single-panel behaves similarly to the downstream bay. After the jump, the downstream bay's amplitudes escalates and the upstream bay, which had lower amplitudes before, starts to behave likely the single-bay. The symmetric laminate, $[0, 90, 0]$, presents a similar behavior as reported in Figure 31, with the post-jump condition having a considerably greater kinetic energy than the asymmetric laminate.

Figure 29 – Jump phenomenon of the multibay panel with $H/h = 1$ and $[0, 90, 0]$.

(a) Phase difference angle between the motions of points P1 and P2.

(b) Fluttering shapes ($y = a/2$).

(c) Phase portrait at point P1.

(d) Phase portrait at point P2.

Source: Elaborated by the author.

Figure 30 – Fluttering shapes of the multibay and single-bay for $[0, 90]_3$, $H/h = 1$, pre- and post-jump conditions.

Figure 31 – Fluttering shapes of the multibay and single-bay for $[0, 90, 0]$, $H/h = 1$, pre- and post-jump conditions.

Source: Elaborated by the author.

5.1.4 $H/h = 2$ cross-stream curvature

The two-bay panel with a curvature ratio of $H/h = 2$ is examined here. The bifurcation diagram of the $[0, 90]_3$ and $[0, 90, 0]$ cross-ply laminates are presented in Figure 32, which includes results for the single panel approach. Similar to the previous curvature ratios, the downstream bay presents larger amplitudes compared to the upstream bay. Moreover, as the dynamic pressure increases, the symmetric laminate starts to present a more complex behavior with multiple peak amplitudes, which will be discussed later. Analogous to the previous curvatures, the single panel behavior resembles the second

Figure 32 – Bifurcation diagram for a cross-stream two-bay panel with $H/h=2$ curvature.

Source: Elaborated by the author.

bay in the flutter boundary's vicinity. As the dynamic pressure increases, the upstream bay starts to behave progressively similar to the single panel. However, the multibay symmetric laminate panel starts to have a complex behavior for $\lambda > 550$ with multiple peak amplitudes. To have a better understanding of the dynamic behavior of both panel arrangements, Figures 33 and 34 present fluttering shapes and the phase portrait of the $[0, 90]_3$ and $[0, 90, 0]$ laminates. Near to the flutter boundary, the single-bay behaves like the second bay. With the dynamic pressure increase, the upstream bay amplitudes raise to amplitudes similarly to the single-bay, while the downstream bay amplitudes undergo a larger increase.

As mentioned before, the multibay symmetric laminated panel presents a more complex oscillation pattern for higher dynamic pressures. Figure 35 shows the time responses and phase portrait for $\lambda = 550, 600$, and 625 of the $[0, 90, 0]$ laminate. The motion patterns for $\lambda = 550$ and 600 are similar, besides the change from single-amplitude to multiple peak LCO. However, the behavior becomes more complex when the dynamic pressure increases to $\lambda = 625$.

Figure 33 – Fluttering shapes of the multibay and single-bay for $[0, 90]_3$ and $H/h = 2$.

Figure 36 depicts the times responses for the multibay and single-bay arrangements for $\lambda = 650$. Although the single-bay and the upstream bay have similar oscillation amplitudes, the later presents a complex motion pattern with multiple peak amplitudes, while the single-bay presents single-amplitude LCO. Figure 36c presents the fluttering shapes of the panel's deformed surface regarding the positive LCO peak at the reference point P2. For the multibay panel, the fluttering shapes of a higher and a lower peak amplitude are drawn. These fluttering sections confirms that the single-bay dynamic behavior resembles the upstream bay for the post-jump condition. Moreover, comparing the shapes of the multibay panel presented in Figure 34b ($\lambda = 450$) and Figure 36c ($\lambda = 650$), it is possible to see the change in the motion pattern from in-phase motion to roughly out-of-phase motion.

Therefore, it is possible to conclude that the single panel behavior approximates reasonably well near the flutter onset. However, as the dynamic pressure increases, the interaction between adjacent bays leads the system to a more complex behavior and a drastically increase in the oscillation amplitudes of the downstream bay of the multibay panel arrangement. Accordingly, the single panel approach severely underestimates the

downwind bay's transversal displacement, demonstrating to be unsafe for structural flight vehicle design and to ensure fatigue-life estimation for supersonic flutter suppression.

