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Optimal Patent Duration in a Lab-equipment Model

Duração ótima de patentes em um modelo Lab-equipment

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Versão Corrigida

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*“Confutatis maledictis,
Flammis acribus addictis,
Voca me cum benedictus.
Oro supplex et acclinis,
Cor contritum quasi cinis,
Gere curam mei finis.”*

(Wolfgang Amadeus Mozart, Requiem Mass in D Minor, 1791).

Resumo

Neste artigo estuda-se a duração ótima de patentes em um modelo *lab-equipment* de inovação horizontal no qual bens patenteados são imitados - e os direitos de Propriedade Intelectual, como consequência, são derogados. Dada uma tupla de parâmetros de preferência e produção, pode ser ótimo para o agente público escolher a duração de patente de tal forma que não haja investimento em P&D, em detrimento de trajetórias com pesquisa e caminhos de crescimento balanceados, em contraste com a visão de que, em um *framework* como este, a redução de proteção a patentes constitui uma perda social. Ademais, para valores do coeficiente de aversão relativa ao risco maiores que um, pode-se encontrar uma solução interior para o problema de duração ótima de patentes, correspondendo à duração dos chamados modelos de utilidade - um tipo de Direito de Propriedade Intelectual criado no escopo da proteção de bens com aplicação industrial - em países da OCDE.

Palavras-chave: Duração de Patentes. Crescimento Endógeno. Lab-equipment. Inovação Horizontal

Abstract

In this article we study optimal patent duration in a lab-equipment horizontal innovation growth framework in which patented intermediate goods are imitated – and intellectual property rights (IPR) are as a result overruled. Given a tuple of parameters faced by the policy-maker, it may be optimal to choose patent duration so that no R&D takes place, to the detriment of BGP trajectories, in contrast to the view that in such a framework reducing patent protection constitutes a social loss. Moreover, as we allow for higher values of the relative risk aversion coefficient, we might find interior solutions to the optimal patent duration problem corresponding to utility models - a type of IPR designed for protecting intermediate goods - terms in OECD countries.

Keywords: Patent Duration. Endogenous Growth. Lab-equipment. Horizontal Innovation

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1 Introduction

The issue of Intellectual Property Rights (IPR) protection sets up a widely acknowledged trade-off to the policymaker between static dead-weight losses due to monopoly power and dynamic fostering of welfare-improving innovations (Reinganum (1989), Barro e Martin (1995)).

In this paper, we provide a striking result, in a horizontal-innovation growth model, regarding the desirability of R&D: given a tuple of parameters faced by the policymaker, it may be optimal to choose patent duration so that no R&D takes place, to the detriment of BGP trajectories. Mathematically, such result stems from the explicit consideration of physical constraints binding the evolution of product varieties to be never strictly decreasing and from the derivation of trajectories and calculation of utility levels not only in the steady-state, but from $t = 0$.

The issue of optimal patent duration has been looked over by the literature; an early strand of academic works concluded for the optimality of infinitely-lived patents – Judd (1985) and Gilbert e Shapiro (1990), for example -. In opposition, a more recent group of papers, such as Horii e Iwaisako (2007), Futagami, Iwaisako e Tanaka (2007) Bessen e Maskin (2009) Chen e Iyigun (2011) points in the direction of the optimality of finite-term patents.

Our paper, by its turn, is more closely linked to another strand of the literature, headed by Kwan e Lai (2003) and Cysne e Turchick (2012), which, akin to us, deploy the lab-equipment specification of the R&D based growth model of Rivera-Batiz e Romer (1991) endowed with exogenous imitation of patents in order to calculate optimal patent lengths. The lab-equipment story assumes that the intermediate goods firms use the final good in order to produce the intermediate good. Innovation in horizontal-innovation models consists of an expansion of product variety. Besides, we assume that valid blueprints are subject to an exogenous imitation process (as in Barro e Martin (1995), or Gancia e Zilibotti (2005)) which brings about the overruling of IPR rights over the affected blueprint.

In particular, in the framework described above, Kwan e Lai (2003) and find that under logarithmic utility it is optimal to provide an infinite patent protection, meaning that the aforementioned trade-off cuts always in the same direction.¹ Nonetheless, in contrast to their paper, we find that, under logarithmic utility, it may be optimal either to set infinite patent duration or to just extinguish IPR from the legal framework, conceding no protection at all for innovations. That is, trajectories with no R&D may be socially better than BGP trajectories with research.

Such discrepancy in results stems from the explicit consideration, in our setting, of the

¹ This view is advocated also in Barro e Martin (1995), which state that, in a setting akin to ours, "a reduced rate of innovation constitutes a social loss"

physical constraints binding the measure of valid and overruled patents to never decrease, ruling out the possibility that ideas be reversible. In fact, the balanced growth path trajectories found in Cysne e Turchick (2012), where such restrictions are not imposed, cannot be sustained for a subset of the parametric space without violation of the imposed physical restrictions. On the contrary, in this subset, once the constraints are incorporated in the consumer's problem, trajectories implying no investment in R&D arise in equilibrium, which becomes particularly relevant once it is shown that these trajectories may yield higher utility levels than trajectories encompassing research. That is, even when the policymaker, through its choice of policy variable - patent length - can implement either type of trajectory, it may be optimal to provide no patent protection and implement an equilibrium without R&D investment. This result links the desirability of R&D investments to the legal and economic framework of an economy, placing under duress the (naive) view that R&D investments are *a priori* beneficial. In particular, we prove that given a sufficiently low level of costs associated with R&D activity, implementing an equilibrium with research will be optimal.

