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Diferentes abordagens à Composição e ao Ambiente das Estrelas de Nêutrons

*Different approaches to the Composition of Neutron Stars and their
Environment*

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*Different approaches to the Composition of Neutron Stars and their
Environment*

Tese apresentada ao Departamento de Astronomia do Instituto de Astronomia, Geofísica e Ciências Atmosféricas da Universidade de São Paulo como requisito parcial para a obtenção do título de Doutor em Ciências.

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To Rosi and to my family: without them this Thesis would not have been possible.

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“Seeing. One could say that the whole of life lies in seeing - if not ultimately, at least essentially. To be more is to be more united (...) But unity grows, and we will affirm this again, only if it is supported by an increase of consciousness, of vision. That is probably why the history of the living world can be reduced to the elaboration of ever more perfect eyes at the heart of a cosmos where it is always possible to discern more. Are not the perfection of an animal and the supremacy of the thinking being measured by the penetration and power of synthesis of their glance? To try to see more and to see better is not, therefore, just a fantasy, curiosity, or a luxury. See or perish. This is the situation imposed on every element of the universe by the mysterious gift of existence. And thus, to a higher degree, this is the human condition.”

“The Human Phenomenon” - Pe. Teilhard de Chardin.

“A great discovery does not issue from a scientist’s brain ready-made... it is the fruit of an accumulation of preliminary work.”

Marie Curie

Resumo

Mesmo depois de 80 anos de pesquisas intensas, a composição das estrelas de nêutrons permanece desconhecida, uma vez que tanto a matéria densa do interior desses objetos compactos quanto a matéria se movendo em torno deles encontram-se em condições físicas extremas, irreproduzíveis em laboratórios terrestres.

Nessa Tese, seguimos quatro diferentes caminhos interconectados para abordar esses objetos extremos. Primeiramente, exploramos a estrutura matemática das estrelas de nêutrons, construindo soluções parametrizadas unicamente pela densidade central, apropriadas para estudar o comportamento estrutural dessas estrelas em diversas situações. Em seguida, adotamos uma abordagem nova, a teoria da informação, para inferir uma hierarquia de equações de estado, mostrando que as estrelas de quarks seriam, por sua conformação, favorecidas na Natureza. Estudando a emissão em raios-X advinda do sistema binário de baixa massa 4U 1608–52, que contém uma estrela de nêutrons, limitamos o tamanho físico da fonte emissora, mostrando que não deve estar longe da superfície da estrela compacta. Para isso, empregamos uma técnica inédita no cálculo dos *time lags*. Por fim, mostramos que é possível obter restrições à massa do quark estranho e ao *gap* de energia da CFL diretamente das observações.

Concluimos essa Tese com a afirmação de que a matéria estranha é, estrutural e energeticamente, favorecida pela Natureza, muito embora exista uma barreira entrópica a ser superada e uma densidade central mínima a ser atingida logo após o colapso da estrela progenitora. Se essa barreira foi superada na Natureza, apenas observações futuras mais refinadas dirão.

Abstract

Even after 80 years of intense research, the composition of neutron stars remains unknown, since both the dense matter in the interior of these compact objects and the matter moving around them are in extreme physical conditions, unreproducible in terrestrial laboratories.

In this Thesis, we follow four different interconnected ways to approach these extreme objects. First, we explore the mathematical structure of neutron stars, building solutions parametrised solely by the central density, what are very appropriate to study the structural behaviour of these stars in different situations. Then, we adopted a novel approach, the information theory, to infer a hierarchy of equations of state, showing that quark stars would be, by its configuration, favoured by Nature. Studying the X-ray emission arising from the low-mass binary system 4U 1608-52, which contains a neutron star, we limit the physical size of the emitting source, showing that it should not be far from the surface of the compact star. For this, we employed a technique to calculate the time lags never used before. Finally, we show that it is possible to obtain restrictions on the strange quark mass and the energy gap of the CFL directly from observations.

We conclude this Thesis with the statement that the strange quark matter is, structurally and energetically favoured by Nature, though there is an entropic barrier to be overcome and a minimum central density to be reached just after the collapse of the progenitor star. If this barrier is actually overcome in Nature, only refined observations will tell.

List of Figures

1.1	Concept of pulsar.	22
1.2	SED distribution of the Crab system.	23
1.3	Bimodal distribution of pulsars.	24
1.4	Recent neutron stars mass measurements.	26
1.5	Mass-radius diagram.	29
1.6	LMXBs components.	30
1.7	Roche lobes.	30
1.8	Atoll and Z sources.	33
1.9	Variability-luminosity-spectra relations.	33
1.10	Colour-colour diagram: variability components.	35
1.11	kHz QPO frequencies <i>vs</i> spectral hardness.	36
1.12	“Colour-colour-colour” diagram: rms amplitude.	37
1.13	“Colour-colour-colour” diagram: 2-200 keV luminosity.	38
1.14	Colour coordinate representing the position along the atoll track.	39
1.15	RMS-RMS diagram.	40
1.16	The structure of the atom.	41
1.17	Elementary particles predicted by the Standard Model.	42
1.18	The building blocks of matter.	42
1.19	Baryons and mesons.	43
1.20	Properties of the interactions.	43
1.21	Isospin symmetry.	44
1.22	Common species inside a neutron star.	47
1.23	SLy4 equation of state.	51

1.24	Mass-radius relation for the SLy4 equation of state.	51
1.25	Equation of state of MIT Bag Model.	54
1.26	Mass-radius relation for MIT Bag Model equation of state.	54
3.1	Intuitive definition of complexity.	84
4.1	Atmospheric windows for the electromagnetic spectrum.	94
4.2	Mirror assembly to focus X-rays.	95
4.3	Variability components.	97
4.4	Constraints on the EoS by kHz QPOs.	99
4.5	PDSs of eight data segments.	101
4.6	Average of the eight PDSs: the PDS for the whole observation.	101
4.7	Average of the eight PDSs after the shift-and-add technique: the PDS for the whole observation.	102

Contents

1. <i>Introduction</i>	17
1.1 What is going on in this Thesis	20
1.2 Observation: pulsars and the neutron stars	22
1.2.1 The problem of the masses and the necessity of radii measurements	26
1.2.2 Low Mass X-ray Binaries	29
1.3 Nuclear and Particle Physics: theory of neutron star matter	41
1.3.1 The Equations of State used in this Thesis	49
1.3.1.1 SLy4	49
1.3.1.2 Strange Quark Matter	52
2. <i>Analytical solutions in the construction of strange quark stars models</i>	57
2.1 Exact and quasi-exact models of strange stars	61
3. <i>Information theory and measurements to infer a hierarchy of equations of state .</i>	81
3.1 Entropy, complexity and disequilibrium in compact stars	87
4. <i>X-ray astrophysics as a tool to study kilohertz quasi-periodic oscillations and time lags of the X-ray emission and to probe the environment of neutron stars</i>	93
4.1 X-ray astrophysics	93
4.2 Shift-and-add	99
4.3 The coherence function and the phase and time lags	102
4.4 Time lags in the kilohertz quasi-periodic oscillations of the low-mass X-ray binary 4U 1608–52	104

5. *Conclusion* 117

Bibliography 121

Appendix 127

A. *QCD Parameters* 129

 A.1 Self-bound models of compact stars and recent mass-radius measurements . 129

Introduction

Neutron stars, one of the most exotic objects in Universe. Inside them, a sea of still unknown particles in the densest form in Nature is supported against further gravitational collapse by the pressure of degenerated matter (pure neutrons, at least in the simplest form). Their very existence, though, was hypothesised by the great Russian physicist Lev Landau in 1932, while Baade and Zwicky (1934) were the first to propose a mechanism for neutron star formation, during their study about supernovae, all this in the two years following the discovery of the neutron by Chadwick (1932a,b). However, in spite of being one of the possible end points of stellar evolution, the true composition of these compact objects remains uncertain after almost 80 years of intense studies.

The birth of neutron stars is marked by the death of a massive progenitor star with mass between $8M_{\odot}$ and $25M_{\odot}$ in a cataclysmic explosive event that releases at least $\sim 10^{53}erg$ of energy, sufficient to overshadow the bright of an entire galaxy. After the contraction of the iron core of the progenitor star, a neutron star with $M \sim 1,4M_{\odot}$ and $R \sim 10km$ remains. The central density is certainly above the nuclear saturation value, or $\sim 2.4 \times 10^{14}g/cm^3$.

Neutron stars are natural laboratories to test two of the most fundamental theories of physics: the Einstenian theory of gravitation, the General Relativity, in the strong field regime and the physics of matter at very high densities and temperatures. Both regimes cannot be achieved in laboratories on Earth.

However, the first evidence of their realization in Nature came from the observation of the pulsar PSR 1919+21 (Hewish et al., 1968). A pulsar is a fast spinning object that releases pulses of energy at a very precise and constant rate, generally seen in radio wavelengths, but also in other wavelengths like optical, X-rays and γ -rays.

From the 1930's (the Baade and Zwick's work) to the 1960's (the first pulsar discovery) the neutron star physics remained as an exotic and speculative research field, mostly because of the lack of methods and technology to access information about the interior from direct observations. It was realized from the beginning that in order to theoretically study the neutron stars one needed of General Relativity to solve for the interior structure of these objects, since the value of the compactness factor, $GM/c^2R \sim 0.2$, is very high compared to common stars: roughly 1,000 times higher than the value for a white dwarf and 100,000 higher the same ratio for the Sun. The compactness factor is a measure for the strength of the gravitational field around the star. In addition, the natural energy scale is in the X-ray band, not in the visible.

The difficulty of using the General Relativistic equations comes from the fact that we have to use the Einstein's Field Equations for a Spherically Symmetric Perfect Fluid from which derive the relativistic equation for the hydrostatic equilibrium. The latter is a first order non-linear partial differential equation that must be solved coupled with the mass equation and the equation of state of the fluid.

The first real attempts for solving the internal structure of neutron stars were the works of Tolman (1939) and Oppenheimer and Volkoff (1939). They studied the solutions for the following set of equations:

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \left(1 + \frac{p(r)}{c^2\rho(r)}\right) \left(1 + \frac{4\pi r^3 p(r)}{c^2 m(r)}\right) \left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}, \quad (1.1)$$

$$\frac{dm}{dr} = 4\pi r^2 \rho(r), \quad (1.2)$$

$$\rho = \rho(r) \quad (1.3)$$

where the first is the relativistic hydrostatic equilibrium equation, the second is the mass integral and the third is some functional form of the density profile (or of one of the other quantities or some combination of them).

We will see that one of the most general ways of solving the Tolman-Oppenheimer-Volkoff (TOV) equations is substituting the third equation by the Equation of State (EoS) of the dense neutron star matter. It is the equation of state that provides the microphysics or the composition of the star (see chapter 2).

And here we face the second major difficulty: the behaviour of matter is relatively well known until densities of about $\sim 5 \times 10^{14} g/cm^3$, which is pretty high. However, neutron stars' densities can reach values as high as $\sim 20 \times 10^{14} g/cm^3$ and we cannot reproduce such densities in laboratory. In other words we do not know the true equation of state for neutron stars. For further discussion about this problem we refer to subsection 1.3 and to chapter 2.

On the other hand, neutron stars do appear in Nature and we can detect their signatures by means of astrophysical measurements, now developed and quite reliable. Astrophysics, then, provides a window to address many of the aspects of the dense matter we cannot address on Earth. Here we face the third major difficulty: because their small size and the extreme physical conditions regarding their own existence, it is a real technological challenge to our telescopes and detectors (see chapter 4 for some discussion about the problematic).

Neutron stars appear in different astrophysical systems: as isolated neutron star, generally detected as pulsars; in binary systems with an ordinary star (high mass and low mass X-ray binary systems, depending on the mass of the companion); and in binary compact star systems (a neutron star and another compact object either another neutron star or a white dwarf or a black hole). Hence, we can study neutron star in several wavelengths, sometimes simultaneously, with different techniques.

1.1 What is going on in this Thesis

...and the supremacy of the thinking being measured by the penetration and power of synthesis of their glance?...

Synthesis. From this word you readily perceive our intent in this Thesis. You readily get a sense of our beliefs and soon you will see how we convey our idea of unity.

Neutron stars have not been deciphered yet, being at the limits of our theories and far out from the scope of the direct experiments. For more than 80 years, scientists around the world have been studying neutron stars, one of the most exotic objects in the Cosmos by means of general relativity, particle and nuclear physics and astrophysics. Independently, each approach bore amazing fruits toward a better understand of the physics of dense matter, the nature of space-time and stellar evolution, with consequences that extend far to other fields of astrophysics and physics like, for example, the chemical abundances and the formation of the elements of periodic table, the chemical evolution of our galaxy and even the origin of the elements necessary to life, to quote only a few.

Our technology has evolved a lot since the 1930s. We are now at the position we are able to reach the edge of present knowledge and effectively test the limits of the theories that mould the physical Reality, the theory of General Relativity and the Nuclear Physics.

It is now time for a new synthesis. In this Thesis, we “hedge our bets” on neutron star composition by applying different methods of studying compact objects, some orthodox, some completely new, forging them into a coherent whole, leading us a step further in the comprehension of this startling neutron stars.

We first present a historic survey through the astrophysical events and hypothesis about neutron stars (1.2), the association neutron star/pulsar/supernova and the problematic of how we could pick up hints about the internal composition of neutron stars through observations (1.2.1). We also give some basis to one kind of system that contains a neutron star, showing how the astrophysical observables are linked together in an inseparable whole (1.2.2). Then we talk about the theory of neutron star matter, necessary to understand what we can expect about the composition, which is given by the Equation of State (1.3).

After these introductory sections we present our unified view, starting with the mathematical framework of neutron stars. In chapter 2 we show how to construct quark star models with a minimum of numerical calculations, exploring exact and quasi-exact solu-

tions in order to give way to derivation of physical properties. We emphasize the role of one of most promising exact solutions (the anisotropic solution) and of the quasi-exact solution in allowing more profound theoretical studies. In fact, we selected the anisotropic solution as the best for what we present in the following chapter.

Thus, in chapter 3 we examine the possibility that quark stars are preferred in Nature provided an effective formation channel, the nucleation of SQM, is confirmed. The methodology employed there, *information theory*, is unusual and its use in astrophysics is in the very beginning. However, the outputs are promising, pointing towards an affirmative answer in favour of the exotic (strange) quark stars. Strange quark stars and their siblings hadronic stars are similar in mass and radius and, isolated, it is difficult to overrule one in favour of the other. Then, it is necessary to study their environments (and now we understand the title of this Thesis) to look for characteristic signals of these two types of stars and to decide which kind of them we are really talking about. The real meaning of what we have just said is obvious: we are deciding the internal composition.

Finally, in chapter 4 we explore how to probe the immediate environment of a class of binary sources (the low mass X-ray binaries) to give one method to extract the radii using the the so-called kilohertz quasi-periodic oscillations (kHz QPOs) frequencies to further test the mass-radius relation, as done in other studies. We also study the time lags and the coherence of the X-ray emission to find out where they are produced: if in an electron corona or in the disc, for example. Constraints on the emission region opens a direct window to the mass and radius of the neutron star in these systems through models of disc reflection/thermalisation and models of scattering off in Compton clouds, for example.

An alternative form, which employs just the mass-radius of exotic models including pairing between quarks, is included in the Appendix. There, we extract limits on the values of the parameters of quantum chromodynamics directly from observations, discussing in the ways some caveats of radii determinations.

Other approaches are available and being developed like the link of the thermodynamic entropy and the leakage of information during the contraction of the iron core just after the supernova explosion. Also, we have a larger dataset of another low mass X-ray binary system where we have much better statistics to detect and study the quasi-periodic oscillations and time lags.

1.2 Observation: pulsars and the neutron stars

As we have already said, the first evidence that neutron stars appear in Nature came from the observation of the pulsar PSR 1919+21. In what follows we identify pulsars as neutron stars, but recall that not every neutron star is a pulsar. “Pulsar” denotes a compact source that emits radiation in pulsed signals, or in well defined time intervals, like a beacon.

It is generally accepted, as elaborated over the years (Woltjer, 1964; Pacini, 1967; Gold, 1968), that pulsars are neutron stars with high magnetic field spinning at rates that ranges from $\sim 1\text{ms}$ to $\sim 2\text{s}$. However, a pulsar is only theoretically characterized if there is a tilt of a few degrees between the rotation axis and the magnetic field axis. What we see in such situation is the electromagnetic radiation emitted by the magnetic cone swept out by the star (although we do not have any self-consistent model for the emergence of the pulses). Otherwise there is no pulsar at all. See figure 1.1 for a schematic view.

(a) Schematic pulsar

(b) Observed pulses. The average is the period.

Figure 1.1: The concept of pulsar. We observe pulses only when the radiation cone is tilted in relation to the rotation axis and if it passes through our sight of view. We see, in reality, what is displayed in panel *b*.

They emit mostly and were discovered by their radio emission, but they can emit in the full electromagnetic spectrum, like the well-studied example in the Crab nebula (see figure 1.2).

Figure 1.2: Notice that the SED distribution of the Crab system covers all the electromagnetic spectrum.

By the conservation of magnetic flux from the progenitor to the compact object during the stellar core contraction, the magnetic fields increases, ranging from $\sim 10^8 G$ to $\sim 10^{14} G$ in the final configuration. On the other hand, the rotation periods decreases, because the star spun-up by conservation of angular momentum, lying in the range from $\sim 1 ms$ to $\sim 2 s$. A very interesting feature regarding the pulsar periods become apparent when we plot the derivative of the rotation period versus the rotation period together with the magnetic field strength lines: we clearly see a bimodal distribution or two populations of pulsars (see figure 1.3).

The majority of pulsars we see has periods about $0.7 s$ (but see below some biases and caveats), high magnetic fields ($\sim 10^{12} G$) and are usually isolated. The second population are the so-called millisecond pulsars with periods $\sim 1 ms$ and weak magnetic fields ($\sim 10^8 G$), appearing commonly in binaries. The first class is believed to be a younger population or classical pulsars, formed just after the supernova explosion.

An evolutionary path to explain this bimodal distribution is certainly believed to exist. During the pulsar's life, angular momentum is lost to radiation and the pulsar slows down and emits less intensely with time. The pulsar slowly approaches the so-called Death Line,

Figure 1.3: The bimodal pulsar distribution: the classical pulsars have high magnetic fields and larger periods, while the millisecond pulsars have weak field and spin very fast. The $\log(P) - \log(\dot{P})$ distribution of pulsars (black dots) (ATNF Pulsar Catalogue data). Binary pulsars are marked by a circle. SGRs and AXPs are marked by stars and triangles, respectively. The lines are: constant surface dipole magnetic field strength (dashed) and characteristic ages (dotted). The arrows indicate a measurement of the braking index. The death line is the pair-creation limit for generating radio pulses. Figure from F. Lamb and W. Yu, astro-ph/0408459.

fading in the process until certain combination of the strength of the magnetic field and the rotation period is attained. Then the pulsar turns-off. However, when there is a low mass companion star, an accretion disc forms at a time when the pulsar has long turned-off. This event marks a new era in the pulsar's evolution. The accreted matter transfers angular momentum to the neutron star, which spins up. When, again, a certain combination of the magnetic field strength and rotation period is attained, the pulsar revives as a millisecond pulsar (Glendenning, 2000).

The subject of the astrophysical part of this Thesis is the so-called Low Mass X-ray Binaries (LMXBs), thought to be the ancestors of the millisecond pulsars. We will discuss

them later on this chapter.

Pacini (1967) and Gold (1968) point to the fact that highly magnetized rotating compact stars must radiate an enormous amount of energy ($\sim 10^{38} \text{erg/s}$) and even more must be stored as rotation energy ($\sim 10^{48} \text{erg}$). Calculating the energy input that must be accelerating the Crab remnant (as observed in several wavelengths) we can derive important relations among the quantities involved in the radiation mechanisms from the pulsar. For instance, we get from the Crab pulsar:

$$P \sim \frac{1}{30} \text{s}, \quad \dot{P} \sim 4 \times 10^{-13} \text{s/s} \quad (1.4)$$

and average density of

$$\rho \sim 1.5 \times 10^{14} \text{g/cm}^3 \quad (1.5)$$

for an estimated radius $\leq 19 \text{km}$ and for an assumed mass of about $2M_{\odot}$.

The existence (and observation) of the quantity $\dot{P}$ eliminates vibrations as the origin of the pulsations. Because this average density, together with the calculated compactness and the decreasing monotonicity of the density with radius (learned from TOV equations) and together with the fact that such compact object must be charge neutral (composed of neutrons, therefore), we conclude that pulsars must be the so long sought neutron stars.

On the other hand, it is hard to associate pulsars/neutron stars with supernova remnant due to the large kick velocity that the pulsar acquires just after its formation, but there is one classical example where this association is positive (the Crab) from which we can conclude that, in fact, pulsars are neutron stars.

It is important to notice that there are detection biases when one is trying to detect pulsars, specially the fast ones. Because the signal intensity is diminished by an inverse-square law of the distance and because the electrons in the interstellar medium disperse the radio signal, there are practical limits in the detection. The dispersion is a particularly important effect. Longer wavelengths suffer a greater delay in reaching us than the shorter. Thus, the distribution of pulsar periods does not reflect properly the relative population of classical and millisecond pulsars.

Another bias comes from the fact that pulsars radiate for a timescale of 10 million years. This timescale is short compared to the age of the galaxy and the lives of massive

stars that are believed to create the pulsars. Then, from the estimated 10^5 active pulsars, maybe the majority of them cannot be seen.

1.2.1 The problem of the masses and the necessity of radii measurements

Another striking characteristic of the observed neutron stars is their mass distribution (see figure 1.4).

Figure 1.4: Recent measurements suggest a bimodal distribution in the mean mass: $1.4M_{\odot}$ and $1.6M_{\odot}$. Original reference is Lattimer and Prakash (2005). This updated figure will appear in Annual Review of Nuclear and Particle Physics, Vol. 62, 2012.

Recent theoretical studies suggest a mass distribution at least bimodal: there would be a peak around $1.3M_{\odot}$ and another one around $1.7M_{\odot}$, with minimum and maximum around $1.15M_{\odot}$ and $2.1M_{\odot}$, respectively.

Now, with much more data available, the mean mass, which was in the past around $1.4M_{\odot}$, can be something around $1.68M_{\odot}$. Or even we could suppose a bimodality although the error bars are a factor of obscurity (the situation is a bit different as discussed by Valentim et al. (2011) with a bayesian approach). The mass distribution is intimately related to the events that lead to the core collapse of massive stars ($M \gtrsim 8M_{\odot}$) and to the subsequent supernova explosion that, then, lead to the formation of the compact star itself.

Independently of the case, the formation of a neutron star begins when a massive star exhausts its nuclear fuel up to the point of iron, the most bound element. The iron core of this pre-supernova star is partially supported by the degeneracy pressure of relativistic electrons ($\propto \eta_e^{4/3}$) and partially by thermal pressure (while there are elements to the fusion process to occur). The layers above the core are still fusing elements and depositing more iron in the core.

When the iron core exceed its effective Chandrasekhar mass¹(Timmes et al., 1996), the electron degeneracy pressure no more can support the core against the collapse and a proto-neutron star begins to form. The reactions that occur are basically two, both of them diminish the pressure and make the collapse even faster:

which, by electron capture, release a huge amount of neutrinos and diminish the degeneracy pressure and

The latter removes almost all of the thermal support. Thus, the velocity of the collapse increases and, about $100ms$, the young neutron star takes its final configuration. This is

¹ $M_{Ch0} = 5.83Y_e^2$ (original Chandrasekhar mass; Y_e is the electron fraction).