Figure 34 – Fluttering shapes of the multibay and single-bay for the $[0, 90, 0]$ and $H/h = 2$.

Source: Elaborated by the author.

Figure 35 – Nonlinear dynamic behavior of the two-bay symmetric laminated panel with $H/h = 2$ under cross-stream flow.

Source: Elaborated by the author.

Figure 36 – Fluttering shapes of the multibay and single-bay for the $[0, 90, 0]$ with $H/h = 2$ and $\lambda = 650$.

(a) Time histories response of the multibay panel.

(b) Comparison between the single-bay and the upstream bay of the two-bay arrangement.

(c) Comparison between the fluttering shapes of the multibay (showing a high and low LCO peak amplitude) and the single-bay panel.

(d) Phase portrait of both multibay and single-bay panel arrangements.

Source: Elaborated by the author.

5.2 Streamwise curved panels

Figure 3 illustrates the streamwise cylindrical shallow shell model, whose properties are already furnished in the beginning of this chapter 5. The initial condition for the time solution procedure are the deformations due to the application of the static aerodynamic loads. The same time step $\Delta t = \frac{10^{-1}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$ and 48×12 mesh were applied for all the asymmetric laminated analysis. However, they showed to be inadequate for the symmetric laminated with $H/h=1$ and 2 curvature ratios, who presented a more complex aeroelastic behavior. Therefore, a mesh and time convergence analysis were performed, whose details will be discussed later in this section.

5.2.1 $H/h = 0.025$ streamwise curvature

Figure 37 depicts the bifurcation diagram for the symmetric and asymmetric laminate two-bay panel with $H/h = 0.025$ curvature under streamwise flow direction. The bifurcation diagram comprises the peak-to-peak LCO amplitude values computed at points P1 and P2 versus the dimensionless dynamic pressure. The static displacements due to static aerodynamic loads are set as initial conditions for the time integration of the aeroelastic equations of motion. This initial condition is applied for all simulations, leading to non-zero steady-state deformation in the pre-flutter dynamic pressure region. Differently from cross-stream panels and flat panels, the streamwise panels have damped motion with a gradual static displacement prior to the flutter onset (*cf.* Figure 37). As expected, the very small curvature ratio promotes the critical flutter dynamic pressure of the streamwise multibay shallow shell to be closer to a flat two-bay panel. As verified to cross-stream

Figure 37 – Bifurcation diagram for a streamwise two-bay panel with $H/h=0.025$ curvature.

Source: Elaborated by the author.

panels in section 5.1, both symmetric and asymmetric streamwise case analyzed here also present a jump discontinuity at approximately $\lambda = 397$ and 209 , respectively (cf. Figure 37). In order to investigate such discontinuity, the phase difference angle and the fluttering shapes are drawn. The phase difference angle of Figure 38a and 39a reports the motion pattern concerning the first and second bays of the $[0, 90]_3$ and $[0, 90, 0]$ laminates, respectively. It is possible to see that there is a clear shift in the phase angle from close to in-phase motion to roughly out-of-phase motion for both laminates panels.

Figure 38 – Jump phenomenon of the multibay panel with $H/h = 0.025$ and $[0, 90]_3$ under streamwise flow.

Figure 39 – Jump phenomenon of the multibay panel with $H/h = 0.025$ and $[0, 90, 0]$ under streamwise flow.

The fluttering shapes at the panel midsection ($y = a/2$), Figures 38b and 39b,

for a pre- and post-jump condition confirm the change in the flutter mechanism for the asymmetric and symmetric laminates, respectively.

To better understand the panel's dynamic behavior, the time response and phase portrait of the LCO motion are drawn for different dynamic pressures as shown in Figure 40.

Figure 40 – Nonlinear dynamic behavior of the two-bay symmetric laminated panel with $H/h = 0.025$ under streamwise flow.

Source: Elaborated by the author.

Figures 40 and 41 refer to the symmetric and asymmetric laminated, $[0, 90, 0]$ and $[0, 90]_3$, respectively. It is possible to see the change in dynamic behavior after the jump discontinuity, denoted by the change in oscillation amplitude and velocity.