Moreover, as we allow the coefficient of relative risk aversion to assume values greater than unity, we are able to find interior solutions to the optimal patent duration problem. In fact, for reasonable values of the underlying parameters, the optimal patent terms fit reasonably well to utility models terms in OECD countries. An utility model is a simplified and less stringent form of IPR's, better fitted to innovations possessing an industrial scope, in comparison to patents involving final goods. This is particularly relevant in our framework since here innovation concerns (an expanding variety of) goods used as inputs in the production of a final good, which represents the scope of such IPR.

The equilibrium behaviour of agents in our framework, therefore, provides some interesting conclusions:

1. Given a tuple of parameters faced by the policymaker, it may be optimal to choose patent duration so that no R&D takes place, to the detriment of BGP trajectories.
2. Economic agents will prefer to abdicate from R&D activities if proper structural conditions for innovation are not there
3. For logarithmic utility, it is optimal to implement either minimal or maximal patent protection, depending on the underlying structural parameters of the problem, contrasting with in Cysne e Turchick (2012). Results are particularly sensitive to the level of costs associated with R&D activity - lower costs attached to a higher desirability of total IPR protection - which can be lowered through proper governmental action.
4. For some relevant parametric values, the optimal patent term has an interior value which is very close to utility models terms in OECD countries

The paper is organized as follows. In Section 2 we introduce our framework and describe the model. In Section 3, the equilibrium trajectories of the state and control variables are presented and proven correct. Section 4 is devoted to analysis of the optimal patent length in our framework. In chapter 5, a discussion of the results and their relation to previous literature is provided. Section 6 concludes.

2 The Model

We assume there is a continuum P of identical infinitely-lived agents whose measure is constant and normalized to one, and who are endowed with L units of labor in each period. On the production side of the economy, there are both competitive final good firms and monopolistically competitive intermediate goods firms that produce according to blueprints created by individuals. In particular, each intermediate good firm j is owned by the individual who conceived variety j and to whom the corresponding IPR was granted.

At each period, each intermediate good variety can either be monopolized, in that a patent upon its design is in effect, or copied, in which case the inventor's market power has already eroded. The proprietorship of final good firms is divided evenly among agents.

Denote the measure of intermediate good varieties in the economy at instant t by $A(t)$, and the measure of intermediate good varieties designed by individual $i \in P$ by $A_i(t)$. Analogously, denote by $B(t)$ ($B_i(t)$) the measure of intermediate good varieties (conceived by individual i) which have already been copied as of instant t .

Final good firms produce competitively at each time t using intermediate varieties $j \in [0, A(t)]$ and labor as inputs. Following Romer (1987), we assume the following production function:

$$Y(L, x) = L^{1-\alpha} \int_0^{A(t)} x_j^\alpha dj, \quad (2.1)$$

where L is labor input, and x_j is the amount of intermediate good j being used, with elasticity $\alpha \in (0, 1)$. In equilibrium, copied intermediate goods must share the same price p_c and the same demand level x_c . Likewise, each monopolized intermediate good has price p_m and demand level x_m . Taking the final good as numéraire, final good firms should then choose x_m , x_c and L to maximize

$$L^{1-\alpha}((A(t) - B(t))x_m^\alpha + B(t)x_c^\alpha) - p_c B(t)x_c - p_m(A(t) - B(t))x_m - wL,$$

where w is the wage rate. Assuming $B(t) \in (0, A(t))$, the first-order conditions ensure that each factor will be paid its marginal product:¹

$$p_c = \alpha L^{1-\alpha} x_c^{\alpha-1} \quad (2.2)$$

$$p_m = \alpha L^{1-\alpha} x_m^{\alpha-1} \quad (2.3)$$

$$w = (1 - \alpha)L^{-\alpha}((A(t) - B(t))x_m^\alpha + B(t)x_c^\alpha) \quad (2.4)$$

¹ If $B(t) = 0$ ($B(t) = A(t)$), there exist no copied (monopolized) intermediate varieties at t , whence the issue of interiority or not of x_c (x_m) is immaterial for the computation of equilibrium.

In line with Rivera-Batiz e Romer (1991) lab-equipment specification, we assume that intermediate goods firms use one unit of the final good – and no labor – in order to produce one unit of intermediate good.

As will be explained shortly, patents will generally not last forever, in that some of once monopolized intermediate good varieties become copied.

Intermediate goods firms producing copied varieties choose x_c in order to maximize $p_c x_c - x_c$. Thus, in equilibrium, we must have $p_c = 1$, and such firms will earn zero profits. Plugging this price in (2.2) yields

$$x_c = L\alpha^{\frac{1}{1-\alpha}} \quad (2.5)$$

Intermediate goods firms producing monopolized varieties choose x_m in order to maximize their profit $\pi = p_m x_m - x_m$ given the inverse demand schedule (2.3). Substituting this equation into the objective function yields $\alpha L^{1-\alpha} x_m^\alpha - x_m$, a concave function. Therefore, x_m can be obtained from the first-order condition $\alpha^2 L^{1-\alpha} x_m^{\alpha-1} - 1 = 0$:

$$x_m = L\alpha^{\frac{2}{1-\alpha}} \quad (2.6)$$

Substituting this back into (2.1) gives $p_m = 1/\alpha > 1 = p_c$, as expected. Once a variety is copied, its inventor loses his market power.

Plugging equations (2.6) and (2.5) in (2.4), we obtain an expression for the wage rate that does not depend on L . Furthermore, since $\alpha < 1$, $x_c < x_m$, whence, by equation (2.2) the marginal productivity of labor – as well as wages – are increasing in $B(t)$. Plugging (2.6) and $p_m = 1/\alpha$ in the expression of π yields

$$\pi = L\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \quad (2.7)$$

We assume the imitation/patent-breaking process to be of the Poisson type, and, in accordance with Grossman e Helpman (1991), the corresponding imitation rate $\dot{B}_i(t)/(A_i(t) - B_i(t))$ to be a constant, say m .