$M_{Ch} = M_{Ch0} \left[1 - 0.0226 \left(\frac{Z}{6} \right)^{2/3} \right]$ (with electrostatic corrections due to the non-uniformity in the electron distribution).

$M_{Ch} = M_{Ch0} \left[1 + \left(\frac{\pi k_B T}{\epsilon_f} \right)^2 \right]$ (due to the finite entropy of the iron core; $\epsilon_f = 1.11(\rho_7 Y_e)^{1/3}$ is the Fermi energy).

$M_{Ch} = M_{Ch0} \left[1 + \left(\frac{s_e}{\pi Y_e} \right)^2 \right]$ (written in terms of the electronic entropy per baryon; $s_e = 0.5\rho^{-1/3}(Y_e/0.42)^{2/3}$).

the timescale that takes to the neutrons become degenerate and, because they are fermions, their pressure halts the collapse.

The physics that leads to the formation of neutron stars is different depending on the mass interval of the progenitor stars (Fryer et al., 2011). Theoretically there are three cases:

1. progenitor mass between $\gtrsim 7.5M_{\odot}$ and $\sim 11M_{\odot}$: the main source of uncertainty comes from the stellar evolution theory. The models do not make an accurate division of the edge for the formation of neutron stars and white dwarfs. These stars are moderately massive and abundant in the galaxy and probably produce reminiscent compact stars with low average masses, $\sim 1.1M_{\odot}$ to $\sim 1.3M_{\odot}$.
2. progenitor mass between $\sim 11M_{\odot}$ and $\sim 25 - 30M_{\odot}$: the main source of uncertainty comes from the explosion mechanism itself, e.g., the mass of the reminiscent is determined mainly by the fallback material onto the proto-neutron star. These stars are massive and less abundant in the galaxy and could provide neutron stars with masses around $\sim 1.7 - 1.8M_{\odot}$.
3. progenitor mass greater than $\sim 25 - 30M_{\odot}$: the main source of uncertainty comes from the mass loss rate. They probably form black holes, not studied here.

On the other hand, even more important than mass determinations are the radii determinations, because the mass can be accurately determined when the neutron star is in binary systems. In the end, the radii are the crucial factor and they are the most troublesome to determine. Here, the major problem is the error bars. The confidence interval always captures a big region in the mass-radius diagram, encompassing many equations of state. Thus, it is important to combine as many methods of determining the radius as we can. For an extensive work on masses and radius determinations combining many techniques see Özel (2006) and Özel et al. (2009). Another way of determining the radius is combining timing and spectroscopy in LMXBs (see subsection 1.2.2). The latter uses the association of kilohertz quasi-periodic oscillations with the inner radius of the disc or with the innermost stable circular orbit and the general relativistic distortions of the iron line (generally seen in these systems) (Hiemstra et al., 2011; Sanna et al., 2011). This provides a much smaller confidence range which implies smaller error bars in radius determinations.

Figure 1.5: The mass-radius diagram; original figure is from reference Demorest et al. (2010). The horizontal strips are observational constraints. The three black dots are observed neutron stars' radii and their respective error bars (the black dots are from the references [20], [21] and [22] quoted in our paper in the Appendix).

All we need is an equation of state that passes through the observed masses in the plot. Or, in other words, the equation of state must support neutron stars' masses at least equal to the effective Chandrasekhar limit of the core of the progenitor star. Although the maximum mass allowed by this equation of state may be greater than $2M_{\odot}$, it is necessarily lower than the causal limiting mass of $\sim 3.0 - 3.5M_{\odot}$. So, it seems that we are stuck in our search for the true equation of state of neutron stars. We need another approach to address this question (see chapter 3).

We now move to a brief description of the astrophysical systems studied in this Thesis, the Low Mass X-Ray Binaries.

1.2.2 Low Mass X-ray Binaries

Low mass X-ray binaries are gravitationally bound star systems in which one of them is a compact object (a neutron star or a black hole) and the other (the companion star) is an ordinary low mass ($\lesssim 1.4M_{\odot}$) star, in contrast to high mass X-ray binary in which the companion star mass can be as high as $\sim 20M_{\odot}$. These systems are very bright, generally non-pulsating X-ray sources and very soon after the first observations of them (Giacconi et al., 1962) it became clear that the energy source is provided by accretion of matter onto the compact object. See figure 1.6 for a schematic view.

Figure 1.6: Notice all the components. We study here phenomena at the inner edge of the disc, near the surface of the central compact star.

The accretion of matter onto the compact object in LMXBs proceeds via accretion disc (Frank et al., 2002) which forms when the companion evolves to the giant phase and its matter fills the Roche lobe (see figure 1.7). The angular momentum possessed by the matter induces the formation of the disc and as matter loses angular momentum, its gravitational energy is converted into kinetic energy that is radiated away mostly in the X-ray band.

Figure 1.7: Roche lobes gravitational equipotentials. Here we see five Lagrangian points or points of stable equilibrium in the gravitational potential.

The first work regarding the structure and radiative processes of the disc was the seminal work of Shakura and Sunyaev (1973) where they derived that the disc around the black holes is geometrically thin and optically thick, emitting energy as a collection

of black bodies (or multi-colour black body). With these assumptions we can use the Stephan-Boltzmann law and derive the approximated temperature of the disc as function of the radial coordinate:

$$T(r) \approx \left[\frac{3GM\dot{M}}{8\pi r^3 \sigma} \left[1 - \left(\frac{R_{inner}}{r} \right)^{1/2} \right] \right]^{1/4}. \quad (1.8)$$

Notice that when $R_{inner} \rightarrow R_*$ and r is not far from the surface, the temperature reaches $\sim 10^7 K$ for typical values of the mass accretion rate, $\dot{M} \sim 10^{-8} M_\odot / yr$, and mass of the compact object $M = 1.4 M_\odot$. That is what we detect as the X-ray radiation of a few keV of energy.

In this Thesis we studied LMXBs with a neutron star and our concern is to address the internal composition of this startling compact object. We have for this purpose some tools of high energy astrophysics. These systems show up spectroscopic features, variability of the light curve and thermonuclear bursts in the energy range from few keV up to hundreds of keV .

The theory of accretion discs predicts the formation of a boundary layer on the surface of the neutron star where the accreting gas has its azimuthal velocity slowed down to match the spin velocity of the star in a very abrupt process. This very thin layer has a high density and is supposed to emit like a blackbody. Recalling that the accretion luminosity is given by the rate of release of gravitational energy

$$L_{acc} = \frac{GM\dot{M}}{R_*} \quad (1.9)$$

and that the disc luminosity is

$$L_d = 2 \int_{R_*}^{\infty} \sigma T(r)^4 2\pi r dr = \frac{GM\dot{M}}{2R_*}, \quad (1.10)$$

we see immediately that the boundary layer is responsible for at least half of the total energy release:

$$L_{BL} = \frac{GM\dot{M}}{2R_*}. \quad (1.11)$$

Based on equation 1.11 we calculate the blackbody temperature of the boundary layer in $\approx 3 \times 10^7 K$ for typical values of the quantities involved, comfortable in the soft X-ray

band $1keV$ to $10keV$.

Actually we observe the X-ray spectra of LMXBs in that range of energies and, based on theoretical grounds, we represent the spectra as a sum of two components: a disc multicolour soft blackbody plus a hard blackbody from the boundary layer/neutron star surface, although this decomposition is often ambiguous (Gilfanov et al., 2003). Besides, another spectroscopic feature is an emission line from iron at $\sim 7keV$ (Hirano et al., 1987; Miller, 2002; Bhattacharyya et al., 2006; Hiemstra et al., 2011) that is supposed to be distorted by general relativistic effects. In this case, the distortion of the line itself becomes a window to access the parameters of the compact object.

Notice that the disc plays an important role on the spectra through the mass accretion rate $\dot{M}$. It is supposed that variations in $\dot{M}$ drive changes in spectral and temporal states, apart from changes in luminosity.

On the other hand, LMXBs show a myriad of temporal features, e.g., variability on scales from millihertz (mHz) to kilohertz (kHz) called quasi-periodic oscillations or QPOs, besides some types of noise. There are models that try to explain the origin of the QPOs, from disc instabilities and/or oscillations to orbital motion-based models like the Sonic-point Model (Miller et al., 1998) or the Relativistic Precession Model (Stella and Vietri, 1998). So far, none of them were able to fully explain or predict the appearance of these characteristic frequencies. Regarding the kHz QPOs, they are generally associated to orbital motion of matter at the innermost stable circular orbit or ISCO and if it is the case, we can derive the radius of the central neutron star (see chapter 4).

Finally, we observe in many sources the so-called thermonuclear bursts, sudden and huge eruptions where the X-ray flux increases by several factors in a short time, $\sim 2s$, and decay time that can last several minutes. We will not enter in more details here, but it suffices to say that thermonuclear bursts have thermal blackbody spectra and the measurements of the bolometric fluxes and blackbody temperatures can be used to estimate radii if the distance to the source is known (Özel et al., 2009).

There are two major classifications of neutron star LMXBs: the atoll and Z sources (van der Klis, 1989b,a). This division is due to the shape the sources acquire when plotted in the colour-colour² or colour-intensity diagrams. See the figure 1.8. They can also be

² Colour in X-rays are defined to be the ratio of photons in a predefined high energy band relatively to

transient or persistent: while the latter have been showing detectable X-ray emission for most of the time, the transients show long periods of inactivity.

Figure 1.8: Colour-colour and colour-intensity diagrams. Panel *a* names the atoll sources and panel *c* names the Z sources. This figure was extracted from van der Klis (2006).

To be more specific, we studied here the transient atoll source 4U1608–52 (see chapter 4). Let us now briefly describe some general features common to many atoll sources. For a far more complete description see the reference Linares (2009).

To begin with, a cyclic relation between luminosity, variability and spectra is assumed to exist, as shown in figure 1.9.

Figure 1.9: The observed quantities are supposed to relate each other.

In what follows we show a set of figures nicely provided by M. Linares that will serve to our purpose of summarizing the general properties of atoll sources. We will state the results but not interpret them. For interpretations we refer the reader to Linares (2009). The author studied these relations in detail for nine atoll sources, namely, 4U 1608–52, Aql X–1, 4U 1705–44, 4U 1636–53, 4U 0614+09, 4U 1728–34, 4U 1820–30, 4U 1735–44, GX 3+1. Keep in mind the source 4U 1608–52, the object studied in this Thesis.

a lower energy band.

Regarding the relation variability and spectra, let us take a look in figures 1.10 and 1.11. In figure 1.10 we see which frequency component appears in which spectral state. Important to us in this Thesis are the kHz QPOs, in blue, which appear mostly in the intermediate and soft states. Except by the very low frequency noise complex (VLFN) that appears in the softest states, the overall trend is an anti-correlation between frequency and hardness, e.g., the lower the hardness, the higher the frequency. Figure 1.11 shows an important characteristic regarding the kHz QPOs in atoll sources. Both kHz QPOs, the lower and the upper, anti-correlate with hardness. However, they follow different paths in the frequency *vs* colour diagram as can be easily seen. Because of this, when one sees only one kHz QPO in a given observation, the identification of which one it is becomes obvious when taking into account the typical range of frequencies of the lower and the upper kHz QPO and the correspondent hard colour.

To end the relation between the variability and the spectra, we see in figure 1.12 that the strength of the variability, given by the fractional rms, increases if the hardness increases.

The atoll sources display luminosities from 0.1% to 50% of the Eddington luminosity (3×10^{35} to $1 \times 10^{38} \text{ erg/s}$) in the energy range $2 - 200 \text{ keV}$. Because the hard colour is independent of the interstellar absorption, it traces very well the changes in the spectral state. In figure 1.13 we see that, on average, the soft states are more luminous, but there are hard states more or equally luminous than the soft ones. Thus, the hard colour is not a good indicative of the luminosity. From figure 1.14 we see that luminosity and spectral hardness are anti-correlated in soft and hard states but not in intermediate states.

Finally, the authors in Linares (2009) have not found any obvious relation between frequencies, spectra and luminosity neither in a given source nor between sources, but they actually found a correlation between luminosity and hardness for a fixed frequency (see figure 1.15). This means that the same temporal states can occur at different luminosities.

By studying specific spectroscopic and temporal features and their relation with the observed quantities, we can find how to calculate properties of the central compact object like the mass, radius, magnetic fields *etc* and determine the equation of state observationally.

With this we conclude this subsection and refer the reader to chapter 4 to our work about the time lags in the source 4U 1608–52.

Figure 1.10: Colour-colour diagrams for the nine atoll sources. Green points are measures of the break frequency of the flat-topped broadband noise, small blue open circles mark the detections of upper kHz QPOs, red points show those observations where VLFN is present and grey points the cases where none of these phenomena were present. Observations combining either VLFN or flat-topped noise with upper kHz QPO show up as blue circles with red or green interior, respectively. The hard ($S_a \equiv 1$) and soft ($S_a \equiv 2$) vertices are marked with a large black open circle on the upper and lower part of the atoll track, respectively. Figure from reference Linares (2009).

Figure 1.11: kHz QPO frequencies versus spectral hardness. Black dots represent upper kHz QPOs and grey triangles lower kHz QPOs. Squares show initially unidentified (single) kHz QPOs. Figure from reference Linares (2009).

Figure 1.12: “Colour-colour-colour” diagrams of the nine atoll sources. The colour scale shows the 1-100 Hz fractional rms amplitude of the variability. Figure from reference Linares (2009).

Figure 1.13: “Colour-colour-colour” diagrams of the nine atoll sources. The colour scale shows the 2-200 keV luminosity, in Eddington units and logarithmic scale. The luminosity ranges from $2.5 \times 10^{35} \text{ erg/s}$ ($0.1\% L_{Edd}$) to $1.3 \times 10^{38} \text{ erg/s}$ ($50\% L_{Edd}$). Figure from reference Linares (2009).

Figure 1.14: Colour coordinate representing the position along the atoll track, S_a , versus luminosity (in the same units as 1.13). The S_a ranges corresponding to soft, hard and intermediate states are separated by the double dashed lines and indicated on the right-hand axis. Figure from reference Linares (2009).

Figure 1.15: RMS-RMS diagrams of the nine atoll sources. The color scale shows the 2-200 keV luminosity, in Eddington units and logarithmic scale. The luminosity ranges from $2.5 \times 10^{35} \text{ erg/s}$ ($0.1\%L_{\text{Edd}}$) to $1.3 \times 10^{38} \text{ erg/s}$ ($50\%L_{\text{Edd}}$). Figure from reference Linares (2009).

1.3 Nuclear and Particle Physics: theory of neutron star matter

In this section we intend to show some aspects of the theory and of the experiments used to study neutron star matter. Our goal in this section is to link what we know empirically about nuclear matter to the matter inside the neutron stars (Krane, 1987; Glendenning, 2000, chapters 4 and 5 and references therein). Are they the same? Can we describe them both with the same theory? As we have already stated, it is not possible to attain densities like those reached in the interior of neutron stars in our laboratories. However, much we have learnt about nuclear physics and the behaviour of matter and its constituents at Earth's laboratories conditions. With facilities like the CERN LHC, the Tevatron and many others particle and heavy-ion accelerators around the world, we shattered the nuclei and its constituents (see figure 1.16) and learnt about the building blocks of matter, the elementary particles, which give rise to the Standard Model of Particle Physics.

Figure 1.16: The structure of the atom. Here we see the electron and quarks which are elementary particles. Notice that protons and neutrons are not elementary particles since they are formed by different combination of quarks.

The Standard Model needs the existence of 17 basic particles as shown in figure 1.17 and so far can account for all the phenomena we see in Nature. The only particle not yet seen is the Higgs boson. Ironically, it is the most fundamental since it is (theoretically)

the responsible for the mass of all other particles.

Figure 1.17: All the 17 elementary particles predicted by the Standard Model. The Higgs boson is the only not yet observed.

There are two classes of particles: the fermions, with half-integer spin ($1/2, 3/2, 5/2, \dots$) are the matter constituents and obey the Fermi-Dirac statistics and the Pauli Exclusion Principle; and the bosons, with integer spin ($0, 1, 2, \dots$), that are the force carriers and obey the Einstein-Bose statistics, but not the Pauli Exclusion Principle. In figure 1.18 we show some properties of fermions and bosons.

FERMIONS			matter constituents spin = 1/2, 3/2, 5/2, ...		
Leptons spin = 1/2			Quarks spin = 1/2		
Flavor	Mass GeV/c ²	Electric charge	Flavor	Approx. Mass GeV/c ²	Electric charge
ν_e electron neutrino	$<1 \times 10^{-8}$	0	u up	0.003	2/3
e electron	0.000511	-1	d down	0.006	-1/3
ν_μ muon neutrino	<0.0002	0	c charm	1.3	2/3
μ muon	0.106	-1	s strange	0.1	-1/3
ν_τ tau neutrino	<0.02	0	t top	175	2/3
τ tau	1.7771	-1	b bottom	4.3	-1/3

(a) Fermions

BOSONS			force carriers spin = 0, 1, 2, ...		
Unified Electroweak spin = 1			Strong (color) spin = 1		
Name	Mass GeV/c ²	Electric charge	Name	Mass GeV/c ²	Electric charge
γ photon	0	0	g gluon	0	0
W^-	80.4	-1			
W^+	80.4	+1			
Z^0	91.187	0			

(b) Bosons

Figure 1.18: The building blocks of matter.

Fermions by themselves form two more sets of (non elementary) particles: baryons with half-integer spin and mesons with integer spin. We show only a few types of baryons and mesons in figure 1.19 and we show which kind of force acts on which type of particles in figure 1.20.

Baryons qqq and Antibaryons $\bar{q}\bar{q}\bar{q}$					
Baryons are fermionic hadrons. There are about 120 types of baryons.					
Symbol	Name	Quark content	Electric charge	Mass GeV/c^2	Spin
p	proton	uud	1	0.938	1/2
$\bar{p}$	anti-proton	$\bar{u}\bar{u}\bar{d}$	-1	0.938	1/2
n	neutron	udd	0	0.940	1/2
Λ	lambda	uds	0	1.116	1/2
Ω^-	omega	sss	-1	1.672	3/2

(a) Baryons

Mesons $q\bar{q}$					
Mesons are bosonic hadrons. There are about 140 types of mesons.					
Symbol	Name	Quark content	Electric charge	Mass GeV/c^2	Spin
π^+	pion	$u\bar{d}$	+1	0.140	0
K^-	kaon	$s\bar{u}$	-1	0.494	0
ρ^+	rho	$u\bar{d}$	+1	0.770	1
B^0	B-zero	$d\bar{b}$	0	5.279	0
η_c	eta-c	$c\bar{c}$	0	2.980	0

(b) Mesons

Figure 1.19: All types of particles (baryons and mesons) are different combinations of quarks. Here we show a pretty small number of baryons and mesons.

PROPERTIES OF THE INTERACTIONS						
Property \ Interaction	Gravitational	Weak (Electroweak)	Electromagnetic	Strong		
				Fundamental	Residual	
Acts on:	Mass - Energy	Flavor	Electric Charge	Color Charge	See Residual Strong Interaction Note	
Particles experiencing:	All	Quarks, Leptons	Electrically charged	Quarks, Gluons	Hadrons	
Particles mediating:	Graviton (not yet observed)	$W^+ W^- Z^0$	γ	Gluons	Mesons	
Strength relative to electromag for two u quarks at:	10^{-41}	0.8	1	25	Not applicable to quarks	
	10^{-41}	10^{-4}	1	60		
	10^{-36}	10^{-7}	1	Not applicable to hadrons	20	

Figure 1.20: We show here the four forces of Nature and their effectiveness acting upon the particles.

By knowing what the building blocks of matter are, we can study how they clump and form the nuclei we see in the periodic table. Two things became apparent from the experiments since the earliest times: that the mass of the proton and the neutron are approximately equal and that Nature seems to prefer nuclei with approximately the same number of protons and neutrons (see figure 1.21). We say that there is isospin symmetry in nuclear matter or that the nuclear matter is isospin-symmetric. Isospin is a word derived

from isotopic spin and is an abstract concept mathematically resembling the ordinary spin. By definition, protons have isospin $1/2$ and neutrons have isospin $-1/2$. Symmetry, for quantum systems, is related to degeneracy of energy levels. In this case, the fact that $m_p = 938.28 \text{ MeV}/c^2 \approx m_n = 939.57 \text{ MeV}/c^2$. Thus, there is an approximated symmetry in the strong nuclear interaction (which binds the proton and neutron, for example), the isospin symmetry meaning that the strong interaction does not depend on the charge. Hence, the proton and the neutron are supposed to be different states of the same particle: the nucleon. However, notice that the isospin symmetry is a theoretical idealization and should be considered as approximate.

(a) Decay times.

(b) Decay process.

Figure 1.21: The stable elements are shown in black. Notice that the isospin symmetry is an idealization: we clearly see the deviation from the line $Z = N$.

For nuclear matter from which our world is composed, the importance arises from the fact that nuclei are made by protons and neutrons which, in turn, are composed of u - and d -quarks. Although there are many combinations of quarks, the isospin symmetry plus the properties of the chemical potential of the particles (we will discuss thoroughly soon) prevents the formation of stable “nuclei” with other combinations of quarks at the actual energy density conditions of the Universe. Thus, the ground state of nuclear matter

is that of matter composed of protons and neutrons, unless some exotic process happens, for example the formation of strange quark matter.

In short, the nuclear matter is hot non-degenerate (the thermal energy is of the same order of the Fermi energy), symmetric (number of protons and neutrons are approximately the same), and do not carries strangeness (because the energy needed to create strangeness is not attained by these systems). It can be described by the nuclear mean-field theory constrained by the charge symmetry and strangeness conservation (Glendenning, 2000).

Now, let us back to the neutron star case. The dawn of neutron star is the supernova explosion in which the star core implodes releasing a huge amount of energy, 99% of it carried by neutrinos. The source of this huge amount of energy is the gravitational binding energy of the star. The neutrino emission cools down the star from $\sim 50\text{MeV}$ ($\sim 10^{11}\text{K}$) to $\sim 1\text{MeV}$ ($\sim 10^{10}\text{K}$) in a few seconds. In some millions of years the temperature will drop below to 10^6K . This temperature, while hot to Earth standards is quite cold in the nuclear scale and because the Fermi energy (or the chemical potential at $T = 0\text{K}$) depends on density, which is now very high, the matter becomes degenerate after the first few seconds. In fact, the chemical potential is the quantity that we must to analyse in order to find out which kind of particles could form, in principle, in a certain medium. The particle will appear if its chemical potential exceeds its mass in that given medium which, in turn, is given by its vacuum mass corrected by the interactions with other particles in that medium. For instance, Λ -baryons will appear when $\mu_\Lambda = \mu_n \gtrsim m_\Lambda \simeq 1115\text{MeV}$.

The chemical potential of any particle can be written as a linear combination of the chemical potential of any conserved charges in the medium – in neutron stars, the baryon and electric charges:

$$\mu = b\mu_n - q\mu_e, \quad (1.12)$$

where b is the baryon charge and q is the electron charge and μ_n and μ_e are the chemical potentials of neutron and electron, respectively.

As the density increases, so do the chemical potential, and the energy of the system can be reduced by sharing the conserved baryonic number with other baryon species. In this way we expect that the super-dense matter inside neutron stars be populated by many baryon species or even by quarks (see subsection 1.3.1). Very important for these compact

objects is the hyperon threshold at around three times the saturation density, a value that can be easily attained after the implosion of the iron core of the progenitor. Hyperon is a baryon that contains a strange s -quark.