Figure 41 – Nonlinear dynamic behavior of the two-bay asymmetric laminated panel with $H/h = 0.025$ under streamwise flow.

Source: Elaborated by the author.

5.2.2 $H/h = 1$ streamwise curvature

In this section, the bifurcation diagrams of the laminates will be plotted separately due to its complex behavior. The bifurcation diagram of Figure 42 refers to the $[0, 90]_3$ laminate and shows steady-state deflections occurring for $0 \leq \lambda \leq 276$, which are related to damping oscillations. The pre-flutter equilibrium point of the first bay (represented by point P1) is in the opposite curvature direction, while the second bay (represented by point P2) deflects in the curvature direction. The steady-state equilibrium deflections and the flutter onset highly increased compared with the previous case with $H/h = 0.025$ curvature ratio. This behavior occurs due to the increase in the multibay panel curvature that raises the panel stiffness and aerodynamic static loads. The flutter onset occurs near $\lambda = 277$ for the asymmetric laminate panel, characterized by a violent jump to large LCO amplitudes. Accordingly, the flutter onset point for this case with $H/h = 1$ reveals that the increase in curvature ratio leads to higher critical flutter dynamic pressures. By looking at the post-flutter peak-to-peak amplitudes of the limit cycle oscillations, it is revealed that the downstream bay has similar positive and negative peak amplitude values. In comparison, the upstream bay has greater deflections in the opposite direction of the panel curvature. Figure 43 shows the aeroelastic response at $\lambda = 300$ and $\lambda = 400$ for

Figure 42 – Bifurcation diagram for a streamwise two-bay panel with $H/h=1$ curvature, $[0, 90]_3$.

Source: Elaborated by the author.

the asymmetric laminate. Although the displacement of the reference points P1 and P2 remains almost at constant amplitude, the amplitude of the velocity still fluctuates as can be seen through the phase portraits.

Figure 43 – Nonlinear dynamic behavior of the two-bay asymmetric laminated panel with $H/h = 1$ under streamwise flow.

Source: Elaborated by the author.

All the results until this point were obtained with the 48×12 mesh and the time response of $\Delta t = \frac{10^{-1}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$, which showed to be enough to guarantee satisfactory results. However, the symmetric laminated panel with $H/h = 1$ presented a more complex behavior and these parameters demonstrated to be inadequate. Figure 44 shows the results of the multibay panel under the same flow condition ($\lambda = 600$) for various meshes and $\Delta t = \frac{10^{-2}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$. The 48×12 and 64×16 meshes show some discontinuities in the fluttering shapes, which indicates some numerical issue that may compromise the accuracy of the solution. The first insight is that the time step was not properly chosen, since the Newmark integrator is conditionally stable, requiring a minimum time step for convergence. Therefore, it is necessary to perform a time convergence analysis to verify if the time step is the source of the identified numerical issue. The finer the mesh, the shorter is the time required for convergence, Eq. (2.105). So, the 32×8 mesh was used to perform this time convergence analysis to reduce the computational cost. Although a coarse mesh may provide less accurate results, it is still capable of reproducing the same behavior

pattern that a convergent analysis with a fine mesh would. Figure 45 compares the results of three different meshes with $\Delta t = \frac{10^{-2}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$ for $\lambda = 700$ (which possesses a simpler behavior with single-amplitude LCO) and show great agreement. With this in mind, the next step was to reassure that the chosen time step was adequate. Figure 46 shows the time responses for the 32×8 mesh with $\Delta t = \frac{10^{-1}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$, and $\Delta t = \frac{10^{-2}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$, whose results show good agreement. Finally, the 32×8 mesh and $\Delta t = \frac{10^{-2}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$ were used to perform the analysis presented in Figure 47. A finer mesh with an adequate time step would provide more accurate results, however, the computational cost would be too high because the time step necessary for convergence significantly decreases as the mesh is refined.

Figure 44 – Convergence analysis of the two-bay symmetric laminated panel with $H/h = 1$ under streamwise flow at $\lambda = 600$.

Source: Elaborated by the author.