Since $A = \sum_{i \in P} A_i$ and $B = \sum_{i \in P} B_i$, we thus obtain

$$\dot{B}(t) = m(A(t) - B(t)). \quad (2.8)$$

It should be noted that m is a parameter inversely related to expected patent duration.² In section 4, we show how its value should be set by the policymaker. For now, we take it as given.

Notwithstanding the fact that the arrival rate of a Poisson process is strictly positive and finite, we set the domain of m to be $[0, +\infty]$. Of course, for the extrema of this interval,

² In fact, Kwan e Lai (2003) show that $m = r/(e^{rT} - 1)$, where r is the interest rate and T the expected patent duration.

there corresponds not a stochastic, yet a deterministic, process, for which we can fathom a clear interpretation. The case $m = 0$ refers to the situation where the patent term is infinite. For $m = \infty$, patent terms are null, i.e., any new intermediate good is immediately copied. This is equivalent to the very absence of patent protection in the legal framework.

Our formulation of the imitation process might seem somewhat unnatural, since the law prescribes fixed patent lengths, while (2.8) embodies a probabilistic nature of patents. Nevertheless, once the structure of the judicial system is accounted for, the probabilistic nature of patent lengths cannot be overlooked (see, e.g., Lemley e Shapiro (2005)). This is the case since the statutes that prescribe patent duration are often general and abstract norms, whose exact content in a concrete case is left to be determined by courts in face of a suit. Thus, it may be the case that a blueprint is copied, but the courts, after analyzing the underlying facts and law, are convinced that the similarity of both inventions is not great enough in order to configure a unlawful imitation, so that, in practice, the IPR are overruled before the term determined by the statute is due.

Consumers are identical, with initial endowments of total and copied intermediate goods $A_0 > 0$ and $B_0 \in [0, A_0]$. Each consumer $i \in P$ is interested in the maximization of

$$\int_0^\infty e^{-\rho t} u(C_i(t)) dt,$$

where ρ is the discount rate, $C_i(t)$ is consumption of the final good and u is of the constant elasticity of intertemporal substitution class:

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \text{if } \theta > 1 \\ \log c, & \text{if } \theta = 1. \end{cases}$$

We restrict the range of θ in order to guarantee boundedness of the above integral in equilibrium. This restriction is also in accordance with the empirical literature.³

Being the owners of final and intermediate goods firms, agents receive a flow of profit accruing from the operation of these firms. Nevertheless, in equilibrium, final good firms earn zero profits, as well as intermediate goods firms producing already copied goods, since the IPR are abrogated in virtue of the copy. Thus, at each period t , individual i receives, besides a wage of wL , a flow of profits given by $\pi(A_i(t) - B_i(t))$. Since there is no other form of capital goods in the economy, this income must be split at each period between current consumption of the numéraire good and R&D activity, as specified by Rivera-Batiz e Romer (1991): upon incurring in a constant cost of invention β^4 (the invention process is not time consuming), agents conceive

³ In fact, for the U.S. economy, Barsky et al. (1997) find an estimate for θ of 4.17, and Hall (1988), using four different datasets, concludes that $\theta \in [5, 10]$. Moreover, in a meta-analysis of 169 studies carried out in 45 different countries, Valickova, Havranek e Horvath (2015) find that, for 35 of those countries, the mean of the estimates of the elasticity of intertemporal substitution is lower than one.

⁴ The choice of a constant cost of invention in the number of blueprints A_i already conceived is due primarily to analytic convenience. Nevertheless, we can attach an intuitive economic interpretation to this. Let the cost β

a new intermediate good variety. Thus, the budget constraint of individual i at period t can be written as

$$C_i(t) + \beta \dot{A}_i(t) = wL + \pi(A_i(t) - B_i(t)). \quad (2.9)$$

In this deterministic formulation of the R&D process, expending β is a sufficient and necessary condition for the conception of a new blueprint, the establishment of IPR concerning it, and an intermediate goods firm following it. In this respect, β should be interpreted in a broader sense, as all the costs underlying the conception of the new intermediate variety: research costs, costs associated with opening a firm, and administrative costs linked to filing for the approval of the patent and establishment of IPR. The latter may include lawyer fees, learning costs and judicial costs.

We further impose that $\dot{A}_i(t) \geq 0, \forall t$, and $\dot{B}_i(t) \geq 0, \forall t$, to which we shall refer as the physical constraints of the economy. Such restrictions serve the purpose of ruling out scenarios where the invention and/or imitation of intermediate varieties is reversible.

Thus, each consumer i solves

$$\max \int_0^\infty e^{-\rho t} u(C_i(t)) dt$$

subject to

$$\dot{A}_i(t) = \frac{wL}{\beta} - \frac{C_i(t)}{\beta} + \frac{\pi}{\beta}(A_i(t) - B_i(t)), \forall t$$

$$\dot{B}_i(t) = m(A_i(t) - B_i(t)), \forall t$$

$$A_i(0) = A_0$$

$$B_i(0) = B_0$$

$$\dot{A}_i(t) \geq 0, \forall t$$

$$\dot{B}_i(t) \geq 0, \forall t.$$

From the symmetry and strict concavity features of the model, its equilibrium must be symmetric, whence the subscript i can be dispensed with.

be a function of the measure of already conceived patents A_i . On the one hand, as more ideas are conceived, inventors run out of ideas, which would imply that $\beta(A_i)$ increases. On the other hand, already established concepts may make easier for inventors to come up with new ideas, which would imply that $\beta(A_i)$ decreases. We assume that both effects cancel. In this respect, see Barro e Martin (1995)

3 System Dynamics

3.1 Preliminary considerations

In this section we shall derive the equilibrium trajectories of A , B and C . To fix notation, let our parametric space be

$$\Phi := \{(\alpha, \rho, L, \beta, \theta, A_0, B_0, m) : 0 < \alpha < 1, \rho > 0, L > 0, \beta > 0, \\ \theta \geq 1, A_0 > 0, 0 \leq B_0 \leq A_0, m \geq 0\}$$

Given $\phi = (\alpha, \rho, L, \beta, \theta, A_0, B_0, m) \in \Phi$, define the functions:

$$\kappa_1(\phi) := \frac{L}{\beta} \left(\left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right) - \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \right) \right)$$

and

$$\kappa_2(\phi) := -\frac{L}{\beta} \left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right).$$

Although we have defined κ_1 and κ_2 as functions of $\phi \in \Phi$, whenever it assumes the role of a parameter, we drop the functional dependence, for notational simplicity.