In order to know the baryon species that can be formed inside a neutron star we must to study the evolution of that object during its hot phase until it cools down to $10^6 K$, the point from where, for all the purposes, the star is frozen regarding the nuclear scale. For example, when the star is still hot in that scale, reactions like 1.13 occur:

$$N + N \rightarrow N + \Lambda + K, \quad (1.13)$$

where N stands for nucleon, Λ is an uds -baryon (a hyperon) and K (kaon) is a $s\bar{u}$ -meson.

As we can see, the strangeness begin to increase inside the neutron star, although the K meson will rapidly decay in muons, photons and anti-neutrinos unless it condensates (a recent exciting possibility). When the temperature is low enough, the kaon cannot form any more because there is no available energy. However, another reaction can occur that increases the strangeness:

$$n + N + \mu^- \rightarrow N + \Sigma^- + 2\gamma + \nu, \quad (1.14)$$

where n stands for the neutron, μ^- for the muon, Σ^- for the dds -hyperon, γ for the photon and ν for the neutrino. See in figure 1.22 a schematic chart showing some common species.

In this way, processes like the above one will occur until the system reaches its ground state, minimizing the energy since it keeps the charge neutrality for the given number of baryons inside the star. The result is that the neutron star is not made by neutrons as at first the scientists thought, but it is probably a rich zoo of many baryons.

In short, the neutron star matter is cold degenerate (the thermal energy much lower than the Fermi energy), asymmetric (various baryon species), and carries strangeness (because the formation of hyperons). It can be described by the nuclear mean-field theory for baryons plus for leptons, constrained by the charge neutrality and the general beta equilibrium without strangeness conservation (Glendenning, 2000).

Notwithstanding what was said before, we do not know the exact behaviour at supra-nuclear densities $\gtrsim \rho_{sat}$. We need extrapolate the now well-established theory of nuclear matter to the high densities of the neutron star interior. The extrapolation cannot be

random, but must follow some strict rules (Glendenning, 2000):

1. Lorentz covariance,
2. general relativity,
3. causal equation of state ($v^2 = dp/d\epsilon \leq 1$),
4. microscopic stability known as Le Chatelier's Principle ($dp/d\rho \geq 0$),
5. baryon and electric charge conservation,
6. Pauli Principle,
7. generalized beta equilibrium,
8. phase equilibrium,
9. asymptotic freedom of quarks,
10. properties of matter at saturation density.

The most used theory is the relativistic mean-field approximation because it incorporates naturally the properties of matter at the saturation point and the effective mass corrections. To do so we write the Lagrangian of matter inside the neutron stars:

$$\begin{aligned}
\mathcal{L} = & \sum_B \bar{\psi}_B \left(i\gamma_\mu \partial^\mu - m_B + g_{\sigma B} \sigma - g_{\omega B} \gamma_\mu \omega^\mu - \frac{1}{2} g_{\rho B} \gamma_\mu \boldsymbol{\tau} \cdot \boldsymbol{\rho}^\mu \right) \psi_B \\
& + \frac{1}{2} \left(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2 \right) - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu \\
& - \frac{1}{4} \boldsymbol{\rho}_{\mu\nu} \cdot \boldsymbol{\rho}^{\mu\nu} + \frac{1}{2} m_\rho^2 \boldsymbol{\rho}_\mu \boldsymbol{\rho}^\mu - \frac{1}{3} b m_n (g_\sigma \sigma)^3 - \frac{1}{4} c (g_\sigma \sigma)^4 \\
& + \sum_\lambda \bar{\psi}_\lambda \left(i\gamma_\mu \partial^\mu - m_\lambda \right) \psi_\lambda .
\end{aligned} \tag{1.15}$$

We will not enter in details, but it is important to recall that the sum is over the baryon species B from the baryon octet ($p, n, \Lambda, \Sigma^+, \Sigma^-, \Sigma^0, \Xi^-, \Xi^0$) (Glendenning, 2000).

The Euler-Lagrange equations are obtained when we substitute the meson fields by their mean values in the static uniform matter and the nucleon currents by ground-state expectations generated in the presence of the mean meson fields (that is why we call it mean-field approximation).

At the end of this hard work we are in the position to obtain the so-called Equation of State (EoS) for the neutron star matter:

$$T^{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \sum_\phi \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial^\nu \phi, \tag{1.16}$$

where ϕ stands for the fields and $T^{\mu\nu}$ is the energy-momentum tensor given by

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Solving the $7 + N$ variables system results in $\epsilon(\rho)$ and $p(\rho)$ which, together, provide the equation of state. Due to the complexity of the problem, most of the equations of state are given in tabular form, being calculated numerically, but there are a few cases in which a closed expression for the equation of state is available (for example, the equation of state prescription from MIT Bag Model; we will discuss this equation of state in subsection 1.3.1). Now, with the equation of state in hands we can proceed to solve for the structure of the neutron stars by solving the TOV equations, integrating the system subject to the initial conditions $\epsilon(r = 0) = \epsilon_c$ and $M(r = 0) = 0$ (the energy density and the mass,

respectively). There are only a few analytical solutions for this problem (see the chapter 2 for further discussion on this problem).

For each value of ϵ_c allowed by the EoS we have a stellar model, a point in the mass-radius diagram. Thus, varying ϵ_c we can construct a sequence of stars in the mass-radius diagram correspondent to this particular EoS (see figure 1.5). And thus we close the circle.

A question that naturally arises is: what is the true equation of state of neutron star matter? In this Thesis we studied models within two classes of equation of state. One is the SLy4 prescription and is for a neutron star of hadronic composition. The second class is a strange quark star following the MIT Bag Model prescription. On theoretical grounds both classes seem to be feasible in Nature. On observational basis, it has been very hard to access information from the interior of neutron stars, although the search for their modes of oscillation, the evolution of the braking index of pulsars and glitches, for example, are very exciting promises.

We refer the reader to chapter 3 for another way to study the equation of state where we suggest that can exist a hierarchy of equations of state to be realized in Nature.

1.3.1 The Equations of State used in this Thesis

We used here two equations of state, a hadronic equation of state (SLy4) and a strange quark equation of state (MIT Bag Model). We will discuss a bit them both in what follows.

1.3.1.1 SLy4

Here we will briefly describe the equation of state for the hadronic neutron stars we used in this Thesis. This equation of state seems to be very appropriate to calculate the structure of neutron stars composed by very rich neutron matter.

This equation of state is based on Skyrme-Lyon (SLy) effective nucleon-nucleon interaction given as input to the calculation, once the many-body approximation is fixed. This effective interaction is two-body in nature, but contains a term that is an average of an original three-body problem. Besides, there is a strong dependence of a neutron excess which makes it suitable for calculation involving neutron star matter.

The equation of state we used here is an “unified” one in the sense it accounts for the outer and inner crust and for the liquid neutron core and it was derived by Douchin and

Haensel (2001).

For the crust, they used the Compressible Liquid Drop Model (CLDM) of nuclei (Douchin et al., 2000). Within this model, the total energy density is a sum of the bulk contribution of nucleons, the surface contribution of nucleons, the Coulomb interaction contribution and the electron energy contribution. All the contributions are calculated for the SLy forces (Douchin and Haensel, 1999, 2000).

$$E = E_{N,bulk} + E_{N,surf} + E_{Coul} + E_e \quad (1.17)$$

In their calculations for the outer crust, the authors limited the density lower limit to $\rho > 10^6 g/cm^3$ and for densities below the neutron drip ($4.3 \times 10^{11} g/cm^3$) they used not the SLy equation of state, but the Haensel and Pichon equation of state (Haensel and Pichon, 1994). This equation of state is a reliable extrapolation using maximal experimental data for nuclear masses. Then, for densities above the neutron drip, in the region of the inner crust, the equation of state is matched with the SLy equation of state.

The transition from the (inner) crust to the liquid core occurs at constant pressure and is accompanied by a density jump and takes place because the nuclei get closer and closer.

The liquid core is assumed to be a homogeneous plasma of neutrons, protons, electrons and negative muons (the latter, for densities above the threshold density for the appearance of muons). At higher densities the authors extrapolated the npe μ matter, instead of taking into account the appearance of hyperons. The total energy density of the npe μ matter is given by the energy density of the nucleons and of the leptons in addition to the rest energy of the matter constituents:

$$E(n_n, n_p, n_e, n_\mu) = E_N(n_n, n_p) + n_n m_n c^2 + n_p m_p c^2 + E_e(n_e) + E_\mu(n_\mu) \quad (1.18)$$

where the equilibrium condition with respect weak interaction is given by

$$\mu_n = \mu_p + \mu_e, \quad \mu_\mu = \mu_e \quad (1.19)$$

where, in turn,

$$\mu_j = \frac{\partial E}{\partial n_j}, \quad j = n, p, e, \mu. \quad (1.20)$$

Then, the equation of state for the $npe\mu$ is given by

$$\rho(n_b) = \frac{E(n_b)}{c^2}, \quad P(n_b) = n_b^2 \frac{d}{dn_b} \left(\frac{E(n_b)}{n_b} \right). \quad (1.21)$$

Summarizing, this equation of state covers three main regions of the neutron star: the outer crust, the inner crust and the liquid core. In figures 1.23 and 1.24 we show the equation of state covering the whole neutron star structure and the resulting mass-radius relation, respectively:

Figure 1.23: SLy4 equation of state of Douchin and Haensel. Dotted vertical line corresponds to the neutron drip and the dashed one to the crust-liquid core interface.

Figure 1.24: Mass-radius relation for the SLy4 equation of state.

Haensel and Potekhin (2004) derived an analytical representation of the SLy4 equation of state. Analytical representations are preferred over the tabulated form because they avoid two major problems of the latter: first, the ambiguity in the interpolation when calculating the neutron star structure (because this leads to ambiguity in the calculated parameters). Second, the procedure have to take into account thermodynamic relations, which is difficult. Besides, analytical representations can have their derivatives calculated precisely.

In this Thesis we used the analytical representation in our calculations and we talk a bit about it in chapter 3.

1.3.1.2 Strange Quark Matter

When it was realized that the quarks are asymptotically free, the idea of a star composed entirely or in part by quark matter seemed natural. Quark matter, in this context, is a much larger colourless³ region than the hadronic volume through which quarks are free to move. The asymptotic freedom of quarks leads to the possibility that the true ground state of the strong interaction could be the strange quark matter (Bodmer, 1971; Witten, 1984) instead of the nuclei we are familiar with.

The MIT Bag Model (Chodos et al., 1974) was developed to account for the hadronic masses in terms of their quark content. This model also explains the spatial confinement of quarks, e.g., to explain why we do not see free quarks everywhere. In the model, quarks are confined by a boundary, called the Bag, where even the vacuum is expelled. In reality, the energy density of the physical vacuum exerts pressure on this boundary to impede that quarks cross it. Inside the bag the quarks move freely and we can consider them as a Fermi gas.

The expressions for the pressure, energy density and baryon density for a Fermi gas of quarks in the zero temperature, $T = 0$, approximation (as discussed earlier, the neutron star is frozen at strong interaction scales) read:

$$p = -B + \sum_f \frac{1}{4\pi^2} \left[\mu_f k_f \left(\mu_f^2 - \frac{5}{2} m_f^2 \right) + \frac{3}{2} m_f^4 \ln \left(\frac{\mu_f + k_f}{m_f} \right) \right] \quad (1.22)$$

³ Colour is the charge of the strong interaction in the same sense the electron carries the charge of electromagnetism; a quark change its colour when it absorbs or emits a gluon.

$$\epsilon = B + \sum_f \frac{3}{4\pi^2} \left[\mu_f k_f \left(\mu_f^2 - \frac{1}{2} m_f^2 \right) - \frac{1}{2} m_f^4 \ln \left(\frac{\mu_f + k_f}{m_f} \right) \right] \quad (1.23)$$

$$\rho = \sum_f \frac{k_f^3}{3\pi^2} \quad (1.24)$$

where the sum is over all flavours f of interest and k_f is the Fermi momentum defined in terms of the chemical potential $\mu_f = (m_f^2 + k_f^2)^{1/2}$. B is the energy density of the physical vacuum.

Now, if we take the massless approximation, again justified because of the large value of their chemical potentials at the typical densities found in neutron stars⁴, then the equation of state assumes a simple form that we will use throughout this Thesis:

$$p = -B + \sum_f \frac{1}{4\pi^2} \mu_f^3 k_f \quad (1.25)$$

$$\epsilon = B + \sum_f \frac{3}{4\pi^2} \mu_f^3 k_f \quad (1.26)$$

$$\rho = \sum_f \frac{k_f^3}{3\pi^2} \quad (1.27)$$

which results in the known formula $3p = \epsilon + 4B$, with $\epsilon = c^2 \rho_m$ here where ρ_m is the matter density, see figure 1.25. Notice the striking characteristic of this EoS: these are self-bound stars, bounded not by gravity, but by the strong force. This means that when the pressure is zero at the edge of the star, the density assumes a finite value that falls to zero in a distance 10^{-13}cm .

The model is controlled by three parameters: B (the vacuum constant), m_s (the strange quark mass) and α_c (the strong coupling constant that accounts for the interaction among quarks). The numerical value of B is of high interest for the stability of strange quark matter: for $\alpha_c = 0$ and $m_s = 0$, $B = 57 - 91 \text{MeV}/fm^3$; for $\alpha_c = 0$ and $m_s = 150 \text{MeV}$, $B = 57 - 75 \text{MeV}/fm^3$. In such situations, the energy per baryon of a mixture of three quark flavours (*up*, *down* and *strange*) will be less than the energy per baryon of iron ($E/A_{(56Fe)} \sim 931 \text{MeV}$) and the ground state will be the strange quark matter.

⁴ $\mu_s = 1.9B^{1/4}$ at the edge of the strange star and would be higher than 275MeV .

Figure 1.25: Equation of state of MIT Bag Model.

This equation of state produces a very different mass-radius relation than the SLy4 and other hadronic equations of state; see Alcock et al. (1986) and figure 1.26.

Figure 1.26: Mass-radius relation for strange quark equation of state from MIT Bag Model.

What was discussed above is valid for non-interacting strange quarks; if we take into account the coupling constant of the strong interaction to the first order, the expressions for the energy density and pressure (Farhi and Jaffe, 1984) read

$$\epsilon = B + \sum_i \left(\Omega_i + \mu_i n_i \right) \quad (1.28)$$

and

$$p = -B + \sum_i \Omega_i \quad (1.29)$$

where Ω_i is the thermodynamic potential at $T = 0$ and the sum on i is over the quark flavours and the muon and the electron. The other quantities are:

$$\begin{aligned} \Omega_f = & -\frac{\gamma_f}{24\pi^2} \left\{ \mu_f \sqrt{\mu_f^2 - m_f^2} \left(\mu_f^2 - \frac{5}{2} m_f^2 \right) + \frac{3}{2} m_f^4 \ln \left[\frac{\mu_f + \sqrt{\mu_f^2 - m_f^2}}{m_f} \right] \right. \\ & - \frac{2\alpha_s}{\pi} \left[3 \left(\mu_f \sqrt{\mu_f^2 - m_f^2} - m_f^2 \ln \left[\frac{\mu_f + \sqrt{\mu_f^2 - m_f^2}}{\mu_f} \right] \right)^2 \right. \\ & - 2 \left(\mu_f^2 - m_f^2 \right)^2 - 3 m_f^4 \ln^2 \left(\frac{m_f}{\mu_f} \right) \\ & \left. \left. + 6 \ln \left(\frac{\sigma}{\mu_f} \right) \left(\mu_f m_f^2 \sqrt{\mu_f^2 - m_f^2} - m_f^4 \ln \left[\frac{\mu_f + \sqrt{\mu_f^2 - m_f^2}}{m_f} \right] \right) \right] \right\} \end{aligned} \quad (1.30)$$

where γ_f is the flavour degeneracy, α_c is the coupling constant and σ is the renormalization scale. Finally, the quark number density, baryon number density and the charge density are

$$n_f = -\frac{\partial \Omega_f}{\partial \mu_f} \quad (1.31)$$

$$\rho = \frac{1}{3} \sum_{u,d,s} n_f \quad (1.32)$$

$$q = \sum_i n_i q_i \quad (1.33)$$

It is easy to show that setting $\alpha_c = 0$ we recover the previous expressions for $T = 0$.

How is it possible that strange quark matter could manifest itself in Nature? When a compact object forms from a progenitor star with mass higher than $\sim 8M_\odot$, the densities in the core reach something about $\sim 10^{14-15} \text{g/cm}^3$. At such densities, phase transitions are likely to occur because the weak interaction would convert about one third of the quarks

into strange quarks, since the mixture of the three flavours has a lower energy per baryon than the conventional matter and also lower than the matter composed only by a mixture of two flavours.

And how could one observationally differentiate strange stars from neutron stars? We have seen that both, hadronic neutron stars and strange stars, produce sequences with similar values of mass and radius. One possible way is through very fast pulsars; there is a limit on the rotation period of gravitationally bound hadronic star. So if a pulsar is discovered to have rotation period shorter than that limit, the conclusion that the strange quark matter hypothesis realizes in Nature would be unavoidable.

Other possible way comes from studies of phenomenon at the surface of a strange star. Because the matter in these stars is bound by the strong interaction, the star has a very abrupt edge, with the density falling from $\sim 10^{14-15} g/cm^3$ to 0 in a distance range $\sim 10^{-13} cm$ (the strong force range). For example, the strange star can support outgoing radiant fluxes much greater than the Eddington flux, the photon emissivity would be around 1 and the star could be like a “silver ball” in X-rays, and at last, the magnetosphere could be very different from hadronic stars since the electrostatic forces are not able to remove particles from the surface (Alcock, 1991).

To end this discussion, is it possible to have these two families of neutron stars existing simultaneously? In principle, it could be possible. However, the Universe can be too old for this to happen. The reason is that once a strange star is formed, it is possible that an event like a collision would expel strange matter nuggets. These nuggets of strange matter have a net positive electric charge⁵ and they would be inert except in a neutron-rich environment.

Eventually, the flux of such nuggets would have contaminated the hadronic stars and because they are neutron-rich, the nuggets would absorb the neutrons, leading to a conversion of the whole hadronic star to a strange star. (A estimation of the strange nuggets flux in our galaxy can be done with what is discussed in the works of Clark and Eardley (1977) and Cappellaro and Turatto (1988)).

Thus, although all the neutron stars could be strange stars, more studies have to be performed in order to observationally determine the EoS of neutron stars.

⁵ This is because the mass difference between the strange quark and the up and down quarks leads to slight different amounts of the three in the three flavour mixture.

Analytical solutions in the construction of strange quark stars models

In this chapter we explain our motivation for seeking an analytical solution for the structure of neutron stars. The first attempts in doing this calculations are due to Richard Tolman (Tolman, 1934, 1939) and Oppenheimer and Volkoff (Oppenheimer and Volkoff, 1939). Since it was realized that General Relativity is the necessary framework for studying neutron stars, physicists struggled with the equations. The reason is that for a spherically symmetric and static perfect fluid we must solve three coupled non-linear differential equations with four variables. The equations are shown below:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (2.1)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ describes the metric of the spherically symmetric and static space-time and $T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - pg^{\mu\nu}$ is the energy-momentum tensor for the perfect fluid.

From this, we obtain the following set of equations to solve:

$$\frac{\lambda' e^{-\lambda}}{r} + \frac{1 - e^{-\lambda(r)}}{r^2} = \frac{8\pi G}{c^2} \rho(r), \quad (2.2)$$

$$\frac{\nu' e^{-\lambda}}{r} - \frac{1 - e^{-\lambda(r)}}{r^2} = \frac{8\pi G}{c^4} p(r), \quad (2.3)$$

$$e^{-\lambda} \left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu' \lambda'}{4} + \frac{\nu' - \lambda'}{2r} \right] = \frac{8\pi G}{c^4} p(r). \quad (2.4)$$

with the last equation usually replaced by the contracted Bianchi Identities that express the conservation law:

$$p'(r) + \frac{1}{2}(c^2\rho(r) + p(r))\nu'(r) = 0. \quad (2.5)$$

Alternatively, one can solve another form of these equations, the so-called Tolman-Oppenheimer-Volkoff (TOV) Equations:

$$p'(r) = -\frac{Gm(r)\rho(r)}{r^2} \left(1 + \frac{p(r)}{c^2\rho(r)}\right) \left(1 + \frac{4\pi r^3 p(r)}{c^2 m(r)}\right) \left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}, \quad (2.6)$$

$$m'(r) = 4\pi r^2 \rho(r). \quad (2.7)$$

The TOV equations describe the structure of the star and is the relativistic version of the hydrostatic equilibrium equation. As we can readily see, we need another information to close the system and solve for the quantities $\lambda(r)$, $\nu(r)$, $p(r)$ and $\rho(r)$. A common choice, specially in the past, is to give a functional profile for one of these quantities, say, for instance, $\rho(r) = a - b \times r^2$, a and b constants (Tolman, 1939). With such choice, the system is, in principle, solvable.

Such approach provided us about 127 solutions until 1998. Studying these solutions Delgaty and Lake (1998) found out that only 16 satisfy the minimum criteria for being physically acceptable. And of these 16, only 9 satisfy the additional condition that the sound speed in the medium monotonically decreases with radius.

Being physically acceptable, although, does not mean that the equation of state is meaningful from the point of view of nuclear and particle physics. For example, Tolman IV solution (Tolman, 1939) provides an equation of state of the form:

$$\rho(p) = \rho_c - 5\frac{p_c}{c^2} + 5\frac{p}{c^2} + 8\frac{\left(\frac{p_c}{c^2} - \frac{p}{c^2}\right)^2}{\frac{p_c}{c^2} + \rho_c}. \quad (2.8)$$

Does this equation of state represent some form of dense neutron star matter? Probably not. Besides, a general characteristic of the equations of state coming from a choice for one of the quantities is that the vast majority of them depends on the central density. In other words, the equation of state would represent a new type of matter at each central density allowed, however, maintaining the same functional $\rho - p$ dependence.

To avoid such consternation, physicists began to employ a functional equation of state, for example the MIT Bag Model, $p = (c^2\rho - 4B)/3$ (Witten, 1984), replacing the necessity of a functional form for one of that quantities. Now we can enter the microphysics into the problem. Now we know what kind of matter we are describing. But this time, there are no known exact solution for the equations. The only solution is a numerical solution. Using the MIT Bag equation of state, Alcock et al. (1986) first obtained a numeric solution for a strange star.

The last thing we can do if we want an exact analytical solution is to overdetermine the system, e.g., imposing a choice for one of the quantities and an equation of state simultaneously (Ivanov, 2001). However, this procedure results in two solutions for one of the quantities and it is necessary to force a matching of these two in order to have the system consistently solved.

Why does a physicist want to search for an exact solution for the structure of neutron stars? Because they allow easy calculations and, most importantly, they allow to predict some new phenomena or behaviour where numeric solutions blur our insights.

Now we refer the reader to our paper regarding modelling a strange star, e.g., employing an exotic quark matter equation of state, with some known physically acceptable exact solutions and one quasi-exact solution. In this contribution we have shown that the numerical solution for the complete sequence of bare strange stars first obtained by Alcock, Farhi e Olinto using the MIT Bag equation of state can be accurately described by some exact analytical solutions for the Einstein Equations, all of them simply parametrized by one single quantity, namely, the central density. In particular, the gaussian quasi-exact solution is very useful to describe that numerical solution despite the small error in one of the elements of the metric.