The bifurcation diagram of Figure 47 depicts the LCO peak-to-peak values at the references points P1 and P2 for the symmetric laminated panel. Differently from the asymmetric laminated that presents only one jump from steady-state deflections to high LCO amplitudes, the symmetric laminated also presents a jump discontinuity at a higher

Figure 45 – Convergence analysis of the two-bay symmetric laminated panel with $H/h = 1$ under streamwise flow at $\lambda = 700$.

Source: Elaborated by the author.

Figure 46 – Convergence analysis of the two-bay symmetric laminated panel with $H/h = 1$ and 32×8 mesh under streamwise flow at $\lambda = 610$.

Source: Elaborated by the author.

dynamic pressure. Figure 48 provides a better understanding of the nonlinear dynamic behavior of the two-bay symmetric laminated panel. The pre-jump condition at $\lambda = 610$ shows that the upstream bay presents two-peak amplitudes per oscillation period in the opposite curvature direction, while the downstream bay oscillates mainly in the curvature direction with single-amplitude. The jump characterizes the change in the motion pattern, specially for the upstream bay. The closed orbits presented in the phase portraits plot confirm the periodicity of the solution.

Figure 47 – Bifurcation diagram for a streamwise two-bay panel with $H/h=1$ curvature, $[0, 90, 0]$.

Source: Elaborated by the author.

Figure 48 – Nonlinear dynamic behavior of the two-bay symmetric laminated panel with $H/h = 1$ under streamwise flow.

Source: Elaborated by the author.

5.2.3 $H/h = 2$ streamwise curvature

The bifurcation diagram of Figure 49 refers to the $[0, 90]_3$ laminate and shows steady-state deflections occurring for $0 \leq \lambda \leq 302$, which are related to damped oscillations. The steady-state equilibrium deflections and the flutter onset increased ($\lambda_f = 303$) compared with the previous curvature ratio of $H/h = 1$ ($\lambda_f = 277$). However, the behavior is similar for both curvatures: steady-state deflections prior the flutter onset, followed by a violent jump to large LCO amplitudes at λ_f . Note that for $\lambda = 500$, the aeroelastic response presents multiple peaks amplitudes, the lower amplitudes are represented by unconnected markers in the bifurcation diagram of Figure 49. Figure 50 shows the aeroelastic response

Figure 49 – Bifurcation diagram for a streamwise two-bay panel with $H/h=2$ curvature, $[0, 90]_3$.

Source: Elaborated by the author.

at $\lambda = 350$ and $\lambda = 500$ for the $[0, 90]_3$ laminate, which reassures the similarity with the dynamic behavior of the previous curvature ratio of $H/h = 1$. The symmetric laminate with $H/h = 2$ curvature ratio analyzed with 48×12 mesh and $\Delta t = \frac{10^{-2}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$ presents the same numerical issue found for the previous case with $H/h = 1$. Therefore, a mesh and time convergence analysis were performed to investigate this matter. Refining the mesh to 64×16 and using the time step $\Delta t = \frac{10^{-2}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$ did not provide reliable results. Using a coarse mesh 32×8 and reducing the time step to $\Delta t = \frac{10^{-3}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$ only provides reliable results for the lower dynamic pressures, until $\lambda = 700$. For higher dynamic pressures, the same numerical matter presented in Figure 44c was found.

It was experimentally determined that the minimum time step necessary to provide reliable results for nonlinear analysis is affected by the response period. Akay (1980) studied the effect of the load magnitude in the accuracy and stability of the Newmark's constant average acceleration method. With the load increase, the plate's response period

Figure 50 – Nonlinear dynamic behavior of the two-bay asymmetric laminated panel with $H/h = 2$ under streamwise flow.

Source: Elaborated by the author.

decreases considerably, whereas the adequate time step to provide stable and accurate solutions is affected by the response period. To analyze the effect of the load magnitudes, the plate parameters and the mesh were fixed, varying only the load magnitude, which consists of a uniform pulse load. As the load increased, the minimum time step necessary to provide a stable solution considerably decreased. Bringing that to the problem at hand, whether the dynamic pressure or the panel curvature increase, the static aerodynamic loads increase, consequently the time step should decrease. Therefore, a detailed time convergence analysis would be valuable. However, due to the high computational cost, just the simple convergence analysis mentioned early was performed.