Notice that both $\kappa_1(\phi)$ and $\kappa_2(\phi)$ are negative, for any $\phi \in \Phi$. To see the latter fact, note that since $0 < \alpha < 1$, $2 > 2\alpha$ and we have $\alpha^{\frac{2\alpha}{1-\alpha}} > \alpha^{\frac{2}{1-\alpha}}$, so that $\kappa_2 < 0$. Also, since $\kappa_1 = (L/\beta)(1 - \alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}(1 + \alpha^{-1} - \alpha^{\frac{1}{\alpha-1}})$ and Bernoulli's inequality gives $\alpha^{\frac{1}{\alpha-1}} = (1 + 1/\alpha - 1)^{\frac{1}{1-\alpha}} > 1 + (1/(1 - \alpha))(1/\alpha - 1) = 1 + \alpha^{-1}$, $\kappa_1 < 0$.

It is also convenient to define the space of all parameters other than m , i.e., the space of the parameters the policymaker takes as given:

$$\Omega := \{(\alpha, \rho, L, \beta, \theta, A_0, B_0) : 0 < \alpha < 1, \rho > 0, L > 0, \beta > 0, \\ \theta \geq 1, A_0 > 0, 0 \leq B_0 \leq A_0\}$$

In equilibrium, the budget constraint (2.9), with π substituted in from (2.7) and w substituted in from (2.4), (2.5) and (2.6), gives, for all t ,

$$\begin{aligned} C(t) + \beta \dot{A}(t) &= (1 - \alpha) \left[(A(t) - B(t)) \alpha^{\frac{2\alpha}{1-\alpha}} + B(t) \alpha^{\frac{\alpha}{1-\alpha}} \right] L + \left(\frac{1}{\alpha} - 1 \right) L \alpha^{\frac{2}{1-\alpha}} (A(t) - B(t)) \\ &= L \left[\left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right) (A(t) - B(t)) + \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \right) B(t) \right], \end{aligned}$$

where $C(t)$ denotes total consumption at t . We thus obtain the movement equation for A :

$$\dot{A}(t) = -\kappa_2 A(t) - \kappa_1 B(t) - \frac{C(t)}{\beta}, \quad (3.1)$$

Having established the dynamics for B and A in (2.8) and (3.1), one should expect to complete the equilibrium dynamics for (A, B, C) with the usual Euler equation. Unfortunately, things are not that simple, because of the model's physical constraints. In this section, we will give them due consideration, as this will prove crucial in the solution of the policymaker's problem (that of finding the optimal duration of patents).

Within this scope, it is also convenient to define the following functions of $\phi \in \Phi$:

$$\gamma(\phi) := \frac{\frac{\pi}{\beta} - m - \rho}{\theta}$$

where π is given by equation (2.7) and:

$$g_0(\phi) := \frac{B_0}{A_0}$$

$$\lambda(\phi) := \frac{-(m + \kappa_2(\phi)) - \sqrt{(m - \kappa_2(\phi))^2 - 4m\kappa_1(\phi)}}{2}$$

and, for $m > 0$, $\gamma \geq 0$,

$$\bar{g}_0(\phi) := \frac{m(m + \gamma(\phi) + \lambda(\phi))}{(m + \lambda(\phi))(m + \gamma(\phi))}$$

$\gamma(\phi)$ represents the consumption growth rate in an equilibrium with R&D and g_0 the ratio of copied goods at $t = 0$. $\lambda(\phi)$ is the nonpositive root of $x^2 + (m + \kappa_2(\phi))x + m(\kappa_1(\phi) + \kappa_2(\phi)) = 0$. $\bar{g}_0(\phi)$ will be important when we analyse equilibria with R&D. Whenever they assume the role of parameters, we drop the functional dependence, for notational simplicity.

Two relevant facts about λ :

1. Given $\omega \in \Omega$, $\lambda(\omega, 0) = 0$

Proof. Let $\omega \in \Omega$. Then, $\lambda(\omega, 0) = (-\kappa_2 - |\kappa_2|)/2 = 0$, since $\kappa_2 < 0$. □

2. Given $\omega \in \Omega$ and $m > 0$, then $m + \lambda(\omega, m) < 0$

Proof. Let $\omega \in \Omega$ and $m > 0$, and $\phi = (\omega, 0)$. By adding m to both sides of the definition of λ , and letting we get:

$$\lambda(\phi) + m = \frac{m - \kappa_2(\phi) - \sqrt{(m - \kappa_2(\phi))^2 - 4m\kappa_1(\phi)}}{2}$$

Note that $m - \kappa_2(\phi) > 0$, as well as the square root. Now:

$$(\sqrt{(m - \kappa_2(\phi))^2 - 4m\kappa_1(\phi)})^2 > (m - \kappa_2(\phi))^2,$$

since $-4m\kappa_1(\phi) > 0$.