On the other hand, fully exact analytical solution that take into account the equation of state for the strange stars were also employed and modelled. We call attention for the anisotropic solution that also describes the strange stars equally well the gaussian solution, although this time there is an anisotropy in pressure, which now has two components: the radial and the tangential. The mass-radius relation thus obtained matches almost perfectly the numerical solution. Besides the pressure anisotropy, this solution needs a slightly higher central density to produce the same mass.

At last, if we allow the existence of a radial electric field inside the strange star, another exact solution can be obtained, this time producing masses up to 3 solar masses.

In this contribution we also discuss the problems that results from choosing analytical profiles to quantities other than choosing the equation of state explicitly. To illustrate our point we discuss the Tolman IV and the Buchdahl I solutions when forcing a matching with the Alcock, Farhi and Olinto numerical solution.

2.1 *Exact and quasi-exact models of strange stars*

WARNING: there is a misprint mistake in page 69 of this Thesis which corresponds to page 1944 of our paper, below. The expression 5 actually reads

$$p(r) = \frac{c^2 \rho(r)}{3} - \frac{4B}{3}.$$

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EXACT AND QUASI-EXACT MODELS OF STRANGE STARS

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We construct and compare in this work a variety of simple models for strange stars, namely, hypothetical self-bound objects made of a cold stable version of the quark-gluon plasma. Exact, quasi-exact and numerical models are examined to find the most economical description for these objects. A simple and successful parametrization of them is given in terms of the central density, and the differences among the models are explicitly shown and discussed. In particular, we present a model starting with a Gaussian *ansatz* for the density profile that provides a very accurate and almost complete analytical integration of the problem, *modulo* a small difference for one of the metric potentials.

Keywords: Strange stars; exact solutions; compact objects.

1. Introduction

We study in this paper a variety of models for strange stars, namely compact objects constituted of a hypothetical cold version of a quark plasma featuring (by hypothesis) a lower energy per baryon than normal hadronic matter. Quarks were suspected to constitute a substantial fraction of some neutron star interior since the early 70s,^{1,2} and their main properties were explored. In 1984³ the possibility of a star fully composed of quarks up to its surface (but perhaps covered with a “normal” crust) was considered. The simplest model was constructed with a free quark + vacuum energy equation of state (see Ref. 4). Because the scale of the strong interactions is set by a multiple of the parameter B of the MIT Bag Model, identified as the vacuum energy density and numerically valued around 60 MeV fm^{-3} or so,⁵ these compact objects feature masses $\sim 1 M_{\odot}$ and radii $\sim 10 \text{ km}$, pretty similar to their “normal” cousins. In fact they may constitute all the neutron stars hitherto observed.

The strong interactions (not gravity) that bind these compact objects, and at least for the low-mass range $M \leq 1 M_{\odot}$, are responsible for the striking behavior of their mass–radius plot, namely the continuity of models all the way down

1938 *M. G. B. de Avellar and J. E. Horvath*

to pretty small masses $\sim 0.5 M_\odot$, the minimum allowed by the MIT equation of state. Even though the low-mass stars are clearly Newtonian, gravity is increasingly strong at the higher end and forces the use of general relativistic models when $\frac{2GM}{c^2 R} \geq 0.2$.

As is well-known, the models of these objects can be described, in a first approximation, by the solutions of the Einstein equations for a perfect isotropic, static and spherically symmetric fluid (whose metric is given by $ds^2 = c^2 e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2$):

$$\frac{8\pi G}{c^2} \rho(r) = \frac{\lambda' e^{-\lambda}}{r} + \frac{1 - e^{-\lambda(r)}}{r^2}, \quad (1)$$

$$\frac{8\pi G}{c^4} p(r) = \frac{\nu' e^{-\lambda}}{r} - \frac{1 - e^{-\lambda(r)}}{r^2}, \quad (2)$$

$$p'(r) + \frac{1}{2}(c^2 \rho(r) + p(r))\nu'(r) = 0, \quad (3)$$

where the last equation (contracted Bianchi identity) expresses the conservation of the energy–momentum of the fluid.

Mathematically, there are at least three different strategies to solve these equations:

- If the pressure, or the density or one metric element is given (for example, by an *ansatz*), an exact or numerical solution can be found by integration. However, this does not guarantee to have any control over the equation of state $p(\rho)$, and similar solution functions.
- Instead, if the equation of state $p(\rho)$ is given (i.e. the fluid is characterized from the beginning), the integration can be performed (at least numerically) and the properties of the stellar models follow.
- If both the equation of state and one additional function (p , ρ or one of the metric) are known, a match of the overdetermined system can be attempted, being in general possible for only certain values of the involved parameters.

It is not difficult to envision that the third route can be also employed *without* an overdetermined system since one provides more degrees of freedom (for example, electric field or pressure anisotropy) at the expense of modifying the originally posed physical problem. We shall return to this point below.

In this work, we have performed three types of studies:

- The modeling of a strange star by using some of the well-known exact solutions (only the ones that meet physical requirements) for the Einstein field equation for static and spherically symmetric perfect fluid, e.g. the solutions obtained by the first strategy;
- The modeling of a strange star by the third routine, e.g. overdetermining the system, in order to have not only the microphysics involved (by means of the

equation of state), but also to have control over the density profile, chosen to be Gaussian as a first reasonable guess;

- A comparison of the above studies to the true solution, or profiles, for a strange star. Here, by true solution, we mean that one obtained closing the system with the equation of state of MIT bag model (see below) and numerically integrating the system.

With this we show that the applicability of the well-known analytical solutions are rather limited for modeling these stars, and that the best one can do by having in mind the original system is to construct an analytical quasi-exact solution with an overdetermined system, at the expense of introducing an error in the geometry.

Then we show how the introduction of more degrees of freedom (specially pressure anisotropy and the existence of an electric field) can avoid an overdetermined system and at the same time provide self-consistent analytical solutions for strange stars.

2. Calculations

2.1. General considerations

As a benchmark for our calculations we have adopted the simplest numerical integration of the massless noninteracting quarks presented, for example, by Alcock, Fahri and Olinto (hereafter AFO).⁶ This is mainly motivated by the shapes of the profiles they obtained, that are reproduced in Figs. 1–3. It is clear that any other model in which interactions or finite-quark masses are needed can be obtained analogously (see Ref. 7 for an exploration of the full parameter space of the models).

Since the main goal of this work is to simplify the description of the stellar models as much as possible, possibly neglecting small ($\leq 1\%$) differences arising from

Fig. 1. Mass-central density relation of the strange star sequence calculated by Alcock, Farhi and Olinto.⁶

1940 *M. G. B. de Avellar and J. E. Horvath*

Fig. 2. Mass–radius relation of the strange star sequence calculated by Alcock, Farhi and Olinto.⁶

Fig. 3. Density profiles of four selected strange stars, calculated by Alcock, Farhi and Olinto.⁶

the fact that numerical and exact (or quasi-exact) models do not match exactly, it is useful to parametrize all the features in terms of a single input parameter, chosen to be the central density ρ_c which must satisfy $\rho_c > 4B/c^2$ in order to be consistent with the description of the matter in the MIT bag model, where $p = (c^2\rho - 4B)/3$.

To fix ideas, assume that $f(x) = ag(x) + b$ is a solution of the density profile. If an *ansatz* is given for $g(x)$, with that profile it is possible to obtain solutions for the pressure and for the metric elements. Then we fit the parameters $a = a(\rho_c)$ and $b = b(\rho_c)$, so that the mass–radius relation of AFO is reproduced as accurately as possible. After this is achieved, we can proceed to derive all the features of any given model.

However, not all exact (or even quasi-exact, see below) solutions describe a viable star: physically acceptable models should fulfil the following conditions, as discussed in the work by Delgaty and Lake⁸:

(i) $e^{\lambda(0)} = 1$

To integrate the Einstein equations, two initial conditions are needed, generally taken as $m(r = 0)$ and $p(r = 0)$. This is why this condition is so important to define the central region of the star. A small sphere of radius $r = \epsilon$ has a length of $2\pi\epsilon$ and proper radius $e^{\lambda/2}\epsilon$. A small circle around $r = 0$ has a ratio of the length to its radius given by $2\pi e^{-\lambda/2}$. However, since the space-time is locally flat, this ratio should be, for this small circle around $r = 0$, equal to 2π . Thus, $e^{\lambda(0)} = 1$ and if $r \rightarrow 0$, then also $m(r) \rightarrow 0$ (cf. Ref. 9).

(ii) $e^{\nu(0)} = \text{const} < 1$

A particle characterized by the geodesic with a given constant $\mathcal{E}$ has an energy $e^{-\nu/2}\mathcal{E}$ relative to a locally inertial observer at rest in this space-time. Since $e^{-\nu/2} = 1$ far away from the object, $\mathcal{E}$ is the energy a distant observer would measure if the particle was distant (i.e. the energy at infinity). Since $e^{-\nu/2} > 1$ for all other regions not at infinity,^a the observer will measure an energy higher than $\mathcal{E}$ in those regions. This extra energy is the energy a particle gains when it falls into the gravitational field of the object. Thus, $e^{-\nu/2} > 1 \Rightarrow e^{\nu(0)/2} < 1$ (cf. Ref. 9).

(iii) $\rho(r)$ regular at the origin and positive-definite

A region around $r = 0$, small enough to be called “center of the star” and large enough to maintain the physical features of the fluid at constant (uniform) density, must have a positive value for the fluid to be real.

(iv) $p(r)$ regular at the origin and positive definite

The pressure at a central core which is approximated by a constant (uniform) density must have a constant positive value, as a result of the properties of real matter.

(v) $\frac{dp}{d\rho} \leq c^2$

The speed of sound in the fluid should be smaller than the velocity of light.

Our first approach to the problem is described below, using the well-known Tolman IV¹⁰ and Buchdahl I,¹¹ which satisfy all properties (i)–(v) necessary for realistic models.

2.2. Exact and quasi-exact solutions

2.2.1. Tolman IV and Buchdahl I

Tolman IV and Buchdahl I are two well-known exact solutions featuring mathematically simple expressions for static self-gravitating fluid spheres. Moreover, they

^aFrom the continuity condition at the boundary of the object, $e^{\nu(r=R)} = 1 - \frac{2GM}{c^2 R} < 1$. But $e^{\nu(r>R)} = 1 - \frac{2GM}{c^2 r} \rightarrow 1$ if $r \rightarrow \infty$.

1942 *M. G. B. de Avellar and J. E. Horvath*

are physically sound and fulfil the requirements listed in the former section. We have tested the applicability of these simple models to the strange star problem, comparing them with the numerical results of AFO. A previous attempt can be found in Ref. 12.

We choose these two solutions also because the density profile can be well-fitted in order to reproduce the AFO density profile.

Tolman IV

This solution was found by Tolman in his seminal paper (10). To solve the Einstein equations, he made an *ansatz* $e^\nu \nu' / 2r = \text{const}$, a method that rendered an exact integration of the problem. The functions thus obtained read

$$e^\lambda = \frac{1 + \frac{2r^2}{A^2}}{\left(1 - \frac{r^2}{R^2}\right) \left(1 + \frac{r^2}{A^2}\right)}, \quad e^\nu = Q^2 \left(1 + \frac{r^2}{A^2}\right),$$

$$\frac{8\pi G}{c^2} \rho(r) = \frac{1}{A^2} \frac{1 + \frac{3A^2}{R^2} + \frac{3r^2}{R^2}}{1 + \frac{2r^2}{A^2}} + \frac{2}{A^2} \frac{1 - \frac{r^2}{R^2}}{\left(1 + \frac{2r^2}{A^2}\right)^2}$$

and

$$\frac{8\pi G}{c^4} p(r) = \frac{1}{A^2} \frac{1 - \frac{A^2}{R^2} - \frac{3r^2}{R^2}}{1 + \frac{2r^2}{A^2}}.$$

From these expressions, an equation of state, a physical radius (boundary, r_b) the constant Q and the total mass of the sphere are, respectively

$$\rho(p) = \rho_c - 5\frac{p_c}{c^2} + 5\frac{p}{c^2} + 8\frac{\left(\frac{p_c}{c^2} - \frac{p}{c^2}\right)^2}{\frac{p_c}{c^2} + \rho_c},$$

$$r_b = \frac{R}{3^{1/2}} \left(1 - \frac{A^2}{R^2}\right)^{1/2}, \quad Q^2 = \frac{1 - \frac{r_b^2}{R^2}}{1 + \frac{2r_b^2}{A^2}},$$

$$M_{\text{bound.cond.}} = \frac{c^2 r_b}{2G} \left(1 - \frac{\left(1 - \frac{r_b^2}{R^2}\right) \left(1 + \frac{r_b^2}{A^2}\right)}{1 + \frac{2r_b^2}{A^2}}\right),$$

with

$$\frac{8\pi G}{c^2} \rho_c = 3 \frac{R^2 + A^2}{R^2 A^2}, \quad \frac{8\pi G}{c^4} p_c = \frac{R^2 - A^2}{A^2 R^2}$$

with A , R and Q three arbitrary constants.

To match numerical strange star models as calculated by AFO with the Tolman IV solution, we seek to reproduce the mass and the radius of each model, that is, to determine A , R and Q so that the mass and the radius of the numerical calculations are reproduced. However, the central density could not be adjusted simultaneously (the latter depends on A and R , which in turn are determined by the radius and mass). A relevant question is whether the resulting equation of state is compatible with the linear expression of the MIT bag model. We can easily check that Tolman IV is not appropriate, first because it is not linear, but more importantly because it depends on the A and R present in the metric elements. When A and R are adjusted to reproduce accurately the masses and radii of the AFO calculations, they happen to depend on the central density of the model. Therefore, the equation of state varies from model to model (from star to star), despite its functional form remaining the same. This behavior is illustrated in Fig. 4.

It is clear that an equation of state that varies from model to model is not useful to describe a compact star, since it should reflect the fundamental properties of cold matter and should not behave in this way.

Buchdahl I

Much in the same way as in Tolman IV, we examined the Buchdahl I¹¹ solution. The expressions are given by

$$e^\lambda = \frac{2(1 + Cr^2)}{2 - Cr^2}, \quad e^\nu = A[(1 + Cr^2)^{3/2} + B\sqrt{2 - Cr^2}(5 + 2Cr^2)]^2,$$

$$\frac{8\pi G}{c^2}\rho(r) = -\frac{-\frac{2Cr}{2 + 2Cr^2} - \frac{4Cr(2 - Cr^2)}{(2 + 2Cr^2)^2}}{r} + \frac{1 - \frac{2 - Cr^2}{2 + 2Cr^2}}{r^2},$$

Fig. 4. Variation of the equation of state with the constants A, R present in Tolman's models.¹⁰

1944 *M. G. B. de Avellar and J. E. Horvath*

$$\frac{8\pi G}{c^4}p(r) = \frac{2 \left[3Cr\sqrt{1+Cr^2} - \frac{BCr(5+2Cr^2)}{\sqrt{2-Cr^2}} + 4BCr\sqrt{2-Cr^2} \right] (2-Cr^2)}{[(1+Cr^2)^{3/2} + B\sqrt{2-Cr^2}(5+2Cr^2)](2+2Cr^2)r} - \frac{1 - \frac{2-Cr^2}{2+2Cr^2}}{r^2},$$

where A , B and C are arbitrary constants.

Two fits were possible: in the first one we fitted the mass and radius of each model, and as a consequence $\rho_c \neq \rho_{c,AFO}$ as before. We also attempted to fit both the mass and central density, leading to different radii, depending on the masses. However, we eventually came to the same problem of the nonlinearity of the equation of state. Again the obtained equation of state depends on variable quantities and is not useful to model the quark matter.

2.2.2. Quasi-exact Gaussian model

An inspection to the density profiles presented by AFO (Fig. 3) suggests a simple and popular parametrization of the density dependence. As a simple *ansatz* we assumed a Gaussian profile, a procedure that led to an overdetermination of the system (since the linear MIT equation of state was also imposed). With these assumptions, the following expressions were immediately obtained in closed forms

$$\rho(r) = \left(\rho_c - \frac{B}{c^2} \right) e^{-r^2/r_o^2} + \frac{B}{c^2}, \quad (4)$$

$$p(r) = c^2 \rho(r) - \frac{4B}{3}, \quad (5)$$

$$\nu(r) = \frac{r^2}{2r_o^2} + a, \quad (6)$$

$$e^{-\lambda(r)} = \frac{4\pi G \rho_c e^{-r^2/r_o^2} r_o^2}{c^2} - \frac{2\pi^{3/2} G \rho_c r_o^3 \text{erf}\left(\frac{r}{r_o}\right)}{rc^2} - \frac{4\pi G B e^{-r^2/r_o^2} r_o^2}{c^4} + \frac{2\pi^{3/2} G B r_o^3 \text{erf}\left(\frac{r}{r_o}\right)}{rc^4} - \frac{8\pi G r^2 B}{3c^4} + 1, \quad (7)$$

$$e^{-\lambda(r)} = \frac{r_o^2 (8\pi G r^2 e^{-r^2/r_o^2} \rho_c c^2 - 8\pi G r^2 e^{-r^2/r_o^2} B - 24\pi G r^2 B + 3c^4)}{3c^4 (r^2 + r_o^2)}. \quad (8)$$

The problem posed in this way reduces to proving whether the two expressions for $\exp(\lambda(r))$, Eqs. (7) and (8), are equivalent. Alternatively, even if the functions

are different, they could be almost identical inside the star, and therefore a small error would be introduced by employing either. This is why we speak about quasi-exact solutions for the linear equation of state to stellar problem.

We used as benchmark the same four density profiles shown in AFO for definite values of the total mass $M = 1.99 M_\odot$, $M = 1.95 M_\odot$, $M = 1.4 M_\odot$ and $M = 0.53 M_\odot$.

The idea was to fit expressions for a and for r_o (the scale of decay of the density profile) as a function of the central density ρ_c which could reproduce the masses and radii.

After that, given any ρ_c , one may obtain any stellar model along the sequence because one can calculate the total mass, the radius, the metric elements and the density and pressure profiles.

However, it proved impossible to fit all parameters simultaneously without making the system inconsistent. Letting the radius to vary (i.e. relaxing the condition for them to reproduce the AFO results), but keeping the masses and central densities, the stellar radius R , the parameter a and the scale r_o could be obtained. As an important remark, we stress that by “fixing the mass” we mean that the integral of the density function and the boundary condition derived from the metric $\nu(r)$ must be the same, with their values the same as those found by AFO.

The fit resulting from only the four profiles did not allow for a good determination of a and r_o . Thus, the grid of models was extended to 20 profiles (i.e. 20 values of the central density) to cover wider mass and radii intervals. The values of r_o and a are shown in Table 1.

Table 1. Values of r_o and a obtained by imposing the same AFO mass value.

$\rho_c(10^{14} \text{ g/cm}^3)$	$M/M_\odot(\text{AFO})$	$r_o[\text{km}]$	a
21	2.014042750	7.631010387	-1.780521357
20.5	2.014801608	7.714337813	-1.762999185
20	2.015326779	7.801045927	-1.744982347
19.6	2.015436678	7.872850429	-1.730140677
18	2.013326957	8.185473708	-1.666867564
16	2.002284428	8.646108988	-1.577075908
14	1.975792515	9.212929519	-1.47142208
12	1.921914333	9.936363686	-1.343215145
11	1.877755062	10.38285995	-1.267268177
10	1.815572219	10.90882098	-1.180584905
9	1.726746594	11.54174779	-1.079793495
8	1.597234612	12.32531382	-0.959831395
7.5	1.510155564	12.79466558	-0.890275006
7	1.402444810	13.33296263	-0.81247546
6.5	1.267766569	13.95933096	-0.724422383
6	1.097507111	14.70148514	-0.62334087
5.7	0.973622557	15.21887892	-0.554849121
5.2	0.721604642	16.24225101	-0.423820856
4.885	0.529204878	17.02459731	-0.327483126
4.6	0.333488500	17.85691977	-0.228380811

1946 *M. G. B. de Avellar and J. E. Horvath*

Having these values for a and r_o , we proceeded to derive an accurate functional fit of the form

$$a(w) = -2.4 + 9.96e^{-w/0.19} + 2.426e^{-w/0.59} + 1.92e^{-w/3.45}, \quad (9)$$

$$r_o(w) = 5.56 + 22.46e^{-w/0.50} + 10.35e^{-w/2.42} + 169.46e^{-w/0.17} \quad (\text{in km}), \quad (10)$$

where $w \equiv \frac{8\pi G}{c^2} l^2 \rho_c$ and $l \equiv 10 \text{ km}$ is a dimensionless quantity of the order unity that embodies physical constants (c , G etc.) and the length scale of the object ($\sim 10 \text{ km}$).

With these values for r_o and a , we may wonder about the accuracy of the metric element $e^{\lambda(r)}$. We have seen that an ambiguity for this quantity exists. Its first expression was derived from the first Einstein equation (1) and the second from the second one (2). How are they really different? Actually it can be checked that the agreement is very good for the lower densities (Fig. 5), and that it worsens for the highest densities (Fig. 6) along the stellar sequence. This behavior is a consequence of the Newtonian character of the gravitational field for the lower densities, already discussed in Sec. 1.

In summary, we have shown that with only the central density value (quite the same as any other calculation of stellar structure that begins by specifying this value), the values of a and r_o can be calculated from the analytical fit. They determine in turn the profiles of density and pressure within the Gaussian *ansatz*, and with them the physical radius of the model R . Finally, the mass can be calculated analytically by integrating the Gaussian density profile and/or using the boundary condition, yielding the same values within $\lesssim 10\%$ because of the deviations of the fitted expressions from the exact numerical results. We believe that this quasi-analytical model Eqs. (4)–(8) can be useful for a (very) accurate evaluation of the stellar model structure in a variety of situations.

It is worth to mention at this point that Cheng and Harko¹³ presented a very similar approach, starting with the relativistic conditions of thermodynamic equilibrium and the MIT bag equation of state. They found very accurate approximate mass and radius formula for strange stars, in the static case and, amazingly, in the rotating case. Their errors are less than 1% for the former case and 3% for maximally rotating stars.

Their formulae, however, become increasingly inaccurate when $\rho_c \rightarrow 4B$. Here we give our derivation for mass and radius, the analog to their formulae,¹³

$$R = \sqrt{-[r_o(w)]^2 \ln\left(\frac{3B}{c^2 \rho_c - B}\right)} \quad (\text{in km}), \quad (11)$$

$$M = 5 \times 10^{-29} \frac{\left(1 - e^{\left(\frac{1}{2} \frac{R^2}{[r_o(w)]^2} + a(w)\right)}\right) c^2 R}{2G} \quad (\text{in } M_\odot). \quad (12)$$

We stress two important points about these two expressions. First, they provide the “right” value for the zero-pressure situation, although this feature should

Fig. 5. The difference between the two expressions of the metric element $e^{-\lambda(r)}$ inside the star for the model with $\rho_c = 5.2 \times 10^{14} \text{ g/cm}^3$. The radius of the star is marked by the vertical line at the end of the curves. Top: the two expressions. Bottom: the relative error as a function of the radial coordinate.

not be taken too strictly, because the expressions are only approximations. Second, although they appear quite generic, one should remember that we used $B = 57.5 \text{ MeV/fm}^3$ to derive $a(w)$ and $r_o(w)$. A different value of B would force the recalculation of the latter.