Figure 51 presents the peak amplitudes obtained with the 64×16 mesh and $\Delta t = \frac{10^{-2}}{\omega_0} \left(\frac{\Delta x}{2a} \right)^2$ time step. This figure records the trends of maximum and minimum peaks without capturing all the relative peaks because it would be impossible to give a clear peak-mean trend due to the great number of relative peaks. Depending on the

dynamic pressure, results show chaotic or quasi-periodic motion. Figure 52 depicts the time responses and the phase portraits for the following λ values: 800, 900, 950, and 1200. For $\lambda = 800$, there is no periodicity and the behavior resembles chaotic motion. The damping is not sufficient to get the system back to equilibrium and the panel maintains a disorderly oscillatory motion with a high kinetic energy level. The upstream bay has negative amplitudes (opposite to the curvature direction), while the downstream bay vibrates in the curvature direction. A similar behavior is encountered for $\lambda = 900$. Note that for $\lambda = 950$, the oscillation amplitudes of the downstream bay highly increases, while the upstream bay maintains a chaotic motion with lower oscillation amplitudes. For $\lambda = 1200$, the bays seem to have entered in quasi-periodic motion. From Figure 52 it is possible to see that both the vertical displacements and velocity fluctuates.

Figure 51 – Bifurcation diagram for a streamwise two-bay panel with $H/h=2$ curvature, $[0, 90, 0]$.

Source: Elaborated by the author.

Figure 52 – Nonlinear dynamic behavior of the two-bay symmetric laminated panel with $H/h = 2$ under streamwise flow.

6 FINAL REMARKS

6.1 Conclusions

The present study has presented a numerical investigation on the nonlinear aeroelastic behavior of reinforced double cylindrical panels subjected to supersonic flow. The analysis addresses the effect of some important parameters such as the flow direction, curvature ratio, and lamination scheme in the panel's aeroelastic behavior. The main goal is to study how the nonlinear coupling between adjacent panels influences the panel flutter phenomenon. To achieve this, the panel flutter behavior of double cylindrical shallow shells was analyzed through the aeroelastic model implemented.

The nonlinear Mindlin theory is employed to describe the cylindrically curved panel. Geometric nonlinearity is attained with the Von Kármán strain-displacement relations. The first-order piston theory is applied to account for the steady pressure due to the panel curvature and the unsteady aerodynamic pressure loading acting transversely on the structure. The structural and aerodynamic models are combined by the Principle of Virtual Work and discretized with the Finite Element Method (FEM). The set of aeroelastic differential equations are integrated by means of a Newmark method. The model and solvers have been verified through comparison with several benchmark results from the literature and the commercial finite element software ABAQUS. The comparative study has been performed in multiple levels: linear analysis (vertical deflections and fundamental frequencies), structural nonlinear plate vibration, flutter boundary prediction, and nonlinear aeroelastic simulation. Results have demonstrated that the developed code is capable of yielding accurate solutions for thin and thick, isotropic and laminated panels, in both linear and nonlinear regimes.

Regarding the isotropic analysis, cross-stream and streamwise two-bay panels with $H/h = 1.25$ were analyzed. Two jump discontinuities were found for the cross-stream panel with $H/h = 1.25$, and different LCO amplitude branches were detected by applying distinct initial conditions. Investigations through the fluttering shapes and phase difference angle between the bays showed that the first discontinuity is simply due to the system's sensitivity to initial conditions. In contrast, the second jump represents a change in the flutter mechanism. The streamwise isotropic two-bay panel presents static deflections in the pre-flutter region due to the static aerodynamic loads. It has demonstrated strong sensitivity to initial conditions and a complex nonlinear behavior. Various jump discontinuities and sudden changes in stability were found through the application of the *STD2* initial condition, while the *ST* initial condition presents only one jump in the transition between static solution and periodic solutions with high amplitudes. The results obtained through the application of the *STD2* initial condition has revealed

different regions of LCO limited by jumps. A small range of dynamic pressures within the LCO region has shown to present steady-state equilibrium. Therefore, the curved multibay panels has revealed complex nonlinear dynamics and a more detailed study is necessary to enhance the understanding on this kind of system.