□

In what regards $\bar{g}_0$, three relevant facts:

1. Given $\phi \in \Phi$, $\gamma(\phi) = 0 \iff \bar{g}_0(\phi) = 1$

Proof. Let $\phi \in \Phi$ such that $m > 0$. Then:

$$\bar{g}_0(\phi) = 1 \iff m(m + \gamma(\phi) + \lambda(\phi)) = (m + \lambda(\phi))(m + \gamma(\phi)) \iff \lambda\gamma = 0 \iff \gamma = 0,$$

since $m > 0$ and then $\lambda < 0$

□

2. Given $\phi \in \Phi$, $\gamma(\phi) = -m - \lambda(\phi) \iff \bar{g}_0(\phi) = 0$

Proof. Let $\phi \in \Phi$ such that $m > 0$. Then $\bar{g}_0(\phi) = 0 \iff m(m + \gamma(\phi) + \lambda(\phi)) = 0 \iff \gamma(\phi) = -m - \lambda(\phi)$, since $m > 0$.

□

3. Given $\phi \in \Phi$, $m > 0$, $0 \leq \bar{g}_0 \leq 1 \iff 0 \leq \gamma(\phi) \leq -m - \lambda(\phi)$

Proof. Let $\phi \in \Phi$ such that $m > 0$. We have $\gamma \geq 0$ and since $m > 0$, $\lambda(\phi) < 0$, so that $\bar{g}_0(\phi) \geq 0 \iff m + \gamma(\phi) + \lambda(\phi) < 0 \iff \gamma(\phi) \leq -m - \lambda(\phi)$. On the other hand, $\bar{g}_0(\phi) \leq 1 \iff \gamma(\phi)\lambda(\phi) \leq 0 \iff \gamma(\phi) \geq 0$

□

With due consideration of physical constraints, we shall see that, depending on the specific values of the parameters, the solution path for (A, B, C) will be of one of three possible different types: one with no R&D whatsoever (which we coin type-1 equilibrium), another with R&D starting at time zero (type-2 equilibrium), and yet another in which R&D starts at an instant $t^* > 0$ (type-3 equilibrium). The distinction between these cases depends primarily on the values of γ and g_0 defined above.

This distinction can be captured through the partition $\{\Phi_1, \Phi_2, \Phi_3\}$ of Φ , defined in what follows:

$$\Phi_1 = \{\phi \in \Phi : \gamma(\phi) \leq 0\}$$

Figure 1 – Typology of Equilibria

$$\Phi_2 = \{\phi \in \Phi : \gamma(\phi) > 0 \text{ and } (\gamma(\phi) \geq -m - \lambda(\phi) \text{ or } g_0(\phi) \geq \bar{g}_0(\phi))\}$$

$$\Phi_3 = \{\phi \in \Phi : 0 < \gamma(\phi) < -m - \lambda(\phi) \text{ and } g_0(\phi) < \bar{g}_0(\phi)\}$$

We shall prove in this section for $i \in \{1, 2, 3\}$, if $\phi \in \Phi_i$, then in an economy described by ϕ , a type- i equilibrium arises.

It is noteworthy that, given $\omega \in \Omega$, $(\omega, 0) \notin \Phi_3$, since $\lambda = 0$. Thus, $(\omega, 0) \in \Phi_1 \iff \gamma(\omega, 0) \leq 0$ and $(\omega, 0) \in \Phi_2 \iff \gamma(\omega, 0) > 0$.

Figure 1 illustrates how the values of g_0 and γ jointly determine the kind of equilibrium that arises in the economy, with the blue region corresponding to type-1 equilibria, the pink one to type-2 and the yellow one to type-3. The line connecting the points in the Cartesian plane $(0, 1)$ and $(-m - \lambda, 0)$ is the $\bar{g}_0(\phi)$ locus. Notice that this construction is consistent with the three relevant facts regarding $\bar{g}_0$,

From equations (2.8) and (2.9), the optimal control problem described in the last section could be rewritten as

$$\max \int_0^\infty e^{-\rho t} u(C(t)) dt \quad (3.2)$$

subject to

$$\dot{A}(t) = \frac{wL}{\beta} - \frac{C(t)}{\beta} + \frac{\pi}{\beta}(A(t) - B(t)), \forall t \quad (3.3)$$

$$\dot{B}(t) = m(A(t) - B(t)), \forall t \quad (3.4)$$

$$A(0) = A_0 \quad (3.5)$$

$$B(0) = B_0 \quad (3.6)$$

$$\frac{wL + \pi(A(t) - B(t)) - C(t)}{\beta} \geq 0, \forall t \quad (3.7)$$

$$A(t) - B(t) \geq 0, \forall t. \quad (3.8)$$

We are now in position to define an equilibrium for an economy described by $\phi \in \Phi$.

Definition 3.1. An equilibrium for an economy described by $\phi \in \Phi$ is a triple (A^*, B^*, C^*) such that

1. (A^*, B^*, C^*) solves (3.2) - (3.8)
2. w and π in (3.3) and (3.7) satisfy (2.4) - (2.7).

With this formulation, it is easy to see that for any $\phi \in \Phi$ the equilibrium is unique. Given $\phi \in \Phi$, w and π are uniquely defined from (2.4) - (2.7). Suppose $(A_1, B_1, C_1) \neq (A_2, B_2, C_2)$ and that both solve (3.2) - (3.8). We assert that $C_1 \neq C_2$. If $C_1 = C_2$, from the Picard-Lindelöf Theorem applied to the Initial Value Problem given by (3.3) - (3.6), $(A_1, B_1) = (A_2, B_2)$, contradiction. Now, the triple $((A_1 + A_2)/2, (B_1 + B_2)/2, (C_1 + C_2)/2)$ satisfies (3.3) - (3.8), from the linearity of these restrictions and of the derivative operator. However, since $C_1 \neq C_2$, from strict concavity of u , higher levels of utility are attained through $(C_1 + C_2)/2$ than through C_1 or C_2 , contradicting the optimality of (A_1, B_1, C_1) and of (A_2, B_2, C_2) .