The advantage of our approach is that we have the profile for $\rho(r)$, $p(r)$, $\nu(r)$ and $\lambda(r)$ in closed form for each possible value of the central density ρ_c . Besides that, our formula are in complete agreement with the formula derived by Cheng and Harko.¹³

We also point out that recently, Narain, Schaffner-Bielich and Mishustin¹⁴ have made a numerical calculation integrating the dimensionless TOV equation with a

1948 *M. G. B. de Avellar and J. E. Horvath*

Fig. 6. The same as in Fig. 5 for the model with $\rho_c = 19.6 \times 10^{14} \text{ g/cm}^3$. The differences are bigger, as mentioned in the text.

linear equation of state for generic fermionic matter. The possibility of scaling the solutions of the TOV problem was already known and exploited, for example, in Witten's paper.³ Using this property, Narain, Schaffner-Bielich and Mishustin found a very useful general scaling solution and discussed how to rescale these equations in order to find the solutions for arbitrary fermion masses and interacting strengths. In spite of this generality, we point out that our process of making the equations dimensionless is different from the work and, moreover, the choice of a Gaussian form induces the presence of the length r_o which controls the spatial decay of the density. As a result, it is not possible to compare easily our quasi-exact models with their results, which remain more general but require the knowledge of the dimensionless M – R curve.

2.2.3. Polytropic models

A recent interesting approach was presented by Lai and Xu,¹⁵ who model a quark star with a polytropic equation of state. This equation of state is generally stiffer than the conventional linear one, like the bag model. Unfortunately, exact solutions are difficult to obtain for the polytropic case. However, their numerical results are interesting because they deal with two models: with and without QCD vacuum energy (Λ), and show how they compare to linear bag-like equations of state. (See their numerical results in Ref. 15.)

As a general feature we may say that the work of Lai and Xu shows that a polytropic description of strange stars is possible and accurate, but it does not provide an easy and economical form for the generation of models, at least not easier than a full numerical work.

2.2.4. Exact anisotropic model

The exact anisotropic model of Sharma and Maharaj¹⁶ is another accurate approximation for the stellar structure and needs, as above, only one parameter (the central density). As a physical motivation to consider this anisotropy, they quote Usov¹⁷ who supposes a strong electric field formed into a thin layer at the quark surface of a bare strange star. This is possible due to the depletion of s-quarks in this region.

The exact expressions of these models are

$$\rho(r) = \frac{b(3 + ar^2)}{8\pi(1 + ar^2)^2}, \quad (13)$$

$$m(r) = \frac{r^3 b}{2(1 + ar^2)}, \quad (14)$$

$$p_r(r) = \frac{c^2 b(3 + ar^2)}{24\pi(1 + ar^2)^2} - \frac{4B}{3}, \quad (15)$$

$$e^{\lambda(r)} = \frac{(1 + ar^2)c^2}{(c^2 a - Gb)r^2 + c^2}, \quad (16)$$

$$e^{\nu(r)} = e^{Xr^2} (1 + ar^2)^{1/3} (c^2 + c^2 ar^2 - Gr^2 b)^Y K, \quad (17)$$

$$p_t(r) = p_r(r) - \frac{1}{2} \left[-\frac{c^2 bar}{12\pi(1 + ar^2)^2} + \frac{c^2 bar(3 + ar^2)}{6\pi(1 + ar^2)^3} - \frac{\left(\frac{c^2 b(3 + ar^2)}{12\pi(1 + ar^2)^2} + \frac{4B}{3} \right) \left(\frac{Gr^3 b}{2(1 + ar^2)} + \frac{4\pi Gr^3 p_r(r)}{c^2} \right)}{r \left(c^2 r - \frac{Gr^3 b}{1 + ar^2} \right)} \right] r, \quad (18)$$

1950 *M. G. B. de Avellar and J. E. Horvath*

Fig. 7. Radial and tangential pressures for two anisotropic stellar models. Top: $\rho_c = 4.8 \times 10^{14} \text{ g/cm}^3$; bottom: $\rho_c = 24 \times 10^{14} \text{ g/cm}^3$.

and are enough to obtain the mass–radius relation, the density profile, and the pressures (tangential and radial). Here, $b = \frac{8\pi\rho_c}{3}$ and $a = \frac{1}{r_o^2}$. We have checked that the anisotropy is small (see Fig. 7) in the interval of central densities. Once the free parameters r_o and K are fitted as functions of the central density ρ_c , the models are completely specified. We show the resulting mass–radius relation in Fig. 8.

The similarity of the curve in Fig. 8 and the numerical results by AFO is apparent. However, in spite that masses and radii are practically the same, anisotropic models are systematically denser at the center by 25% or so. For example, the maximum mass model along the sequence has a central density of $\simeq 24 \times 10^{14} \text{ g/cm}^3$ whereas in the AFO numerical calculation the value is just $\simeq 20 \times 10^{14} \text{ g/cm}^3$. The density profiles are shown in Fig. 9. The similarities with AFO are also apparent.

Fig. 8. Mass-radius curve for the anisotropic model of Sharma and Maharaj.¹⁶

Fig. 9. Density profiles of the anisotropic models of Sharma and Maharaj.¹⁶

Ruderman¹⁸ argued that the anisotropies could be important for densities $\rho > 10^{15} \text{ g}/\text{cm}^3$. This is consistent with our results. It should be pointed out, however, that the anisotropy could affect some parameters like the maximum mass (the effect is small in our approach) and the redshift, as discussed long ago by Bowers and Liang.¹⁹

Only the analytical solution for the anisotropic star is not sufficient for a complete description of such an object. This is because it is important to explain where the anisotropy comes from in order to produce the differences we have shown. Mak

1952 *M. G. B. de Avellar and J. E. Horvath*

and Harko²⁰ pointed out that a source of anisotropy could be an anisotropic velocity distribution of the particles inside the star due, for example, a magnetic field, turbulence or convection. Perhaps the biggest challenge of the anisotropic model is the stability criterion. Chan *et al.*²¹ showed that, in the onset of instabilities, even small anisotropies might drastically change the stability of the system.

We have shown that the pressure anisotropy is small for low central densities but becomes larger and larger as the central density increases. Further studies are necessary in our approach to verify if all of the sequence (e.g. in all central density ranges) is stable.

2.2.5. Exact electric field model

In the same way as before, we explored the model developed by Komathiraj and Maharaj²² to model strange stars parametrized by a single parameter, the central density. This model also has all the desirable properties cited above. However, the electric field is an explicit function of the position coordinate, starting from zero at the center and growing up to the surface. The effect of this field on the mass–radius relation is to increase the masses and their respective radii. The mass–radius relation is shown in Fig. 10.

In this model, we deal with a charged strange star. It is important to notice that this is quite different from the assumption made by Usov, since the charge is distributed inside the whole star. Charged compact stars also have been studied in many ways by Ray *et al.*^{23,24} In this work they have found that a star can have an electric field of about $\sim 10^{21}$ V/m in a particular case of the polytropic equation of state. It is interesting to notice that the exact solution by Komathiraj and Maharaj with electric field also provides electric fields at boundary $> 1.5 \times 10^{20}$ V/m.

Fig. 10. Mass–radius curve for the electric model of Komathiraj and Maharaj.²²

3. Conclusions

We have used a wide set of mathematical solutions (exact and approximate) in order to model strange quark stars. All these solutions were parametrized by a single quantity, the central density. Both simple exact and quasi-exact models were addressed, with and without extra degrees of freedom (anisotropy and electric field).

No simple analytical model was found to describe these stars by keeping the equation of state functionally unchanged (see Sec. 2.2.1). Solutions other than the ones discussed here should be examined with this description in mind.

In our particular treatment of the quasi-exact solution, we have used a grid with 20 points, corresponding to 20 central densities, and we have fitted expressions for the free parameters of the problem that solves analytically the Einstein equations for a perfect fluid with spherical symmetry and a linear equation of state. The success of this particular approach can be gauged in Fig. 11, which summarizes the mass–radius relations of all solutions found, and also in Fig. 12 where we show the density profiles of the stars of maximum mass in some solutions (Tolman, Buchdahl, Gaussian-approximated and anisotropic). We remind that this work used the simple numerical models of AFO as a benchmark, but the latter has a small degree of freedom, itself based on the MIT bag equation of state. This linear behavior, however, makes the fluid equation system impossible to be integrated analytically unless we give an extra equation overdetermining the system. If this is done, then a match of the overdetermined system is necessary.

We have shown that it is possible to find a quasi-exact solution that is useful to model a strange star in an easy way. We just give the central density, and all other quantities like mass, radius, profiles *etc.* can be found. With these analytical expressions we can predict many properties of the star defined by that central density. The errors remain $\leq 1\%$ in all quantities, including the geometry (Figs. 5 and 6).

Fig. 11. Mass–radius relations for various models.

1954 *M. G. B. de Avellar and J. E. Horvath*

Fig. 12. Density profiles of the stars of maximum mass.

We can see in Fig. 11 that the Gaussian-approximated solution is valid in the range of low central densities, matching almost exactly to AFO. The anisotropic model is valid for all range of central densities and is not very different from AFO. This is a promising model (see Fig. 9 and compare it with Fig. 3; but see also the remarks in Sec. 2.2.4). The electric field model is also valid, but only in the context of charged quark stars.

This integrability problem does not appear with a system of fluid equations featuring anisotropic pressure or an electric field. These new degrees of freedom makes the system integrable. We have discussed how these results compare to the linear, isotropic, uncharged models and how they affect the actual stellar features in practice.

It should be reminded that, in spite of their deep physical differences, it is too early to dismiss any equation of state for strange stars. The extraction of the radius from the observation of thermal-like emission is still problematic and may contain substantial inaccuracies. Taken at face value, however, the determinations have rendered small numbers, and are certainly incompatible with neutron star models if true. For example, the objects EXO 0748-676²⁵ ($2.10 \pm 0.28 M_{\odot}$, $\geq 13.8 \pm 1.8$ km) and EXO 1745-248²⁶ ($1.4 M_{\odot}$, 11 km) or ($1.7 M_{\odot}$, 9 km) can be described by the anisotropic model (by taking into account the error bars) respectively ($2.0 M_{\odot}$, 11.44 km) and ($1.39 M_{\odot}$, 11.0 km) (this last one with a claimed high precision). We conclude this work by pointing out that simple, economical descriptions of a variety of strange stars can be constructed and may be useful to explore their properties and structural behavior in many static and dynamical situations.

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Information theory and measurements to infer a hierarchy of equations of state

So far, we have discussed some difficulties to address the composition of neutron star matter either theoretically or observationally. Our current technology does not have yet the necessary resolution and sensibility for this task. A beautiful theory is only a beautiful theory, if it cannot be confronted with observations. Nature does not care about theories, the way we describe it, but our goal as scientists is to decode how Nature happens to work.

We believe that in order to circumvent some of these difficulties we need to use as many methods of studying neutron stars as possible. Thus we could, in principle, to narrow the window of acceptance of possible equations of state. One of these new approaches is information theory.

Nowadays, information theory is the “flavour of the month” in physics: from cosmology to biological systems, through condensed matter and communications and also linguistics, information theory has been helping to elucidate many aspects of the behaviour of systems otherwise not amenable of detailed treatment.

This theory relies on the central concept of Shannon entropy (Shannon and Weaver, 1949) (also known as *information entropy* or *logic entropy*) that is related with the information stored in the system. It is a measure of the uncertainty associated with the value of a quantity. In other words, how much one can say about a system with the smallest piece of information: just one *bit*. Shannon entropy is defined by the quantity

$$S = -K \sum_x p(x) \log_b [p(x)], \quad (3.1)$$

where $p(x)$ is the probability distribution (see below) of the quantity to be measured and the basis b and the constant K determine the “type” of entropy being calculated. In the specific case of Shannon entropy, $K = 1$ and $b = 2$, the resulting unit of entropy being a *bit*. If instead $b = e$, the resulting unit is *nat*. It is worth to say that there is a relationship with the thermodynamic entropy. In this case, $K = k_B$, the Boltzmann constant, and $b = e$. However, one must to take some care because the relation with thermodynamic and statistical physics is not straightforward: the meaning of $p(x)$ may be very different. In the latter case, $p(x)$ stands for the the probabilities of a given microscopic state of the system and is related to a specific energy configuration.

Then, information is what we can get from observing the occurrence of an event (how surprising or unexpected or what else) and, with certain reductionism, we define information in terms of the probability of that event to occur. Another feature is that information theory deals with *any* kind of probability. The definition of information, $I(p)$, by Shannon relies in some desired mathematical properties of this quantity:

- $I(p) \geq 0$,
- $I(p_1 * p_2) = I(p_1) + I(p_2)$ (additivity),
- $I(p)$ is monotonic and continuous in p ,
- $I(1) = 0$,

from which we derive that information is:

$$I(p) = \log_b(1/p) = -\log_b(p) \quad (3.2)$$

for some constant b .

Thus, flipping a fair coin once gives you $-\log_2(1/2) = 1$ bit of information.

If a source provides n symbols $\{a_i\}$ each with probability $\{p_i\}$, then we could be interested in the average amount of information in the stream of symbols, which implies in a weighted average. Then:

$$\frac{I}{N} = \sum_{i=1}^N p_i \log_b(1/p_i) = - \sum_{i=0}^N p_i \log_b(p_i) \equiv H(P). \quad (3.3)$$

This quantity is defined as the *entropy* of the probability distribution $P = \{p_i\}$.

A good property is that the maximum of this quantity is achieved when we have equiprobability, $p_i = 1/n$. Example: a student and his grades in three situations. If the grades are A, B, C, D and F, with equal probabilities, then the student will get 2.32 bits of information. However, if the probabilities are $\{1/10, 1/5, 2/5, 1/5, 1/10\}$, then the information that the student will get is 2.12 bits; if $\{0, 0, 0, 0, 1\}$, then he will get 0 bit of information.

Equation 3.1 can be generalized to the continuous case:

$$S = -K \int_x p(x) \log_b[p(x)] dx, \quad (3.4)$$

and we will use it in what follows.

In describing a physical system, another statistical quantity related to information entropy is the complexity. But what exactly is complexity? Let us state that complexity is what does not match the requirements of being simple. It seems a tautology, but we shall see that when dealing with *physical* systems it makes sense. In physics, we always begin by studying ideal systems as a first approximation to Nature, since Nature always happens to be much more sophisticated. We study ideal systems because they are the simplest. We say that these systems have minimum complexity, say zero complexity, by construction.

Let us allow our definition of complexity to encode the order and the disorder (or the self-organization in broader terms) of a system and let us analyse two ideal systems, extremes in all aspects and opposites as well (López-Ruiz et al., 1995):

- the perfect crystal has zero complexity by definition; strict symmetry rules imply probability density centered around the prevailing state of perfect symmetry which, in turn, gives you minimal information, e. g., a small piece of information is enough to describe the whole system. The perfect crystal is completely ordered;
- the ideal gas also has zero complexity by definition; all the accessible states are equiprobable which imply maximal information. The system is totally disordered.

As we see from above, the information alone is not enough to define properly the complexity of a system. Then we define the *disequilibrium* as the distance to the equiprob-

ability. With this in mind we can get a “visual” intuition about what complexity would be, as shown in figure 3.1:

Figure 3.1: Intuitive definition of complexity.

Although there is no unique definition of complexity, we shall adopt the definition given by López-Ruiz, Mancini e Calbet (1995) as modified by (Catalán et al., 2002), because it matches the asymptotic behaviour for that two extreme ideal systems: the ideal gas and the perfect crystal. In their definition, then:

$$C = H \times D, \quad (3.5)$$

where $H = \exp(S)$ and S is the information entropy (or the information content of the system) in natural logarithmic units and D is the disequilibrium (identified with the distance of the system to its state of equiprobable probability distribution). In its original definition, the expressions for S and D are the following:

$$S = - \int \rho(\mathbf{r}) \ln[\rho(\mathbf{r})] d\mathbf{r}, \quad (3.6)$$

$$D = \int \rho^2(\mathbf{r}) d\mathbf{r}. \quad (3.7)$$

The quantity $\rho(\mathbf{r})$ is the normalized probability distribution that describes the state of the system. S describes the uncertainty associated to that probability distribution while D

stands for the information energy as defined by Onicescu (1966), or the quadratic distance to the equiprobability.

These definitions have been used by a number of authors (Panos et al., 2007; Panos and Chatzisavvas, 2009; Panos et al., 2009) to study atomic systems. With astrophysical interest Sañudo and Pacheco (2009) applied the concepts to white dwarfs and Chatzisavvas et al. (2009) applied them to neutron stars with a simple equation of state.

In astrophysical situations concerning the structure of compact objects, the choice for the quantity $\rho(r)$ may be seen as a caveat: these authors use the mass density distribution as the quantity to enter in the integrals. This is justified by noticing that the mass density distribution, that comes from the solution of the equations of structure for a relativistic star (TOVs, see below), is somehow related to the probability of finding some particles at a given location r inside the star. This is something reminiscent from the Liouville's theorem for the density of points in a fluid in the phase space, after the integration in the momentum volume.

And now we face the very relevant question: how do these concepts relate to the composition of neutron stars? Much in the same way a given equation of state determines a particular composition and a particular sequence of stars in the mass-radius diagram, it also yields a particular expression of information entropy through the density profile of each star in that sequence. Hence, we can calculate the disequilibrium and the complexity and say something about which is the state of matter inside the compact object.

In this contribution, we calculate these statistical quantities for two sequences of neutron stars: one of hadronic composition (obtained with the SLy4 equation of state) and the other of strange quark composition (obtained with MIT Bag equation of state in the context of the anisotropic exact analytical solution).

We show that the complexity of the two sequences is very low and that there is a trend for these stars to be at a state of minimum complexity and by comparing the outputs, we suggest a hierarchy of equations of state to be realized in Nature, e.g., we suggest that the quark equation of state would be preferred or would be more probable to occur after the events that lead to the formation of the compact object. However, we also discuss that even if that is the preferred equation of state, the central density barrier could prevent its realization.

To make the link with the thermodynamic of neutron stars is our next goal and one should keep in mind the G. N. Lewis statement about chemical entropy (Lewis, 1930), “Gain in entropy always means loss of information, and nothing more”.

We now refer the reader to our paper on this subject where we discuss our contribution to the problem.

3.1 *Entropy, complexity and disequilibrium in compact stars*

Entropy, complexity and disequilibrium in compact stars

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ABSTRACT

We used the statistical measurements of information entropy, disequilibrium and complexity to infer a hierarchy of equations of state for two types of compact stars from the broad class of neutron stars, namely, with hadronic composition and with strange quark composition. Our results show that, since order costs energy, Nature would favor the exotic strange stars even though the question of how to form the strange stars cannot be answered within this approach.

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1. Introduction

In the recent past, scientists from different areas have looked at information theory to characterize physical and biological systems, their patterns and correlations. The idea is that a statistical measure of *complexity* (to be defined precisely below) encodes the self-organization of a system, and links the information stored in it (or the logic/information entropy) to its “distance” to the state of equilibrium probability distribution [1]. Recently, Sañudo and Pacheco [2] first related such measures to an astrophysical object, a white dwarf star, while Chatzisavvas et al. [3] applied these same concepts to another type of compact stars, collectively known as “neutron stars”, where matter is in the densest form known in Universe and is under even more extreme physical conditions, namely supra-nuclear densities.

The importance of performing information theory studies on compact astrophysical objects results from the fact that the very nature of the matter in such extreme physical conditions is still uncertain, and these studies can shed a new light on this subject from a different point of view. In this Letter, we address “neutron stars” made of nuclear hadronic matter and made of free quarks (the self-bound strange stars, modeled in the context of MIT Bag Model).

The extension of these information concepts to astrophysical macroscopic objects is not straightforward, since the forces involved in the respective equilibrium configurations are very different from that ones in an atomic system. In atomic systems, the factor that determines the self-organization is mainly the Coulomb interaction and the Fermi exclusion principle. On the other hand, in a (ordinary) neutron star, gravity, strong and weak interactions all contribute. Finally, in a quark star the nuclear structure and the nucleons themselves have been bypassed, and the truly fundamental degrees of freedom show up to form a self-gravitating ball which is nevertheless bound by strong interactions, not gravity, although the latter is still very important for the overall structure. It is an open question whether the information quantities can be used for a gross description of these equilibrium configurations.

Motivated by these considerations, we compared the information and complexity stored in these two “neutron” stars of different microscopic composition, and found that these quantities are comparable in general, but sensitive to the composition, because the latter determines the behavior of the *radii* of the stars for the same mass. Since the value of the radius is also an important feature for an observational identification [4,5], information theory may link the formation and structure aspects.

2. Calculations and models

We used the statistical measure of complexity as defined by López-Ruiz, Mancini and Calbet [1], as modified by Catalán et al. [6]:

$$C = H \times D,$$

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where $H = \exp(S)$ and S is the information entropy (or the information content of the system) in natural logarithmic units, D is the disequilibrium (identified with the distance of the system to its state of equiprobable probability distribution). In its original definition, the expressions for S and D are the following

$$S = - \int \rho(\mathbf{r}) \ln[\rho(\mathbf{r})] d\mathbf{r},$$

$$D = \int \rho^2(\mathbf{r}) d\mathbf{r}.$$

The quantity $\rho(\mathbf{r})$ is the normalized probability distribution that describes the state of the system. S describes the uncertainty associated to that probability distribution while D stands for the information energy (as defined by Onicescu [7]), or the quadratic distance to the equiprobability.

In order to study our two types of compact objects in this way, we need an analogue to the probability distribution. Because the energy density distribution, $\epsilon(r)$ [erg/cm³], is related to the probability of finding a number of particles in a given location inside the star, we use the energy density profile as the quantity to enter in the integrals. However, in the case of the structure of our stars, the gravitation is non-Newtonian and we must solve the Tolman–Oppenheimer–Volkoff equations, or the equation of relativistic hydrostatic equilibrium of the star plus the mass integral, both complemented by the equation of state which describes the micro-physics (composition) of the stellar matter, to finally find $\epsilon(r) = c^2 \rho(r)$, where $c = 3 \times 10^{10}$ cm/s is the velocity of light and $\rho(r)$ [g/cm³] is the matter density.

In this work we use the same approach of [3] to solve the TOV equation: first we define the barred quantities as the dimensionless variables scaling as follows

$$M(r) = \bar{M}(r) M_{\odot} \quad \epsilon(r) = \bar{\epsilon}(r) \epsilon_0,$$

$$P(r) = \bar{P}(r) \epsilon_0 \quad \epsilon_0 = 1 \text{ MeV/fm}^3,$$

where $M(r)$ is the mass of the star in solar units ($M_{\odot}$), $P(r)$ is the pressure and ϵ_0 is an energy density scale, which in turn provide us the following form of the TOV equation and the mass equation

$$\frac{\bar{P}(r)}{dr} = -1.474 \frac{\bar{M}(r) \bar{\epsilon}(r)}{r^2} \left(1 + \frac{\bar{P}(r)}{\bar{\epsilon}(r)} \right) \times \left(1 + 11.2 \times 10^{-6} r^3 \frac{\bar{P}(r)}{\bar{M}(r)} \right) \left(1 - 2.948 \frac{\bar{M}(r)}{r} \right)^{-1},$$

$$\frac{\bar{M}(r)}{dr} = 11.2 \times 10^{-6} r^2 \bar{\epsilon}(r).$$

Thus, the integrals to be evaluated are

$$S = -b_0 \int \bar{\epsilon}(r) \ln[\bar{\epsilon}(r)] d\mathbf{r}, \quad (1)$$

$$D = b_0 \int \bar{\epsilon}^2(\mathbf{r}) d\mathbf{r}, \quad (2)$$

where $\bar{\epsilon}$ is the dimensionless energy density (which is just $c^2 \rho / \epsilon_0$) and obtained from the solution of the TOV equation. The parameter $b_0 = 8.89 \times 10^{-7} \text{ km}^{-3}$ is just a properly chosen quantity that makes S and D dimensionless. The integration is performed from 0 to the radius R [km]. We now refer separately to the specific cases of the hadronic star and the (strange) quark star, defined by different micro-physical descriptions.