Regarding the linear and nonlinear analysis of composite laminated panels, the cross-stream and streamwise configurations were analyzed for the following curvature ratios: $H/h = 0.025$, 1, and 2. For the cross-stream model, the aerodynamic load acting upon the panel surface corresponds to the unsteady aerodynamic pressure. It does not have static aerodynamic loads as in the streamwise model. Two laminate schemes were analyzed: symmetric and asymmetric cross-ply laminate, with the same material parameters.

The $H/h = 0.025$ curvature is regarded as a smooth modification of the flat multibay panel, so it behaves more like a plate than a shell. The LCO amplitudes are closely symmetric to $W/h = 0$ for the upstream bay for both symmetric and asymmetric laminate, cross-stream and streamwise flow direction. Jump discontinuities in the oscillation amplitudes were also identified due to the nonlinear dynamic interaction between adjacent bays. These discontinuities are related to the abrupt changes in the flutter mechanism as a result of the phase-shifting from in-phase to out-of-phase oscillations.

The behavior of the multibay cross-stream model with $H/h = 1$ is similar to the previous curvature ratio: a progressive increase in the LCO amplitudes after the flutter onset, followed by a sudden jump in amplitude, for both symmetric and asymmetric laminated. Concerning the jump phenomenon, the symmetric laminated showed greater velocity and vertical displacements than the asymmetric laminate for this curvature ratio. Regarding the streamwise model, the nonlinear aeroelastic behavior of the symmetric and asymmetric laminated is quite different. The asymmetric laminated presented a violent jump from steady-state deflections to large LCO single-amplitudes. Right after the flutter onset, the symmetric laminated presented limit cycle oscillations with two-peak amplitudes per period for the upstream bay, then suddenly jumps to higher LCO single-amplitude. Therefore, the overall behavior of the symmetric laminated showed to be more complex.

Concerning cross-stream multibay panels with $H/h = 2$ curvature ratio, no sudden jump in the LCO was found. However, as the dynamic pressure increases, the symmetric laminated multibay panel starts to present a more complex behavior with multiple peak amplitudes. The behavior of the multibay streamwise asymmetric laminated panel is similar to the previous curvature ratio, with an increase in the critical flutter dynamic pressure from 277 to 303, for $H/h = 1$ and 2, respectively. However, the multibay streamwise symmetric laminated presented a complex behavior. Also, a numerical issue concerning the convergence of the solution was encountered and requires further investigation. The inaccurate preliminary results show mainly chaotic motion.

Moreover, all the multibay cross-stream results were compared with the equivalent

single-bay approach to understand better the similarities and discrepancies between these arrangements concerning the aeroelastic behavior. It was possible to conclude that the single panel approximates reasonably well near the flutter onset. However, it drastically underestimates the amplitudes of the second bay as the dynamic pressure increases. This great increase in the multibay panel's oscillation amplitude is a result of the nonlinear structural interaction between adjacent bays, which also leads the system to complex behavior.

This work has investigated the linear and nonlinear (especially) aeroelastic behavior of reinforced isotropic and composite laminated panels. The multibay model showed a more complex dynamic behavior and demonstrated that the single panel model, which has been widely explored in the literature, cannot predict critical behaviors potentially capable of leading to structural failure. The present results add a new perspective on the panel flutter behavior by assessing some important structural frameworks.

6.2 Suggestions for future works

This work have investigated the effect of the nonlinear structural interaction between adjacent bays on the flutter of curved panels in high speed supersonic flow for two-bay panels arrangements. Would be interesting to investigate the effects of a higher number of bays and also the effects of unequal bay lengths. In addition, it is suggested to investigate the nonlinear aeroelastic response of multibay curved panels under thermal effects, since the temperature effects becomes significant when an aerospace vehicle reaches supersonic speeds.

The dynamic behavior of multibay shallow shells showed to be complex and highly sensitive to initial conditions. Therefore, a full investigation of the complex nonlinear behavior of curved multibay panels in supersonic flow would be valuable, which requires different techniques, for instance, those from numerical continuation.

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