Now we study the solution to the optimization problem (3.2) - (3.8). Since $\dot{A}$ represents investment in R&D, its zero lower bound given by (3.7) translates into an upper bound for consumption, given by $wL + \pi(A(t) - B(t))$. This is a mixed constraint, in that it includes both control and state variables, in contrast to the pure state constraint (3.8). We follow Seierstad e Sydsaeter (1986) in finding the solution to this problem.

For convenience, we rewrite the problem so that there is only one state variable $D := A - B$. We can also dispense with constraint (3.8), once (3.7) is satisfied. To see this, firstly note that restriction (3.8) can be expressed as, $D(t) \geq 0, \forall t$. Creating the auxiliary variable $G(t) = D(t)e^{mt}, \forall t$, $G(t)$ and $D(t)$ have the same sign for all t . Differentiating $G(t)$ with respect to t , one finds $\dot{G}(t) = e^{mt}(\dot{D}(t) + mD(t))$. Also, from (3.3) and (3.4), $\dot{D}(t) = wL/\beta - C(t)/\beta + (\pi/\beta - m)D(t)$. Moreover, (3.7) can be rewritten as $\dot{D}(t) + mD(t) \geq 0$. Once this constraint is satisfied, $\dot{G}(t) \geq 0$, and since $G(0) = D(0) \geq 0$, $G(t) \geq 0, \forall t$, whence $D(t) \geq 0, \forall t$. Hence, the consumer optimization problem can be rewritten as:

$$\max \int_0^\infty e^{-\rho t} u(C(t)) dt \quad (3.9)$$

subject to

$$\dot{D}(t) = \frac{wL}{\beta} - \frac{C(t)}{\beta} + \left(\frac{\pi}{\beta} - m\right)D(t), \forall t \quad (3.10)$$

$$D(0) = D_0 := A_0 - B_0 \quad (3.11)$$

$$\frac{wL + \pi(D(t)) - C(t)}{\beta} \geq 0, \forall t \quad (3.12)$$

As explained above, if (A, B, C) satisfies (3.2)-(3.7), then (D, C) satisfies (3.9)-(3.11), where $D := A - B$. This also works the other way around: if (D, C) satisfies (3.9)-(3.11), then (A, B, C) satisfies (3.2)-(3.7), where B solves $\dot{B}(t) = mD(t), \forall t \geq 0$ with the initial condition $B(0) = B_0$, and $A := B + D$. Thus, both problems are equivalent.

Define the Hamiltonian $\mathcal{H}$ and the Lagrangian $\mathcal{L}$ for this problem

$$\mathcal{H}(D, C, p, t) = e^{-\rho t} u(C(t)) + p(t) \left(\frac{wL}{\beta} - \frac{C(t)}{\beta} + \left(\frac{\pi}{\beta} - m \right) D(t) \right) \quad (3.13)$$

and

$$\mathcal{L}(D, C, p, q, t) = \mathcal{H}(D, C, p, t) + \frac{q(t)}{\beta} (wL - C(t) + \pi D(t)) \quad (3.14)$$

where p is the co-state variable associated with D , and q is the Lagrange multiplier associated with restriction (3.12).

3.2 No research Case

The trajectories below describe an economy where there is no investment in R&D. Since no new blueprints are created in this case, patent laws protect just the $A_0 - B_0$ valid patents inherited by initial conditions. Consumers at every instant spend all their income in consumption, with no savings at all. Thus, $A(t) = A_0, \forall t$ and $B(t) \rightarrow A_0$, although at every finite instant $B(t) < A(t)$. Consumption rises at a decreasing rate, i.e., trajectories encompassing no research do not form a balanced growth path.

Proposition 3.1. Given an economy described by $\phi \in \Phi$, if $\phi \in \Phi_1$ the equilibrium trajectories (A^*, B^*, C^*) are given by:

$$A^*(t) = A_0 \quad (3.15)$$

$$B^*(t) = A_0 + (B_0 - A_0)e^{-mt} \quad (3.16)$$

$$C^*(t) = \beta(A_0(-\kappa_2 - \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt}) \quad (3.17)$$

Proof. Given w and π from (2.4) - (2.7), (A^*, B^*, C^*) from (3.15)-(3.17), checking the validity of constraints (3.3) - (3.8) is straightforward. It should also be noted that $C^* > 0$, since $C^*(0) = -\kappa_2 A_0 - \kappa_1 B_0 \geq -\kappa_2 A_0 > 0$ and C^* is increasing, since $\dot{C}^*(t) = -m\kappa_1(A_0 - B_0)e^{-mt} \geq 0$.

Now we show that the consumption trajectory is in fact optimal. In order to do so, we use Theorem 6.11 in Seierstad e Sydsaeter (1986) to the problem defined by (3.9) to (3.12), which provides sufficient conditions for optimality in an infinite-horizon optimal control problem with mixed constraints. Consider the Lagrangian defined in (3.14). Let the co-state variable and Lagrange multiplier be:

$$p(t) = \frac{\pi}{\beta^\theta} e^{mt} \int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{(A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta} \quad (3.18)$$

$$q(t) = \frac{e^{-\rho t}}{\beta^{\theta-1}(A_0(-\kappa_2 - \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt})^\theta} - \frac{\pi}{\beta^\theta} e^{mt} \int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{(A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta} \quad (3.19)$$

Now we verify that each of the six conditions of the theorem is satisfied. The $*$ below indicates that the derivative is being evaluated at $(D^*(t), C^*(t), p(t), q(t), t)$. Firstly we have to check that, $\forall C > 0$, it holds that:

$$\frac{\partial \mathcal{L}^*}{\partial C}(C - C^*(t)) \leq 0, \forall t$$

From (3.14), we see that:

$$\frac{\partial \mathcal{L}^*}{\partial C} = \frac{e^{-\rho t}}{C^{*\theta}} - \frac{p(t) + q(t)}{\beta}$$

By plugging equations (3.17), (3.18) and (3.19) in the above expression, we see that $\partial \mathcal{L}^* / \partial C = 0$, which leads to the desired result.