A first treatment of the pure hadronic case has been given by [3], using a theoretically-motivated model equation of state. We instead use the so-called Sly4 equation of state in its analytic form [8] directly in the above form of the TOV equations, to obtain the energy density profiles for each initial value of the central

Table 1

Parameters of the fit.

i	a_i (Sly)	i	a_i (Sly)
1	6.22	10	11.4950
2	6.121	11	-22.775
3	0.005925	12	1.5707
4	0.16326	13	4.3
5	6.48	14	14.08
6	11.4971	15	27.80
7	19.105	16	-1.653
8	0.8938	17	1.50
9	6.54	18	14.67

pressure. Analytical representations of the equation of state are preferred over the tabulated ones, because they avoid two major problems of the latter: the ambiguity of the interpolation and impossibility of calculating the derivatives precisely. Furthermore, the analytical form is constructed obeying all the thermodynamic relations [8]. A suitable form of the equation of state Sly4 is

$$\zeta = \frac{a_1 + a_2 \xi + a_3 \xi^3}{1 + a_4 \xi} f_0(a_5(\xi - a_6)) + (a_7 + a_8 \xi) f_0(a_9(a_{10} - \xi)) + (a_{11} + a_{12} \xi) f_0(a_{13}(a_{14} - \xi)) + (a_{15} + a_{16} \xi) f_0(a_{17}(a_{18} - \xi)), \quad (3)$$

where the coefficients are given in Table 1 and $\xi = \log(\rho/\text{g cm}^{-3})$, $\zeta = \log(P/\text{dyn cm}^{-2})$.

This is currently a popular choice for detailed studies of dense matter and has all nuclear features of interest already built-in [9]. Another reason for choosing the Sly4 is that it allows maximum mass around $2M_{\odot}$, a minimum value similar to the quark equation of state discussed below.

The strange star models also need an equation of state describing the (self-bound) quark particles and their interactions. This is notoriously more involved than in the nuclear phase, since deconfinement is not yet properly understood. To calculate the information entropy, the disequilibrium and the complexity for our model of strange quark stars, we used one of the few analytical exact solution of the Einstein equations (which is of course a solution of the static TOV equation) for a spherically symmetric non-rotating perfect fluid. This solution is the anisotropic expression first obtained by Sharma and Maharaj [10] and studied by us in [11]. The advantage of this very accurate model is that in this way we have an analytical expression for the energy density that can be integrated easily, namely

$$\bar{\epsilon}(r) = \frac{1}{3} \frac{\rho_c c^2}{\epsilon_0} \frac{3 + r^2/r_o^2}{(1 + r^2/r_o^2)^2}. \quad (4)$$

In that expression, ρ_c is the central density and $r_o = r_o(\rho_c)$ is a parameter that controls the decay of the density profile. This analytical solution is obtained imposing the MIT Bag model for strange quark matter equation of state

$$p = \frac{1}{3} (c^2 \rho - 4B), \quad (5)$$

where $B \simeq 57.5 \text{ MeV/fm}^3$ is the energy density of vacuum. This simple expression (3) has been widely used because it readily captures the essential features of the deconfined phase. Crucial to our considerations of self-boundness (that is, a bound star even in the absence of gravitation [12]) is the existence and numerical value of the parametric vacuum energy density B , representing non-perturbative confining interactions. It is easily shown that for this massless quarks case

Fig. 1. Information entropy versus mass (left). The inflexion (zoomed in the right inset) marks the maximum mass of the sequence. The dots correspond to SLy4 (Eq. (3)) and the crosses to the strange quark matter (Eq. (5)).

Fig. 2. Information entropy versus radius. The strong forces bounding strange quark matter change the behavior of entropy with the radius, from monotonically growing in the hadronic case (most massive models featuring the smallest radii) to the opposite (less massive models featuring the smallest radii). The symbols are the same as in Fig. 1.

$$r = r_* \sqrt{\frac{c^4}{4BG}}, \quad m = m_* \sqrt{\frac{c^8}{4BG^3}}, \quad (6)$$

where r_* and m_* are the dimensionless radius and mass, respectively, and G is the gravitational constant. The units of r and m come out from the unit system chosen for the constants inside the square roots.

3. Results and discussion

We now present our results showing the quantities of interest and how they depend on the mass of the star and the respective radius:

We see immediately from Figs. 1, 3 and 5 that both types of stars show pretty much the same behavior of the quantities with the mass, and also that this behavior is consistent with the one obtained in [3]. However, the behavior of the same quantities as a function of the radius (Figs. 2, 4 and 6) differ enormously. While, for instance, we find that S is a *decreasing* function of the mass for both cases in the stable region, we can state that for the hadronic star this is due to the fact that when the mass increases, the radius decreases (see Fig. 1 and Fig. 2) and the energy density become

Fig. 3. Disequilibrium versus mass. The symbols are the same as in Fig. 1.

Fig. 4. Disequilibrium versus radius. A feature similar to the one in Fig. 2 is present. The symbols are the same as in Fig. 1.

Fig. 5. Complexity versus mass (left). Notice that the log scale of the vertical axis causes a behavior similar to the entropy. The points around the maximum mass are zoomed in the right inset. The symbols are the same as in Fig. 1.

more localized (smaller radius, smaller S). On the other hand, for the self-bound quark stars, the behavior of the entropy with radius is quite different: the larger the radius, the smaller the entropy S until a certain value of the former, from where we recover a behavior quite analogous to the hadronic star. This is due to the

Fig. 6. Complexity versus radius. Notice that the log scale of the vertical axis causes a behavior similar to the entropy. The symbols are the same as in Fig. 1.

very different nature of the quark stars, made of free quarks and bound together by the strong interactions, in which $R \rightarrow 0$ when $M \rightarrow 0$. Therefore, in spite that the entropy carried by the gravitational field decreases its value when the mass decreases, the admixture more than compensates this and the total entropy increases.

We can now discuss our results in terms of the concepts of information content, distance to the equilibrium probability distribution and complexity or self-organization. It is expected that the complexity vanishes for two ideal cases, in the opposite extremes of the concept of order and disorder: the perfect crystal (a perfectly ordered system) and the ideal gas (a totally disordered system) [1]. If we compare both sequence of models for compact stars, the complexity is very low in the two cases, being smaller for the hadronic star. The authors of [3] concluded that neutron stars (however, as pointed out above, they assumed a different hadronic equation of state) are low-complexity, ordered systems because the complexity parameter is low and the disequilibrium is higher for the stars with the smaller radii, that is, for the higher masses. From these considerations one can conclude that dense matter hadronic stars systems behave similarly to the perfect crystal: higher mass $\rightarrow$ smaller radii, implies in turn a more localized energy density, higher disequilibrium and lower complexity.

On the other hand, we have shown that strange quark stars display the opposite behavior with the radius: the larger the radius (until the one corresponding to the maximum mass point), the higher the mass, implying a more delocalized energy density, higher disequilibrium and lower complexity. We conclude that the strange quark stars are “less ordered” (i.e. more entropic) than the twin hadronic stars, pending to the side of the ideal gas, but still far from it, since the latter shows a low disequilibrium.

It is interesting to compare these cases with the other type of “low-density” compact star, the white dwarfs, studied in [2]. They found for these quasi-Newtonian objects that the complexity grows with increasing mass, reaching a maximum finite value at the Chandrasekhar limit. This behavior is consistent with the ones that have been reported for atomic systems [13–15], if the mass is replaced by the atomic number, and it can be related to the degenerate electron gas features. In contrast, both hadronic and strange quark stars are the result of the interplay between strong interactions and (strong) gravity, the latter being progressively less important for low masses in the second case.

It is also of interest to see the composite dependence of the statistical quantities simultaneously with mass and radius. We show in Appendix A the 3D plots (Figs. A.7, A.8, A.9).

4. Conclusions

In the preceding section we have calculated the information entropy, disequilibrium and complexity for two kinds of compact star sequences: hadronic stars and a quark stars. We observe a similar behavior of these quantities with mass, but a very different one with radius. The question that arises regarding the true composition of the neutron stars is, then, which state would be realized in nature. One could guess that, since order costs energy, then nature should favor exotic strange quark stars, but we are aware that the *path* followed to reach the state (hadronic or quark) is also very important: the parameter that controls the formation of such objects is thought to be the central density attained just after the events that lead to the huge contraction of the iron core.

Let us imagine that a hadronic star has been formed out of the iron core and settles into a equilibrium state. The strangeness barrier mentioned before would preclude the formation of a strange quark star, in spite of being the preferred state, unless the conditions for a path towards the latter can be found. A couple of possibilities arise: the first is the conversion through fluctuations [16], possibly delaying the conversion by an astronomical time-scale, being a strong function of the mass. The other is the formation of a two-flavor (strangeness ~ 0), followed by the decay towards strange matter on weak-interactions time-scale ($\sim 10^{-8}$ s). The difference of free energy per particle, which is by the strange matter hypothesis, released in the process, would come out in neutrinos [17] and a structural adjustment [18,19].

Our calculations show that the complexity of these two types compact stars with different compositions is very low, i.e., there is a trend for these stars to be at a state of minimum complexity. Calbet and López-Ruiz [21,20] have shown that for a system out of equilibrium, or extremum, there is, in fact, a tendency of the complexity to reach an extremum. Thus, if there is a transition from a hadronic star to a strange star, the system as a whole would, in general, be out of equilibrium. In this way it become clear the parallelism between the systems treated here and those addressed by Calbet and López-Ruiz, a fact which calls for further studies of this subject.

Finally we must remind that the transition will *not* conserve the star mass (the binding energy, which is negative, will be larger after the process), but possibly just the total baryon number, and therefore care should be taken in the comparison of the hadronic and strange stars, in spite that the actual difference is not large and has been ignored above. Therefore, the static analysis of equilibrium configurations cannot reveal the actual situation of matter inside compact stars, but suggests that strange quark stars should be preferred.

The question of the entropy trend as a function of the compactness for a given value of the mass also points towards the self-bound quark stars, but one should say that there is an entropy barrier between the two states that has to be overcome (a full analysis from the stellar evolution point of view is in progress and will be published elsewhere). This analysis provides a complementary look to the question of compact star composition and, in particular, to the accessibility of a (still hypothetical) self-bound quark state, somewhat reminiscent of Carathéodory view of irreversible processes [22].

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Appendix A. 3D plots of information entropy, disequilibrium and complexity

Fig. A.7. 3D plot of entropy. Notice the mass-radius relation in the M – R plane. The symbols are the same as in Fig. 1.

Fig. A.8. 3D plot of disequilibrium. Notice the mass-radius relation in the M – R plane. The symbols are the same as in Fig. 1.

Fig. A.9. 3D plot of complexity. Notice the mass-radius relation in the M – R plane. The symbols are the same as in Fig. 1.

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X-ray astrophysics as a tool to study kilohertz quasi-periodic oscillations and time lags of the X-ray emission and to probe the environment of neutron stars

4.1 X-ray astrophysics

The roots of high energy astrophysics are in the beginning of the 20th century, when Victor Hess (Hess, 1912) found out that the average ionization of the atmosphere above $\sim 1.5km$ was higher than the ionization at the sea level. This ionization is due to the *cosmic rays*, as named by Robert Millikan in 1925. After the World War II, the advent of sounding rockets made possible the discovery of sources of high energy radiation from space. In 1949, a flight carrying a Geiger counter showed that the Sun emits X-rays. But it was only in 1962 when Riccardo Giacconi and his colleagues (Giacconi et al., 1962), after setting up an experiment in a sounding rocket whose original purpose was to study X-rays from the moon, that the field of high energy astrophysics was effectively born.

In that flight, the team discovered the brightest X-ray source in the sky, *Sco X-1*, and a completely unexpected diffuse glow of X-rays coming from all direction – the X-ray background. See (Melia, 2009) for a more detailed historic view.

The reason why the high energy astrophysics is so recent when compared with optical astronomy and is mostly (but not exclusively¹) space-based is that the atmosphere is opaque to most of electromagnetic spectrum. See figure 4.1.

¹ At very high energies, the atmosphere itself is a detector since its particles are collisional targets for this very energetic radiation coming from space. When such collision happens, the outcome is a cascade of new produced particles that produces a blue light known as Cerenkov radiation detected on the ground.

Figure 4.1: Atmospheric windows: how deep from the space radiation of different wavelengths can penetrate Earth's atmosphere.

Another feature of high energy astrophysics is that we have much less photons than in other wavelengths since it costs more to Nature to produce them. Because all of this, the field is a challenge from the point of view of experiment and of the theory. First, because of the complexity of the detector (as we shall see below), but also because of the interaction of matter and radiation under extreme conditions is at the limit of the known theories.

The energetics of the processes involved are extreme (Melia, 2009). The release of gravitational energy of an object of mass m falling onto a typical neutron star of mass $M_\star$ and radius $R_\star$ is

$$E_{acc} = \frac{GM_\star}{R_\star} \approx 10^{20} \text{ erg/g}, \quad (4.1)$$

which is about 20 times greater than the energy released in the nuclear fusion of hydrogen.

As an example, during the accretion processes, the strong gravitational field acts more on protons, whereas the radiation release from the fall and from the central object acts more on electrons, because the latter has bigger cross-section. But because the Coulomb interaction loosely maintains the plasma coupled, there is a balance between the gravitational and radiation forces that prevents the accretion. This is the Eddington luminosity,

L_{edd} :

$$L_{edd} \approx 1.3 \times 10^{38} \left(\frac{M_{\star}}{M_{\odot}} \right) \text{erg/s} \quad (4.2)$$

whose black-body effective temperature is $\sim 10^7 K$, in the X-ray band.

Because of this extreme energetics, we need special methods to focus and to properly detect high energy photons (Melia, 2009, for a more complete description). Taking X-rays as example: unlike the optical and UV photons, they reflect from a surface of conducting materials only at high incidence angles (for $1keV$ photons, $i = 87^\circ$). To focus them on the detectors, we need a set of mirrors assembled in a shape of nested barrels (see figure 4.2).

Figure 4.2: The mirror assembly to focus X-rays is composed by a set of paraboloid and hiperboloid mirrors nested. The satellites are usually very long.

On the other hand, the detection is not simple. The first devices for detecting photons up to $\sim 20keV$ were the proportional counters. They are nothing more than a gas-filled discharge tubes with a voltage drop across the gas. When a photon enters the tube, it produces a high energy electron which, in turn, initiate a cascade of electron-ion pairs. This cascade produces a current proportional to the incident photon energy and this information is enough to reconstruct the X-ray spectrum.

Today the devices to detect photons up to $\sim 20keV$ evolved to the microcalorimeters. Here, the photon energy is measured after it has been converted into heat, without worrying about the characteristic charge transport properties of the detector. Generally, it is composed by an absorber, a temperature sensor and a link to a heat sink.

For energies even greater ($\sim 20\text{keV}$ to several MeV), the photons are very penetrative and we need another kind of device: the scintillation counter. Its basic element is a crystal of CsI or NaI. The idea behind is that the energetic γ -ray Compton scatters (inelastically at this energies) several times before the photoelectric absorption occurs. The photon loses energy at each scatter and the ionization energy lost is converted in visible light, which is converted into an electrical signal by a photomultiplier tube.

The late 1990s and the beginning of 2000s were the dawn of a new era in high energy astrophysics with the launch of Rossi X-ray Timing Explorer (RXTE), the XMM-Newton and the Chandra X-ray Telescope. Chandra and XMM are kind of complementary in imaging and in the spectroscopic qualities: while the first has an unmatched angular resolution, $\sim 0.''05$, eight times better than other satellites for imaging, the second has an unprecedented large effective area, $\sim 1500\text{cm}^2$, providing a high energy resolution.

But our work in this Thesis is about timing in low mass X-ray binaries (LMXBs), or the X-ray variability of these sources. For this task we used data from RXTE. Its strength is its unparalleled temporal resolution, capable to discern variability on timescales of months down to microseconds, in an energy range from $2 - 250\text{keV}$. It was the RXTE that discovered the kilohertz quasi-periodic oscillations in compact binaries, the subject of this chapter.

Millisecond variability naturally occurs in processes of accretion onto a compact stellar-mass object and these rapid variations arising from the inner accretion flow are stochastic in nature. Therefore, statistical techniques are necessary to study them and the Fourier analysis is the commonest used tool (van der Klis, 2006, see this reference for a more detailed description).

The *power spectrum* is obtained from the Fourier transform of a X-ray lightcurve, which is a time series, to provide the variance in terms of the Fourier frequency. The power spectrum has many components in a frequency range from millihertz to kilohertz (the highest frequencies so far discovered). Broad structures are called “noise” and narrow ones quasi-periodic oscillations. In figure 4.3 we can see the some components of the power spectra. Some components are described below (van der Klis, 2006):

- Power-law noise: follows a power law $P_\nu \sim \nu^\alpha$ with $0 < \alpha < 2$.
- Band-limited noise: is the noise that steepens towards higher frequency (the local

(a) Estado F

(b) Estado G

(c) Estado H

Figure 4.3: Variability components. The letters F, G and H stand for the position of the source in the colour-colour diagram when it was in the soft state. This figure is from reference van Straaten et al. (2003)

power-law slope, $-d(\log P_\nu)/d(\log \nu)$, increases with ν either abruptly or gradually.

- Quasi-periodic oscillations: is a finite-width peak in the power spectrum, generally described by a Lorentzian, $P_\nu \propto \lambda/[(\nu - \nu_0)^2 + (\lambda/2)^2]$. ν_0 is the centroid frequency and λ is the full width at half maximum. The quality factor is the quantity $Q \equiv \nu_0/\lambda$ and $Q > 2$ denotes a QPO, by convention.

In the spirit of Fourier analysis, we describe in the next two sections the tools and methodology used to study the kHz QPOs and the time lags in the source 4U 1608–52.

As we have already said, the upper kHz QPO is usually associated with the radius of the compact objects through expressions for the orbital frequency and ISCO. Below we show how it is done.

For equatorial circular orbits in a Kerr space-time (the space-time around a punctual spinning mass)² the orbital frequency is given by:

$$\nu_\phi = \frac{\sqrt{GM/r^3}}{2\pi} \frac{1}{1 + j(r_g/r)^{3/2}} = \nu_k(1 + j(r_g/r)^{3/2})^{-1}, \quad (4.3)$$

where ν_k is the keplerian orbital frequency, $r_g = GM/c^2$ and $j = Jc/GM^2$ is the angular momentum parameter. Beside, $J = I\omega$ and $\omega = 2\pi s$, where s is the spin frequency of the compact object and $I = 2MR^2/5$ is its moment of inertia.

To first order in j , the ISCO radius and its corresponding frequency are:

$$R_{ISCO} \simeq 6r_g(1 - 0.54j) \quad (4.4)$$

and

$$\nu_{ISCO} \simeq (c^3/2\pi 6^{3/2}GM)(1 + 0.75j) \quad (4.5)$$

With the equations 4.3 and 4.5 it is possible to constrain the equation of state (see figure 4.4).

² The Kerr space-time is a good approximation to the space-time around a spinning neutron star; more generally, the metric depends on the EoS.

Figure 4.4: The outermost exclusion area is the constraints to M and R from the work with 4U1608-52 for $\nu \rightarrow \nu_{ISCO} = 1064\text{Hz}$ whose spin is 619Hz . The innermost exclusion area is for a fake source for which we imagine $\nu_{ISCO} = 1220\text{Hz}$ with spin 353Hz ; the two equations of state shown in red and green are for strange quark star following the MIT Bag Model and for hadronic stars following the SLy4. The solid black area indicates the black hole line formation ($M=R/2,964$).

4.2 Shift-and-add

Now we describe a very useful technique to deal with the very fast variations in the lightcurve, the kilohertz quasi-periodic oscillations (kHz QPOs) (Méndez, 2001). As we have seen, we use Fourier techniques to obtain the Power Density Spectra (PDS) and to study the temporal features of the lightcurve. This technique is called shift-and-add and we will describe it below.

Suppose you have a lightcurve with $512s$ from one observation. In order to study phenomena at small time scales, we divide the observation in contiguous segments of uniform length to set up the required frequency resolution: for example, we can divide the observation in 8 segments of $64s$. Now, on each segment we perform a Fast Fourier Transform (FFT):

$$S_j \equiv \sum_{k=0}^{N-1} s_k e^{2\pi i j k / N} \equiv S(f_j), \quad (4.6)$$

where s_k is the time series, or our *counts/s*, in a way that we have, in the end:

$$S_{j,0} = \text{Re}[S_{j,0}] + i \times \text{Im}[S_{j,0}],$$

$$S_{j,1} = \text{Re}[S_{j,1}] + i \times \text{Im}[S_{j,1}],$$

...

$$S_{j,7} = \text{Re}[S_{j,7}] + i \times \text{Im}[S_{j,7}],$$

in which each segment spans a wide frequency interval. Thus, we obtain the PDS performing the multiplication by the conjugate complex, $P_k(f_j) = S_{j,k}^* S_{j,k} = S_{j,k}^2$:

$$S_{j,0}^2 = \text{Re}^2[S_{j,0}] + \text{Im}^2[S_{j,0}],$$

$$S_{j,1}^2 = \text{Re}^2[S_{j,1}] + \text{Im}^2[S_{j,1}],$$

...

$$S_{j,7}^2 = \text{Re}^2[S_{j,7}] + \text{Im}^2[S_{j,7}],$$

Thus, we have just obtained the PDS for each segment, as illustrated in the figure 4.5. But we are interested in the PDS of the whole observation that we get by averaging over all segments, e.g., all k from 0 to 7, in the sense that for each frequency (x-axis) in the PDS we average the value of its correspondent power (y-axis):

$$P(f_j) \equiv \langle S_{j,k}^2 \rangle = \frac{S_{j,1}^2 + \dots + S_{j,7}^2}{N}. \quad (4.7)$$

We end up with figure 4.6, the PDS for the whole observation.

If the kHz QPO is strong enough to be measured in short time segments then we can shift the frequencies of all PDSs to a reference frequency just before to perform the operations and the average: $S_{j-j'} \equiv \sum_{k=0}^{N-1} s_k e^{2\pi i(j-j')k/N} \equiv S(f_j - f_{j-j'})$.

Thus, $P_k(f_j - f_{j'}) = S_{j-j',k}^* S_{j-j',k} = S_{j-j',k}^2$ and the average over the intervals becomes

Figure 4.5: Power density spectrum for eight segments of 64s showing one kHz QPO each. Figure from reference Méndez (2001).

Figure 4.6: Average of the eight PDSs of the previous figure: we get the PDS for the whole observation. Figure from reference Méndez (2001).

$$P(f_j - f'_j) \equiv \langle S_{j-j',k}^2 \rangle = \frac{S_{j-j',1}^2 + \dots + S_{j-j',7}^2}{N}, \quad (4.8)$$

where the frequency f'_j is the shift frequency for each interval. For example, if we chose the reference frequency to be 450 Hz and the kHz QPO is in 400 Hz, $f'_j = -50Hz$; if, instead, the frequency of the kHz QPO of the second interval is in 550 Hz, $f'_j = 100Hz$.

Applying the frequency shift we gain statistics because, since the S/N ratio for the QPO is $\sim (T/W)^{1/2}$, where T is the total length of the observation and W is the width of the QPO, the alignment applied to the power spectra increases the statistical significance of this feature, making the QPO narrower. The result is striking (figure 4.7):

Figure 4.7: Average PDS of the whole observation after the shift-and-add technique. Notice the gain in significance and the consequent appearance of the second peak of kHz QPO. Figure from reference Méndez (2001).

Not only the stronger (lower) kHz QPO becomes even stronger, but now a “hidden” (not significant) kHz QPO becomes apparent (significant): the upper.

In our work, we applied for the first time the shift-and-add technique to calculate the time lags in the frequency range of the kHz QPOs, as we shall discuss soon.

4.3 The coherence function and the phase and time lags

In the same way one calculates the PDS, one can calculate quantities that are defined for two or more concurrent processes. These quantities are related to the cross-spectrum.

Time lags are Fourier-frequency-dependent measures of the time delays between two concurrent and correlated time series, in our case, between two X-ray light curves in different energy bands (Nowak et al., 1999).

Given two light curves, $s(t)$ and $h(t)$, it is possible to find a linear transformation

$$h(t) = \int_{-\infty}^{\infty} t_r(t - \tau) s(\tau) d\tau \quad (4.9)$$

where t_r is the transfer function between s and h . Taking the Fourier transform of this convolution we go to the frequency space where $H(f) = T_r(f)S(f)$ are, obviously, the Fourier transform of $h(t)$, $t_r(t)$ and $s(t)$.

The transfer function is something reminiscent from the signal processing analysis. This function is a linear mapping of the Laplace transform of the input signal (analogue to $s(t)$ in our case) to the Laplace transform of the output signal (analogue to $h(t)$ in our case).

On the other hand, the Fourier coherence is a Fourier-frequency-dependent measure of the linear correlation between that two time series or light curves. The coherence function (Nowak et al., 1999) is defined as

$$\gamma^2(f) = \frac{|\langle S^*(f)H(f) \rangle|^2}{\langle |S(f)|^2 \rangle \langle |H(f)|^2 \rangle}. \quad (4.10)$$

The coherence function, then, is a measure of the degree to which the transfer function $T_r(f)$ is constant for the data segments and frequencies over which we took the average. If $T_r(f)$ is constant throughout the the measurements of $s(t)$ and $h(t)$, then the process is said to be perfectly coherent (unit coherence). This means that we can predict the output signal from the input signal and vice-versa. On the other hand, small coherence implies a net phase lag.

The Fourier phase lag is the phase of the average cross spectrum, $C(f)$. Hence, $\phi(f) = \arg[C(f)] = \text{atan}\left(\frac{\text{Im}[C(f)]}{\text{Re}[C(f)]}\right)$, where $C(f) = \langle S^*(f)H(f) \rangle$.

In the same way as with the PDSs, we have the cross density spectra (CDSs) which are the products $S_0^*H_0(f)$, $S_1^*H_1(f)$, ..., $S_7^*H_7(f)$ (if $N = 7$, or eight segments as in our example). Taking the average of these products we get the average cross spectrum $C(f) = \langle S^*H(f) \rangle$. The corresponding time lag is $\tau(f) = \phi(f)/2\pi f$.

The importance of these statistical measurements is that they can provide strong constraints on models relating the energy spectra and the aperiodic variability features like the kHz QPOs. In particular, near perfect coherence can rule out models that assume spatially extended sources or emitting regions like thermal flares *etc* (Vaughan and Nowak, 1997; Nowak et al., 1999).

4.4 *Time lags in the kilohertz quasi-periodic oscillations of the low-mass X-ray binary 4U 1608–52*

This paper have not been submitted yet.

Time lags in the kilohertz quasi-periodic oscillations of the low-mass X-ray binary 4U 1608–52

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ABSTRACT

Since (part of) the high-energy radiation emitted from the surface of the neutron star (NS) is reprocessed and re-emitted at lower energies in the accretion disk, one can constrain the size of the (re-)emitting region measuring the time lags between low- and high-energy photons of the kilohertz quasi-periodic oscillations (kHz QPOs). We studied the system 4U 1608–52, in which the frequency of the kHz QPOs ranges from 540 to 1060 Hz, with the peak separation being $\sim 300Hz$. We find a significant dependence of the time lags with energy ($0.015msec$ at $6keV$ to $0.075msec$ at $18keV$), but a weak (if any) dependence with frequency and we use these results to constrain the location at which these QPOs are produced. Based on our results, we concluded that the size of the emitting region should be small and very near the neutron star if we use a down-scattering model for the delays.

Key words: X-ray, LMXB, neutron star, time delays.

1 INTRODUCTION

In binary systems with a weakly magnetized neutron star (NS) as primary and a low-mass secondary star, there is an accretion disc that extends very close to the surface of the NS (Lewin & Joss 1983). The kilohertz quasi-periodic oscillations (kHz QPOs) in this kind of

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2 *M. G. B. de Avellar, M. Mendez, A. Sanna and J. E. Horvath*

systems are thought to be produced at (or close to) the inner edge of the accretion disk (for a brief review on these two assertions see van der Klis 2006; Kaaret et al. 1999, and references therein). In this paper we study the neutron star X-ray binary 4U 1608–52 using data from the Rossi X-ray Timing Explorer (RXTE) of the 1998 outburst of the source.

4U 1608–52 is a low magnetic field, low luminosity ($\sim 0.1L_{\text{edd}}$) transient atoll (Hasinger & van der Klis 1989) source in which the kHz QPOs appear in pairs, and showing outbursts with recurring times varying from 80 days to years (Lochner & Roussel-Dupre 1994). This source was also found out to exhibit time lags of order of tens of microseconds (Vaughan et al. 1997).

Energy-dependent time/phase lags are one of the most powerful, promising, and so far less used tools to constrain the X-ray emission mechanisms and geometry in these systems (Nowak et al. 1999; Vaughan et al. 1997; Kaaret et al. 1999; J. Poutanen 2000). In this paper, taking into account that our dataset is larger than that in previous works, we selected our data in energy channels and QPO frequency, using the centroid frequency of the lower and upper kHz QPOs, to keep track of the QPO parameters as a function of frequency.

The paper is structured in the following way: in section 2 we describe the data set and the methodology employed in our study; in section 3 we show the results of the data analysis; in section 4 we discuss the results in the light of a existing downscattering model; and in section 5 we present some conclusions and perspectives.

2 OBSERVATIONS AND METHODOLOGY

We have studied the low mass X-ray binary 4U 1608–52 using data from the Rossi X-Ray Time Explorer (RXTE) satellite of the 1998 outburst, from 24 to 29/03/1998 (see table 1), a subset of the observations used in (Méndez et al. 1998). Our data refers to a period in which the source was in the soft/high state, which means that the source had relatively low hardness ratio (hard energy band/soft energy band) and high X-ray flux. To be more precise, 4U 1608–52 was in the transition period from the hard/low to the soft/high state (van Straaten et al. 2003). The data come in two modes, each with different definitions of the energy channels (32M and 64M, see table 2) of the Proportional Counter Array (PCA) detector onboard RXTE.

We produced Fourier Power Density Spectra (PDS) of the full energy band every 16 seconds, with a Nyquist frequency of 2048 Hz. We intend in this work to calculate the time

Time lags in 4U 1608–52 3

OBS-ID	Date	#PDS $\times$ 16 sec	Data mode
30062-02-01-000	24/03/1998	13280	32M
30062-02-01-00	24-25/03/1998	5296	32M
30062-02-01-01	25/03/1998	11584	32M
30062-02-01-02	26/03/1998	8016	32M
30062-01-01-00	27/03/1998	8352	64M
30062-01-01-01	28/03/1998	2336	64M
30062-01-01-02	29/03/1998	4448	64M

Table 1. Observations

32M		64M	
Channels (abs)	$\langle E \rangle$ [keV]	Channels (abs)	$\langle E \rangle$ [keV]
8-12	3.75	8-12	3.75
13-18	5.56	13-17	5.42
19-26	7.96	18-23	7.34
27-35	10.94	24-29	9.47
36-45	14.33	30-41	12.29
46-51	17.49	42-46	15.86
		47-55	18.10

Table 2. Energy bands - averaged by flux

lags of the lower and the upper kHz QPOs in 4U 1608–52 as a function of QPO frequency. In order to do this, we must first identify both kHz QPOs in this source. Here we follow the same procedure described in (Méndez et al. 1998) and (Méndez et al. 1999). In short, the procedure consisted in three steps: first, to produce PDSs every 16s (here we produced also FFTs every 16s); then to track the kHz QPO frequency in time in all PDSs and the final step was to identify the kHz QPOs (upper or lower) by relating the peak separation to the lower kHz QPO and later to the X-ray color and intensities, empirically identified both features.

We then proceeded selecting our data in energy and frequency bands as shown in the tables 2 and 3. The criterium for the frequency selection was defined to span a sufficiently small range in order to minimize the effects of changes of the properties of the QPO with frequency, and to have a sufficiently large number (balanced among the bands) of FFTs in order to reduce the statistical uncertainties.

It is not possible to track the upper kHz QPO on short time scales (table 3) in the same way we described before, because it is weak and broad (Méndez et al. 1999, 1998). We, therefore, took the plot of (Méndez et al. 1998) relating the peak separation to the lower kHz QPO. We then fitted a quadratic function plus a gaussian to the data. In this way we were able to calculate the frequency of the upper kHz QPO as a function of the frequency of the lower kHz QPO.

4 *M. G. B. de Avellar, M. Mendez, A. Sanna and J. E. Horvath*

Frequency range (Hz)	#PDS 32M	#PDS 64M
540-610	635	0
610-690	553	238
690-770	582	399
770-823	616	309
859-910	612	0
910-993	549	176
993-1040	604	353
1040-1064	622	417

Table 3. Lower kHz QPO (upper panel) and upper kHz QPO (lower panel) - counting all observations in that mode together

After we had the frequencies of the lower and the upper kHz QPOs, we applied the shift-and-add technique, which consists in shifting the centroid frequency of the QPO to a reference frequency in each data segment before we calculate the average cross spectrum of all observation. This technique increases the signal-to-noise ratio around the QPO features. This is the first time that the shift-and-add technique is applied to calculate time lags and the coherence of kHz QPOs.

We then combined all the FFTs within each frequency interval to measure the time lags for the lower and upper kHz QPO via the cross-spectrum of the higher energy bands relatively to the reference energy band, defined to be the absolute channels 8-12 of both modes, with mean energy 3.75 keV. Notice that it was possible to combine the data of both modes, 32M and 64M, because we defined the same reference energy band for both, and simultaneously intercalated the other channels without superposition in energy, much like a ladder.

Figure 1 show the time lags of the averaged cross spectrum of the observation after the shift-and-add have been applied in all frequency space. Notice that the time lags show large error bars and are very noise except in the region of the shifted lower kHz QPO. This region spans approximately twice the full width at half maximum of the kHz QPO.

The last step in calculating the time lags of the kHz QPOs was to average the time lags in that twice FWHM range for the given frequency selection for each energy band.

Time lags are Fourier-frequency-dependent measures of the time delays between two concurrent and correlated time series, in our case, between two X-ray light curves in different energy bands. On the other hand, the Fourier coherence is a Fourier-frequency-dependent measure of the linear correlation between those two light curves in different energy bands. Perfect coherence (unit coherence) means that we can predict the signal of one lightcurve

Figure 1. Time lags calculated for the energy band 7.96 keV relatively to 3.75 keV for the frequency selection 690 Hz to 770 Hz.

from the signal of the other. (See Nowak et al. 1999, for a complete explanation about the coherence and lags and how to calculate them).

The importance of these statistical measurements is that they can provide strong constraints on models relating the energy spectra and the aperiodic variability features like the kHz QPOs. In particular, near perfect coherence can rule out models that assume spatially extended sources or emitting regions (Vaughan & Nowak 1997).

3 RESULTS

In Figure 2 we plotted the time lags for three of the frequency selections for the lower kHz QPO. The figure shows that soft photons lag the hard ones (Vaughan et al. 1997; Kaaret et al. 1999). The solid curves show the best-fitting results for the model we used (see below). From this Figure it is apparent that there is a strong dependence of the lags with energy, but there is no (or very weak) dependence on the frequency of the kHz QPO.

Figure 3 shows that there is no significant dependence of the lags with energy for the upper kHz QPO; we note, however, that the upper kHz QPO is weaker and broader and therefore less significant than the lower kHz QPO in each selection. However, given the large error bars, we cannot discard that the energy dependence of the lags of the upper kHz QPO is similar to that of the lower kHz QPO.

Next, we separated the data in two groups, depending on the energy: 4-12 keV and 12-20 keV, and we plotted the mean frequency in the interval vs the mean delay for each group with respect to the reference energy band (see Figures 4 and 5). It is apparent from the plots

6 *M. G. B. de Avellar, M. Mendez, A. Sanna and J. E. Horvath*

Figure 2. Lag [msec] vs energy [keV], lower kHz QPO in frequency bands 610-690 Hz (red), 690-770 Hz (green) and 770-823 Hz (blue). The lines are the best fit for the density number of the emitting region (the color indicates the frequency band the fitted curve belongs to.)

Figure 3. Lag [msec] vs energy [keV], upper kHz QPO in all frequency selections.

that there is no significant dependence of the time lags on the frequency of the kHz QPO. The f-test for the 4-12 keV gives $\text{prob} = 0.10$ and the f-test for the 12-20 keV gives $\text{prob} = 0.26$. In both cases the linear fit is not significant better than a constant.

Finally, we calculated the intrinsic (noise subtracted) Fourier coherence between each of the energy bands to the 3.75 keV band (see Figure 6) in the same way we did for the time lags. Because of the poor statistics, the coherence of the upper kHz QPO is unconstrained. In Figure 7 the average intrinsic coherence of the averaged cross spectrum in the frequency space is shown. Notice that in all but exactly in the region of the lower kHz QPO the error bars are big and the coherence is noisy. Again we see how the shift-and-add technique is powerful.

Time lags in 4U 1608-52 7

Figure 4. Mean lag [msec] vs mean frequency, lower kHz QPO between 4 and 12 keV.

Figure 5. Mean lag [msec] vs mean frequency, lower kHz QPO between 12 and 20 keV.

Figure 6. Intrinsic Fourier-coherence vs Energy [keV] for each frequency selection; data modes 32M and 64M. The dark horizontal line represents the perfect coherence.

8 *M. G. B. de Avellar, M. Mendez, A. Sanna and J. E. Horvath*

Figure 7. Intrinsic Fourier-coherence calculated for the energy band 5.56 keV relatively to 3.75 keV for the frequency selection 690 Hz to 770 Hz.

4 DISCUSSION

For the first time the shift-and-add technique was used in the cross spectrum to calculate the time lags and the coherence of kHz QPOs (see figures 1 and 7 for a nice example). This allowed us to gain statistics when we divided our observations first in energy and after in frequency bands. We found that on time scales of the kHz QPOs the soft photons lag the hard ones by $\sim 10\mu\text{sec}$ to $\sim 70\mu\text{sec}$ (Vaughan et al. 1997), (Kaaret et al. 1999), confirming previous results, but with a larger data set in more than twice energy bands, besides the division of the data in frequency intervals, which was not done before. The soft lags increase with energy but surprisingly depend weakly (if depend at all) on the kHz QPO frequency. Using a model proposed to explain these delays (Falanga & Titarchuk 2007), we calculate the size of the emitting region and compare with other results (Kaaret et al. 1999). Finally we find that the Fourier coherence on the time scale of the kHz QPO is consistent with unity.

The fits give the equivalent density of the reflector (the disc); we calculated the typical size of the scattering region in the disc for different optical depths for each frequency band except the first frequency band (540-610 Hz) that shows no significant trend of the lags with energy (see table 4).

We assumed, at first, that these lags are due to the inner edge of the disc that down-scatters the photons and thus adopted the geometry in ref. (Falanga & Titarchuk 2007). We found soft lags and for this reason we used the approximation below (valid in this case) in order to fit our data see (Falanga & Titarchuk 2007).

Time lags in 4U 1608–52 9

Frequency range (Hz)	n_e [10^{20}cm^{-3}]	a_1 [km]	a_5 [km]
610–690	1.19 +/- 0.23	0.13 +/- 0.02	0.63 +/- 0.12
690–770	1.11 +/- 0.11	0.14 +/- 0.01	0.68 +/- 0.07
770–823	1.02 +/- 0.11	0.15 +/- 0.02	0.74 +/- 0.08

Table 4. Sizes of the scattering region given by the best-fitting the density, for $\tau = 1$ (a_1 ; threshold of optically thin disc) and $\tau = 5$ (a_5 ; optically thick).

$$\delta t \approx -\frac{1}{\sigma_T n_e^{\text{ref}} c} \left(\frac{1}{b_*} - \frac{1}{b} \right). \quad (1)$$

$$u \equiv \frac{c\tau_{\text{opt}}\delta t}{a} = n_e \sigma_T c \delta t \Rightarrow a = \frac{\tau_{\text{opt}}}{n_e \sigma_T}. \quad (2)$$

The physical quantities in the above formula are the following: δt are the lags; σ_T is the Thomson cross section; n_e^{ref} is the electron particle density of the reflector medium (the disc); c is the velocity of light; b_* and b are the dimensionless energy for the bands (the star stands for the reference energy band and $b = E/(m_e c^2)$, where $m_e c^2 = 511 \text{keV}$ is the rest mass of the electron). The remainder quantities are: u , the number of scatters; τ_{opt} is the optical depth of the reflector and a is the sought size of the (re-)emitting region.

On the other hand, assuming the α -disc model, and assuming mass and radius $[M, R_*] = [(1.74 + / - 0.14) M_\odot, (9.3 + / - 1.0) \text{km}]$ (T. Güver et al. 2010), neutron star spin $\nu_{\text{spin}} = 619 \text{Hz}$ (Hartman et al. 2003) and magnetic field $B = (1.4 - 1.8) \times 10^8 \text{G}$ (van Straaten et al 2006) we have for 4U 1608–52:

- $R_{\text{ISCO}} \simeq 14 \text{km}$;
- $H_{\text{ISCO}} \simeq 82 \text{m}$;
- $\rho_{c, \text{ISCO}} \simeq 3.9 \text{g/cm}^3$;

where R_{ISCO} is the radius of the innermost stable circular orbit, and H_{ISCO} and $\rho_{c, \text{ISCO}}$ are the scale height and the density of the disc at the ISCO (here, the density refers to the central plane of the disc).

We found a dependence of time lags on energy as function of the frequency of the kHz QPO. As figure 2 shows, the best fit is consistent in being the same for the three frequency selections, resulting in similar number densities of the gas on which the photons are scattered off.

An interesting point to be noticed is that we found a particle number density of about $1 \times 10^{20} / \text{cm}^3$ that is something about $1 \times 10^{-4} \text{g/cm}^3$. Even if there is no ISCO and the α -disc extends close to the surface, the density would be $\rho_c \simeq 1.7 \text{g/cm}^3$.

Now, if we calculate the vertical density of the disc (remember we have assumed a

10 *M. G. B. de Avellar, M. Mendez, A. Sanna and J. E. Horvath*

simple model for the disc), we see that $\rho_v \sim \exp(-z^2)$. If we are at ISCO, then the density at $z = H = 82$ meters is $2.4g/cm^3$ and if we are at $r = 9.5km$, then the vertical density at $z = H = 34$ meters is $1.0g/cm^3$. So, even the vertical density at the edge of the disc (where $z = H$ at the respective R) does not match the calculated density from our model.

In this way we see that the reflector cannot be a α -type disc in the vicinity of the star. A possibility is that the α -disc extends to and is truncated at ISCO from where the radial velocity increases dramatically and the density decrease sharply. In this scenario, the reflection might occur somewhere between the star and the inner edge of the disc located at ISCO. Or in a thin corona surrounding the star.

However, assuming a weak dependence on frequency, one can get very different locations on where the delays are produced relatively to the edge of the star once the number density can vary very rapidly inside the ISCO.

The density we obtained is much lower than the (central) density calculated from the α disc model. But this is expected because it seems unlikely that the x-rays would penetrate so deep in the disc to be scattered off. A possibility that deserves more studies is that the x-rays are scattered in the disc atmosphere or surface of the disc where the optical depth is lower than one. We could think in calculate a vertical structure including some dissipation and see if we get a match to our density.

Again, we found that the lags do not exhibit clear dependency on frequency what lead us to think that the region where the lags are produced is very small.

This is corroborated by the coherence of the lower kHz QPO being consistent with unity.

However, there may be a caveat in our methodology: if there is a keplerian relation between the frequency of the upper kHz QPO and the inner radius of the disc then this would be a way to calculate the inner radius and a dependence on the frequency would be apparent. But the kHz QPOs in this source span $\sim 300Hz$. Then, $\nu_{upper} = 840Hz \Rightarrow R_{in} \simeq 20km$, while $\nu_{upper} = 1120Hz \Rightarrow R_{in} \simeq 17km$. The difference in $R_{in} \sim 3km$ seems undetectable.

On the other hand there are sources, e. g. 4U 1636-53, in which the upper kHz QPO spans a much wider frequency range $\sim 800Hz$. For such difference we expect $\Delta R_{in} \sim 15 - 20km$, which could be detected if there is a frequency dependence.

5 CONCLUSIONS AND NEXT STEPS

- The size of the downscattering region is very small, which means that the delays are produced very near the neutron star; this is in agreement with the assertion that the consistency of the Fourier coherence with unity precludes extended sources; the size of the downscattering region is consistent with being the same for the three frequency bands.
- The results are based on the model of downscattering of hard photons from the neutron star surface off the inner edge of the disc (Falanga & Titarchuk 2007), and we are now working to develop a model that considers reprocessing in the disc.
- Another study that is already in progress is to study the time lags and the coherence for 4U 1636-53 using a even much bigger data set.

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12 M. G. B. de Avellar, M. Mendez, A. Sanna and J. E. Horvath

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Conclusion

At long last we reached the end of a first stage of our long term goal of advancing in the understanding of the superdense matter at the interior of neutron star. That is, we have contributed to the big synthesis needed for this goal, as described so enthusiastically in the beginning.

Admitting from the very beginning the interesting possibility of strange matter to be the ground state of matter, which can be formed in the so high density interior of neutron stars, we begun our study by the mathematical framework that describes these compact objects.

We obtained exact and quasi-exact analytical solutions parametrized uniquely by the central density with valuable predictive power for many situations physically and astrophysically relevant. These solutions were obtained from the Einstein equations for a spherically symmetric and static perfect fluid, which lead directly to the relativistic version of the hydrostatic equilibrium equation, when supplemented with the equation of state of dense matter (here, the strange matter equation of state from MIT).

Here, we called the attention to the exact solutions with the anisotropic pressure and with the electric field. These two solutions, while exacts from the mathematical point of view, still need to be physically explained. It is necessary to substantiate theoretically the existence of the anisotropy in the pressure or the existence of a radial electric field running through the whole extension of the star. The case of the radial electric field seems actually unlikely and perhaps the insertion of the radial electric field degree of freedom is only a mathematical artefact. However, several authors seriously considered them (Ghezzi, 2005; Ray et al., 2006; Negreiros et al., 2009), since the electric field allows

stars with very low masses up to the causal limiting mass of $3M_{\odot}$. Their actual occurrence in Nature should be considered as open. The case of the anisotropy in pressure is not strong either, but nevertheless seems to be more plausible, since it could be related with the velocity distribution and other mechanical statistical properties of particles (Gleiser and Dev, 2004). The quasi-exact gaussian solution is also physically relevant, but only in a limited range of central densities.