Now we check the validity of the co-state variable movement equation:

$$-\dot{p}(t) = \frac{\partial \mathcal{L}^*}{\partial D}$$

Evaluating $\partial \mathcal{L}^* / \partial D$:

$$\begin{aligned} \frac{\partial \mathcal{L}^*}{\partial D} &= (p(t) + q(t)) \frac{\pi}{\beta} - mp(t) = \frac{\pi}{\beta^\theta} \frac{e^{-\rho t}}{(A_0(-\kappa_2 - \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt})^\theta} - \\ &\quad m \frac{\pi}{\beta^\theta} e^{mt} \int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{(A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta} \end{aligned}$$

As can be seen applying Leibniz Rule and the Product Rule to 3.18, this equals $-\dot{p}(t)$.

In what regards the complementary slackness of the Lagrange Multiplier, since (3.7) is satisfied with equality, so is constraint (3.12), which will be binding, and we need to check that $q(t) \geq 0, \forall t$. Since $\phi \in \Phi_1$, $\gamma \leq 0$, and $\pi/\beta \leq m + \rho$. Now:

$$\frac{\frac{e^{-(\rho+m)t}}{((- \kappa_1 - \kappa_2) + \kappa_1(1-g_0)e^{-mt})^\theta}}{\int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{((- \kappa_1 - \kappa_2) + \kappa_1(1-g_0)e^{-m\tau})^\theta}} = \frac{e^{-(\rho+m)t}}{\int_t^\infty e^{-(\rho+m)\tau} d\tau} = \rho + m \geq \frac{\pi}{\beta}$$

Moreover, since $((-\kappa_1 - \kappa_2) + \kappa_1(1 - g_0)e^{-mt})^\theta$ increases in t :

$$\frac{\frac{e^{-(\rho+m)t}}{((- \kappa_1 - \kappa_2) + \kappa_1(1-g_0)e^{-mt})^\theta}}{\int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{((- \kappa_1 - \kappa_2) + \kappa_1(1-g_0)e^{-m\tau})^\theta}} \geq \frac{\frac{e^{-(\rho+m)t}}{((- \kappa_1 - \kappa_2) + \kappa_1(1-g_0)e^{-mt})^\theta}}{\int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{((- \kappa_1 - \kappa_2) + \kappa_1(1-g_0)e^{-m\tau})^\theta}},$$

which leads to the desired result. We also have to check that a transversality condition is satisfied:

$$\liminf(p(t)(D(t) - D^*(t))) \geq 0$$

where $D^*(t) = A^*(t) - B^*(t)$, for all admissible $D(t)$ ($D(t) \geq 0, \forall t$). We initially check that $\lim p(t) = 0$. For $m = 0$:

$$\lim_{t \rightarrow \infty} p(t) = \lim_{t \rightarrow \infty} \frac{\pi}{\beta^\theta} \int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{(-A_0\kappa_1 - \kappa_1 B_0)^\theta} = 0,$$

since $\rho + m > 0$. Now, for $m > 0$, we use the L'Hopital Rule, which is possible since $e^{-mt} \rightarrow 0$ and

$$\int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{(-A_0\kappa_1 - \kappa_1 B_0)^\theta} \geq \int_t^\infty \frac{e^{-(\rho+m)\tau} d\tau}{(A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta} \geq 0,$$

so that the integral in (3.18) converges to zero. Using Leibniz Rule alongside L'Hopital:

$$\lim p(t) = \frac{\pi}{\beta^\theta} \lim_{t \rightarrow \infty} \frac{\frac{e^{-(\rho+m)t}}{((A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-mt}))^\theta}}{-me^{-mt}} = 0$$

Moreover, $D^*(t) = (A_0 - B_0)e^{-mt}$ and $\lim D^*(t)$ is bounded, whence $\lim p(t)D^*(t) = 0$ and $\liminf p(t)(D(t) - D^*(t)) = \liminf p(t)D(t)$. Since $D(t) \geq 0$ and $p(t) \geq 0$, for every admissible state and for every t , the result follows

Finally, the left-hand side of restriction (3.12) is quasi-concave in (D, C) , since it is linear in these arguments, and that the Hamiltonian (3.13) is concave in (D, C) , since it is linear in D and concave in C .

□

The constraint $\dot{A}(t) \geq 0$ shall be interpreted as a restriction upon consumption, since it implies $C(t) \leq \beta(-\kappa_2 A(t) - \kappa_1 B(t))$. Thus, the solution described above turns out to be a boundary solution, inasmuch consumption at every instant is set at its maximal admissible level $C^*(t) = \beta(-\kappa_2 A(t) - \kappa_1 B(t)) = \beta((- \kappa_2 - \kappa_1)A_0 + \kappa_1(A_0 - B_0)e^{-mt})$.