Noticing that there are many proposals in the literature for the structure equations that directly employ the strange matter equation of state, we move on to the analysis under the point of view of information theory. Our intent was to find out, from the perspective of the self-organization of the system, if we could infer something about the energetic costs to form a strange star instead of a hadronic star. Our results indicate that strange stars should be energetically favoured, although there is a minimum density that has to be achieved when the contraction of the proto-neutron star happens for SQM to form (Benvenuto and Lugones, 1999). A strange star has low complexity, like the hadronic stars, but because its disequilibrium is lower than for hadronic stars, we conclude that strange stars are “less ordered”, which is easier for Nature to form.

These two theoretical approaches so far discussed are complementary to each other about the strange quark hypothesis and are in accordance with the advances arising from the theoretical physics of strong interactions.

From the observational point of view and assuming again the strange matter hypothesis as true (surely in the colour-flavour-locked version), we studied the mass *vs* radius for four astrophysical systems for which there are such measures. From this “observational” mass-radius relation we showed that, although the error bars are large and we indeed need of more accurate and precise measurements, we could constrain the energy gap of CFL and the mass of strange quark in accordance with what we could expect from independent sources of information. This form of strange matter has a lower energy per baryon than the previous unpaired version and can be even more “fundamental”. In spite of this, we still cannot assure that observations favour the strange matter composition. The crucial factor is not in the discovery of higher masses, but mainly in the detection of “neutron stars” with very low mass and measurements of their radii: radii $R_{\star} \leq 9km$ would definitely indicate that we are dealing with self-bound stars (quarks), while radii $R_{\star} \geq 9km$ would

indicate hadronic stars. And so we move on to the next step: the analysis of the X-ray emission of the low mass X-ray binaries that contain a neutron star.

At this stage very early development, we studied the variability in the X-ray lightcurve of the system 4U 1608–52. For a long time we know that this system presents two kilohertz quasi-periodic oscillations, known as lower and upper kHz QPOs. Generally we associate the upper kHz QPO with the inner radius of the disc or with the innermost stable circular orbit (ISCO). Therefore, it is possible to derive the radius (and sometimes the mass) of the compact object or, in the worst case, some upper limit for this quantity. From the highest upper kHz frequency detected in our dataset, $\nu_{upper} = 1064 Hz$, we put an upper limit of $17 km$ on the radius, for an assumed mass of $1.74 M_{\odot}$ (without including relativistic effects). We also studied the time lags in the frequencies of kHz QPOs and we found out a strong dependence on energy, but a weak dependence (or none at all) on the Fourier frequency. The dependence on frequency would indicate different locations to where the lags are produced, again putting limits on the parameters of the compact object. Not only this: we also studied the intrinsic coherence. For all the energy and frequency bands the intrinsic coherence is of order of unit. If on the one hand the time lags impose constraints to the size of the emitting region and to its density, the intrinsic coherence rules out models of extended emitting sources. Models that explain the lags and the coherence have potential to provide constraints on the neutron stars parameter and also to the geometry of these systems.

Ultimately, from our calculations we can state that the astrophysical measures cannot rule out the strange matter in no way, remaining this form of matter as one of the most fantastic possibilities to be confirmed or excluded in the next years.

For the future, our goals are to study other systems LMXBs with a much bigger dataset in order to analyse the kHz QPOs and time lags and the intrinsic coherence. Additionally we intend to study anomalous X-ray pulsar and soft gamma repeaters, with the aid of the exact solutions from which we intend to derive observable quantities of neutron stars, without forgetting the interesting new approach of information theory. This is a long term project, but we believe that will lead us directly to a revealing synthesis about the true nature of matter inside the neutron star.

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Appendix

Appendix A

QCD Parameters

A.1 Self-bound models of compact stars and recent mass-radius measurements

Here we used the Particle and Nuclear Physics approach discussed in chapter 1 to extract information regarding the parameters of the strange quark matter equation of state. The study was performed with two versions of exotic equation of state, namely, the MIT Bag Equation of State (in which we have unpaired free uds quarks) and the Colour-Flavour Locked Equation of State (in which the three quark flavours can form Cooper pairs which allow further lowering in the energy per baryon in the system making it more stable).

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In this contribution studied strange quark star with pairing (the so-called Colour-Flavour Locked, or CFL, equation of state) and we show that the parameters of the CFL can be effectively constrained only when our instruments are capable of producing measurements of masses and radii with sufficiently small error bars.

We also discuss how the controversy in radii measurements quoted in this paper dramatically affects the results, not only concerning the CFL, but also in general.

PHYSICAL REVIEW D **84**, 043004 (2011)**Self-bound models of compact stars and recent mass-radius measurements**

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The exact composition of a specific class of compact stars, historically referred to as “neutron stars,” is still quite unknown. Possibilities ranging from hadronic to quark degrees of freedom, including self-bound versions of the latter, have been proposed. We specifically address the suitability of strange star models (including pairing interactions) in this work, in the light of new measurements available for four compact stars. The analysis shows that these data might be explained by such an exotic equation of state, actually selecting a small window in parameter space, but still new precise measurements and also further theoretical developments are needed to settle the subject.

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I. INTRODUCTION

The first reasonable ideas about the composition of compact stars stated that matter under extreme densities is composed of hadrons, i.e., mainly neutrons with small fractions of protons and also electrons. Further theoretical developments and modern experimental results opened the window to other possibilities as the existence of hyperons inside such stars due to the high densities, and a whole new class of condensates (see, for example, [1] and references therein). Another possibility arisen in the '70s is the existence of deconfined quarks inside compact objects, either located only in the inner core of these stars or present up to their surfaces. The latter extreme possibility was suggested about three decades ago [2–6], emerging as an astrophysical realization of the stability scenario of the so-called strange matter.

Consequences of the so-called *strange matter hypothesis* for compact stars have been extensively analyzed within the simple framework of the Massachusetts Institute of Technology (MIT) Bag Model [7,8] and the Nambu-Jona-Lasinio model [9–11] (see [12] for an overview on the subject). After an initial round of perturbative considerations [13,14], the possibility of nonperturbative pairing between quarks [9,15–19] allowed a new view on this subject due to a great enhancement on the window of stability for strange quark matter (having the strange quark mass, bag constant, and the novel pairing energy of the quark condensate as parameters in this approach). This state of totally paired 3-flavor quarks is called the color-flavor locked (CFL) strange quark matter, in which quarks form Cooper pairs, thus lowering further the energy of the system.

Much work has been put forward in order to have a better understanding on the actual composition of neutron stars, both observational and theoretical. Nevertheless, there is no absolute conclusion for any model as yet.

We intend in this work to contribute to the understanding of how the new and much more precise astrophysical measurements of the mass and radius of neutron stars [20–23] can help revealing the viability of exotic quark star models. We analyze, in particular, the viability of the class of CFL strange quark matter models. They are likely to be the most favorable candidates for self-bound stars, in spite that more complex models can be devised (for example intermediate phases such as Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) may be present [24–26]), and they would constitute a benchmark in testing exotic compact star composition.

In the rest of this work we quantitatively show that accurate astrophysical measurements can already constrain important parameters of QCD, like the Cooper pair energy gap (Δ) and the strange quark mass (m_s), and show how this is done given a subset of the present data.

II. ANALYSIS

When considering unpaired strange quark matter (*uds* matter) the equation of state (EoS) assumes the simplest form when $m_s \rightarrow 0$: $\epsilon = 3P + 4B$, being ϵ , P , and B , the energy density, pressure, and bag constant, respectively. The nonzero strange quark mass has the effect of increasing the energy of the system [7]. On the other hand, if the system presents pairing interactions between quarks it has the net effect of lowering the energy of the system, making it more stable.

There have been different approaches to analyze this effect. Here, in order to set a framework for the class of models studied, we adopt the MIT bag model thermodynamic potential of CFL strange quark matter [16,18] in

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M. G. B. DE AVELLAR, J. E. HORVATH, AND L. PAULUCCI

bulk to order Δ^2 , being Δ the pairing energy gap for the phase, given by

$$\Omega_{\text{CFL}} = \Omega_{\text{free}} - \frac{3}{\pi^2} \Delta^2 \mu^2 + B. \quad (1)$$

In this expression, the CFL state is taken as a model composed of quark matter without pairing plus a term corresponding to the energy of the diquark condensate. Being so, the pairing energy gap is taken as a free parameter, instead of being defined in a self-consistent manner (see, for example, [27] for further discussions and more consistent models). The thermodynamic potential for the unpaired state is then

$$\Omega_{\text{free}} = \sum_{i=u,d,s} \frac{1}{4\pi^2} \left[\mu_i \nu \left(\mu_i^2 - \frac{5}{2} m_i^2 \right) + \frac{3}{2} m_i^4 \log \left(\frac{\mu_i + \nu}{m_i} \right) \right] \quad (2)$$

with $\mu = (\mu_u + \mu_d + \mu_s)/3$ and the common Fermi momentum, $\nu = 2\mu - \sqrt{\mu^2 + m_s^2/2}$.

The EoS is finally obtained by relating the energy density ϵ to the pressure P as follows

$$\epsilon = 3\mu n_B - P \quad (3)$$

with the particle density given by $n_B = (\nu^3 + 2\Delta^2 \mu)/\pi^2$, and the pressure $P = -\Omega_{\text{CFL}}$.

Because of the fact that $m_s \neq 0$, the EoS have to be computed numerically. However, the CFL EoS shows a remarkably linear behavior [19]. Because of this, much work has been done in order to construct a simple EoS that includes the effects of pairing and a heavy strange quark, keeping the linear behavior to a certain extent. We start our analysis, then, by postulating a linear EoS for CFL matter.

We have parameterized the equation of state for CFL matter as

$$P = \alpha(c^2 \rho - 4B_{\text{eff}}), \quad (4)$$

where α is a parameter related with the stiffness of the EoS and B_{eff} is certainly related to the bag constant B and gap parameter Δ . Within this approximation, it is easy to show that the Tolman-Oppenheimer-Volkoff (TOV) and the EoS equations can be written in dimensionless forms:

$$p'_\star = -\frac{m_\star \rho_\star}{x^2} \left(1 + \frac{p_\star}{\rho_\star} \right) \left(1 + \frac{4\pi x^3 p_\star}{m_\star} \right) \left(1 - 2\frac{m_\star}{x} \right)^{-1} \quad (5)$$

$$m'_\star = 4\pi x^2 \rho_\star \quad (6)$$

$$p_\star = \alpha(\rho_\star - 1), \quad (7)$$

where the starred quantities are the dimensionless mass, pressure, and mass density; x is the dimensionless radius, and $\alpha \equiv \frac{1}{3+\beta}$ is an effective parameter which depends on the quantities Δ and m_s in a complicated but very mild and continuous way.

PHYSICAL REVIEW D **84**, 043004 (2011)

Our main goal here was to devise an easy way to display the location (*locus*) of the maximum mass and its correspondent radius, e. g., fitting a curve $f = f(x, \alpha)$. This *locus* strongly depends, in our parameterization, on α . The higher its value, the higher the maximum mass. Because of this, we needed to construct several mass-radius relationships varying α within an acceptable range, in this case, from 0.20 to 0.36. To comply with observations, we needed to reach at least $2M_\odot$. We then fitted an expression for the maximum mass ($m_{\star, \text{max}}$) of each sequence and their correspondent radius (x_{corr}) dependent on α , yielding the result

$$m_{\star, \text{max}} = -0.163\alpha^2 + 0.201\alpha + 0.00286 \quad (8)$$

$$x_{\text{corr}} = -0.262\alpha^2 + 0.329\alpha + 0.111. \quad (9)$$

Restoring the units, we finally find the relation

$$M_{\text{max}} = R_{\text{corr}} \frac{c^2}{G} \frac{m_{\text{max}}}{x_{\text{corr}}}, \quad (10)$$

given that

$$r = x \sqrt{\frac{c^4}{4B_{\text{eff}}G}} \quad m = m_\star \sqrt{\frac{c^8}{4B_{\text{eff}}G^3}}.$$

Here is the point where the dependence on B_{eff} appears. Notice that the radius and the mass individually depend on B_{eff} but the relation between maximum mass and its correspondent radius (when the physical units are restored) does *not*; it depends only on the value of α . This feature is already present in Witten's original paper and seems to be generic for any linear equation of state.

From the effective parameters found above, we have analyzed the agreement of the mass-radius relation obtained for different sets of parameters with data for the following stars: 4U 1608-522 [20], EXO 1745-248 [21], 4U 1820-30 [22], and PSR J1614-2230 [23]. The results are presented in the following section.

III. RESULTS

The value of α as a function of B_{eff} can be constrained by using the observed mass of PSR J1614-2230, considering that the maximum mass provided by a specific EoS should at least equal the mass of this heavy star. This can be obtained from Eqs. (8)–(10) or by varying the values of α and B_{eff} on Eq. (4) and constructing the corresponding mass-radius sequences. Within this previously determined region and having generated the mass-radius relations for the effective equation of state using these constrained values, we compared them with the other three stars within a 3σ demanded precision. The effective parameters, i.e. all the pairs (α, B_{eff}) , that would be suitable for explaining all four data at the same time, i.e. the observed masses and radii and their respective error bars, are those

SELF-BOUND MODELS OF COMPACT STARS AND RECENT ...

PHYSICAL REVIEW D **84**, 043004 (2011)

corresponding to the line shown on the left panel of Fig. 1. The initial and final points of this line were used to generate the mass-radius relations shown on the right panel of the same Figure.

We finally find a very narrow window for α and B_{eff} that can provide compatibility with all four measurements together. As a consequence, the mass-radius relations shown on the right panel of Fig. 1 are almost completely overlapping.

We would like now to relate the effective parameters determined through this analysis with the free parameters determining the CFL matter state. Unfortunately, although the EoS presents a linear behavior, the linearization of Eq. (1) is not possible analytically, unless some major approximations are made. Without the latter, one cannot directly relate α and B_{eff} to Δ , m_s , and B . Previous attempts to find such a relation include the work of Alford *et al.* [28], where a form of parameterized EoS was proposed, in which QCD corrections to the pressure, represented by a parameter $c \approx 0.3$, have been added. This parameterization relied on a power series expansion in μ for the thermodynamic potential of the quark phase, reading

$$\Omega_{QM} = -\frac{3}{4\pi^2} a_4 \mu^4 + \frac{3}{4\pi^2} a_2 \mu^2 + B_{\text{eff}} \quad (11)$$

with $a_4 = 1 - c$ and a_2 as free parameters. In their approach, the strange quark mass and gap parameter affect a_2 alone and B_{eff} is determined by the density at which the transition between quark and nuclear matter phases occur, as they were interested in addressing hybrid stars. Although they have been able to write $a_2 = a_2(m_s, \Delta)$,

and one can in fact for CFL quark matter in equilibrium (within this framework) separate the thermodynamic potential expression to order m_s^4 with only three terms proportional to μ^4 , μ^2 , and μ^0 , the chemical potential itself has a nonlinear and nontrivial dependence on all other parameters, as shown in details in [19]. We have chosen not to make this approximation here, so that we are not able to write α and B_{eff} as function of m_s , B , and Δ in analytical closed form.

Since it is not possible to obtain the expression relating the parameterized EoS to the CFL one, in order to verify the effectiveness of our own linearization, we have taken the full equation of state for CFL strange quark matter, as written in Eq. (1)–(3), and analyzed the agreement of the mass-radius relation obtained for the same set of stars as before. We have covered acceptable ranges for the free parameters: the bag constant $B > 57 \text{ MeV/fm}^3$, the superconductor gap $0 \leq \Delta \leq 50 \text{ MeV}$, and the strange quark mass $m_s \geq 100 \text{ MeV}$. Recent measurements give a very precise value for the s quark mass at the energy scale of 2 GeV: $92.4 \pm 1.5 \text{ MeV}$ [29]. At a smaller energy scale this value should be even higher (see [30], Particle Properties—quark masses, and references therein) so we decided to adopt the value of 100 MeV as a lower boundary.

After solving the TOV equations for several sets of (m_s, B, Δ) and then comparing the values of mass and radius with the measurements under consideration, the best agreement found is the one giving the mass-radius relation shown in Fig. 2. The “optimal” values of the parameters found are $B \sim 69 \text{ MeV/fm}^3$, $\Delta \sim 50 \text{ MeV}$, and $m_s \sim 120 \text{ MeV}$ (well within the window of stability for CFL strange quark matter, thus justifying the quark star

FIG. 1. On the left panel, effective parameters allowing compatibility within 3σ with recent data (taking into account the observed masses and radii and their respective error bars). On the right, mass-radius measurements are shown as crosses with 3σ error bars. Data from PSR J1614-2230 is shown as a shaded region, since there is only information on the mass for this object. The two mass-radius relations shown correspond to the lower and upper limits of α and B_{eff} that can explain the data (shown on the right).

M. G. B. DE AVELLAR, J. E. HORVATH, AND L. PAULUCCI

PHYSICAL REVIEW D **84**, 043004 (2011)

FIG. 2. Best agreement for the mass-radius relation with recent measurements (data represented as in Fig. 1) considering the CFL equation of state. Values for the parameters are $B \sim 69 \text{ MeV/fm}^3$, $\Delta \sim 50 \text{ MeV}$, and $m_s \sim 120 \text{ MeV}$.

scenario). Notice, however, that for 4U 1820-30 the mass-radius curve is in agreement within $\sim 3.4\sigma$ only. It should also be noticed, nevertheless, that one cannot state that in this case the 3σ boundary is the one corresponding to 99.9% of compatibility, since there are very few degrees of freedom involved in the analysis. Therefore, the agreement can be considered as acceptable.

We see that, although the equation of state for CFL strange quark matter clearly presents a linear behavior in

the pressure versus energy density dependency, when a parameterization of the type $P = \alpha(c^2\rho - 4B_{\text{eff}})$ is made, the coefficients α and B_{eff} cannot be treated as independent from each other. Since we are working with a “toy model,” in which one considers a system without any pairing plus a term corresponding to the condensate energy, Δ surely adds up to the bag constant but has no influence on the “unpaired state.” However, the role of the strange quark mass is more subtle. In the analysis discussed in [28], the authors show that B_{eff} , besides being related to the vacuum pressure (and, in the CFL phase, also to the gap parameter), incorporates an additional term which is proportional to m_s^4 when the power series expansion is made. So even in a simplified scheme, we see that B_{eff} is not dependent on the bag constant alone but should also incorporate additional terms which are important for determining the stiffness of the EoS.

In this way, even having found values for the parameters in the linearization that could fit the data within a $\sim 3\sigma$ region, when a full equation of state is employed, one cannot reach such a good agreement. This kind of problem is also evident in the oldest expression employed by Benvenuto and Horvath [31] $P = \frac{1}{3+a}[c^2\rho - (4+b)B]$, the complicated dependence of the parameters on m_s , B , and Δ (and hence their correlation) affects the accuracy of the $M_{\text{max}} - R$ locus and blurs to some extent an accurate determination of the parameter space.

Another general aspect worth of noticing is the very soft raise of the mass-radius curve for paired strange quark matter. It could be attributed to the simplicity of the model, not considering interactions between quarks other than

FIG. 3. A second evaluation of the reconstructed data as given in [32]. On the left side, mass and radius of 4U 1608-522, EXO 1745-248, and 4U 1820-30 and mass range for PSR J1614-2230 considering $R = R_{ph}$. On the right, the same for $R \gg R_{ph}$ (see text for further details). Two examples of mass-radius relations for CFL stars are shown for the sets of parameters (B , Δ , m_s) given by $(69 \text{ MeV/fm}^3, 50 \text{ MeV}, 120 \text{ MeV})$, and $(58 \text{ MeV/fm}^3, 62 \text{ MeV}, 110 \text{ MeV})$ to highlight the fact that, in this case, the mass-radius relation is constrained solely by the massive pulsar.

SELF-BOUND MODELS OF COMPACT STARS AND RECENT ...

PHYSICAL REVIEW D **84**, 043004 (2011)

pairing and one-gluon exchange (see [7] for a discussion on the scaling of the bag parameter with $\alpha_c \neq 0$). A steeper raise of the mass-radius relation could indeed provide a better agreement with the observed data and perhaps an increase in the maximum allowed mass. This could be an important issue since it would be impossible to reach masses as high as $2M_\odot$, for example, with a single quark matter EoS that would also need to explain the values for measured masses and radii for the stars 4U 1608-522, EXO 1745-248, and 4U 1820-30, if they happen ultimately to be accurate and correct.

There have been claims, however, that the values of masses and radii for 4U 1608-522, EXO 1745-248, and 4U 1820-30 have been mistakenly derived [32]. The computational method used in the previous analysis was criticized pointing out possible systematic errors which substantially reduce the uncertainties in the masses and radii values for these systems. Considering in turn the determinations given by Steiner, Lattimer, and Brown [32], we have repeated the previous calculations to verify whether the CFL equation of state is compatible with the reconstruction of the data presented by them. The results of our analysis for this case are very different, though. We see that the very narrow parameter space previously found now becomes an extremely wide region of possibilities, as depicted in Fig. 3.

The error bars for these systems, according to Steiner, Lattimer, and Brown, are much bigger than previously considered, and all three stars are clustered with close values for their masses and radii ($M \sim 1.5M_\odot$ and $R \sim 11$ km with more visible differences for the mass when the photospheric radius for photon emission is considered to be much larger than the stellar radius, $R_{ph} \gg R$; the color correction f_c is a key ingredient for this behavior). This leads now to a large set of parameters m_s , B , and Δ that could in principle explain the data, basically having the mass of PSR J1614-2230 as the main quantity providing the constrain for the equation of state. In other words, one can find several sets of (B, Δ, m_s) that satisfy the necessary conditions within the values needed in this analysis. The general behavior is to have a lower maximum mass with an increase in the strange quark mass and bag constant when the values of the other two variables are held fixed. As the gap parameter increases, the maximum allowed mass also increases.

IV. CONCLUSIONS

Precise measurements for masses and radii of compact stars are very important for constraining their composition. However, besides being precise, it is necessary that they also be accurate for a clear comparison of the mass-radius relation coming from very different compositions with data.

Given the present controversy of the radius determination quoted in the paper, this appears to be the main issue to be resolved for an advance of the conclusions of studies like ours. Meantime we can only conclude that only with small error bars for both mass and radius determinations (as in the analysis of [20–22]) the parameters of the CFL equation of state can be effectively constrained. Otherwise, several sets of parameters defining the CFL EoS can be used to match the data. In this case, more advanced detection techniques and physical analysis are required to constrain the radius R_{ph} in x-ray burst models.

Since one should not treat α and B_{eff} as independent quantities in a parameterized EoS for CFL strange quark matter like the one proposed here, an analysis performed with a parameterization with correlated coefficients should be explored. We intend to address this subject in a future work.

An additional interesting report is the very low mass of at least one neutron star mass observed ($M = 0.87 \pm 0.07M_\odot$ for 4U 1538-52) [33], actually not well explained by ordinary stellar evolution arguments. This “low-end” mass extreme is also important for the quest of exotic compositions in their interiors. In fact, confirming such a measurement would imply a radius of $R \leq 9$ km within the self-bound hypothesis, while the measurement of, say, $R \geq 11$ km would be totally inconsistent with it, pointing to a normal composition. We are at the edge of finally addressing the fundamental composition of compact stars, at least on a first level that leaves a handful of possibilities left out of a huge number of proposals.

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M. G. B. DE AVELLAR, J. E. HORVATH, AND L. PAULUCCI

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