In what regards the growth rate of consumption, see that

$$\frac{\dot{C}}{C} = \frac{-m\kappa_1(A_0 - B_0)e^{-mt}}{\beta(A_0(-\kappa_2 - \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt})}$$

The numerator is positive and decreasing and the denominator positive and increasing in t , so that the growth rate of consumption is decreasing in t and positive, and thus not associated with the variable γ . In particular, notice that $\lim_{t \rightarrow \infty} \dot{C}^*(t) = \lim_{t \rightarrow \infty} (-m\kappa_1(A_0 - B_0)e^{-mt}) = 0$, and asymptotically - in the long-run - there is no growth in consumption, which reaches its maximal level $\beta A_0(-\kappa_2 - \kappa_1)$. Then, the solution of the system when $\gamma \leq 0$ resembles much more the standard Ramsey-Cass-Koopmans solution, in comparison to a solution consistent with the endogenous growth theory. Relatedly, even though consumption increases over time, the rise in consumption is not due to increased productivity in terms of a larger measure of intermediate goods varieties - since $\dot{A}(t) = 0$ -, as would be consistent with the horizontal innovation endogenous growth literature. In fact, the rise in $C(t)$ is due to the rise in $B(t)$ - note that, if we set $\dot{A}(t)$ in equation (3.1) as zero, and express the resulting level of consumption C as a function of A and B , we have $\partial C(A, B)/\partial B = -\kappa_1 > 0$. This stems from the fact that as $B(t)$ decreases - and $A(t)$ is kept constant - the average price of the intermediate goods decreases, since patents and the consequent monopoly rights are being constantly overruled, the marginal productivity of labor increases, and consequently wages, increase. This increase in wages outweighs the loss in total profits accruing to each consumer, which, from equation (2.9), increases total consumption.

Notice that the value of g_0 does not influence the emergence of a type-1 equilibrium, but only the value of γ . As γ might be interpreted as a measure of the marginal incentives to save, or the returns accruing from R&D investment, when such returns are non-positive, consumers, bound by the physical restrictions of the system, are better-off by just spending all their income in consumption, irrespective of the initial values of the state variables. Thus, $\gamma = 0$ is a threshold for the profitability of savings, below which it is never optimal to invest in R&D.

The horizontal innovation endogenous growth literature, interested in balanced growth paths, works with the case $\gamma > 0$, having overlooked, as a result, the dynamics when the $\gamma \leq 0$. In this paper we fill this gap, which was made possible by the explicit consideration of physical restrictions. Although the story here, stand-alone, might seem a bit uninteresting, its relevance shall be made clear as we find out that the, given a tuple of parameters not including m , utility

levels attained in such equilibria might be higher than utility levels reached when an equilibrium encompassing R&D is implemented, phenomenon to be discussed thoroughly in Section 4.

3.3 Research case

Now we work with equilibrium trajectories with research for all t . Given specific conditions on γ and g_0 , the economy in this kind of equilibrium is characterized by a constant positive growth rate of consumption γ , as well as a constant asymptotic growth rate of the stocks of state variables, forming a balanced growth path. For vectors in Φ generating intermediate values for the growth rate γ , the initial stocks of the state variables influence the kind of equilibrium that arises, but above a certain threshold of γ , the type of equilibrium attained does not depend on g_0 . For mathematical convenience, we present the trajectories for $m = 0$ and $m > 0$ in two separate propositions.

Before presenting the propositions, we prove a lemma, which will be useful a couple of times:

Lemma 3.1. For $\phi \in \Phi$, $\gamma(\phi) + \kappa_2(\phi) < 0$

Proof. First note that

$$\kappa_2 = \frac{L}{\beta} (1 - \alpha^{-2}) \alpha^{\frac{2}{1-\alpha}} = -\frac{1+\alpha}{\alpha} \frac{L}{\beta} \left(\frac{1}{\alpha} - 1 \right) \alpha^{\frac{2}{1-\alpha}} = -\frac{1+\alpha}{\alpha} \frac{\pi}{\beta}$$

So that:

$$\kappa_2 + \frac{\pi}{\beta} = -\frac{1}{\alpha} \frac{\pi}{\beta} < 0.$$

If $\theta = 1$, $\kappa_2 + \gamma = \kappa_2 + \pi/\beta - m - \rho = -(\pi/\beta)/\alpha - m - \rho < 0$. As we rise θ , $(\pi/\beta)/\theta$ gets smaller, whence $\kappa_2 + (\pi/\beta)/\theta$ gets more negative, so that $\kappa_2 + \gamma = \kappa_2 + (\pi/\beta - m - \rho)/\theta < 0$. $\square$

Proposition 3.2. Given an economy described by $\phi = (\alpha, \rho, L, \beta, \theta, A_0, B_0, m) \in \Phi$ such that $m = 0$ and $\phi \in \Phi_2$, the equilibrium trajectories are given by:

$$A^*(t) = -\frac{\kappa_1}{\kappa_2} B_0 + (A_0 + \frac{\kappa_1}{\kappa_2} B_0) e^{\gamma t} \quad (3.20)$$

$$B^*(t) = B_0 \quad (3.21)$$

$$C^*(t) = \beta(-\gamma - \kappa_2)(A_0 + \frac{\kappa_1}{\kappa_2} B_0) e^{\gamma t} \quad (3.22)$$

Proof. We shall prove this proposition using the original formulation (3.2) - (3.8). Since $m = 0$, from (2.8), $\dot{B}(t) = 0, \forall t$, there is no copy process, so that $B(t) = B_0, \forall t$ and we can dispense with $B(t)$. We shall not bother to directly incorporate the $\dot{A}(t) \geq 0$ restriction; instead, we solve the unconstrained problem and verify that for $\gamma > 0$ the physical constraint is satisfied. Thus, each household maximizes:

$$\max \int_0^\infty e^{-\rho t} u(C(t)) dt$$

subject to :

$$\begin{aligned} \dot{A}(t) &= \frac{wL}{\beta} - \frac{C(t)}{\beta} + \frac{\pi}{\beta}(A(t) - B_0) \\ A(0) &= A_0 \end{aligned}$$

Firstly, note that $C^*(t) > 0, \forall t$, by Lemma 3.1, the fact that $\kappa_1, \kappa_2 < 0$ and (3.22). The satisfaction of $A(0) = A_0$ is straightforward. Now, we show that the consumption trajectory is in fact optimal. In order to do so, we show that all conditions in Theorem 3.13 of Seierstad e Sydsaeter (1986), which provides sufficient conditions for optimality in an unconstrained infinite-horizon optimal control problem, are satisfied. Define the co-state variable $p(t)$ associated with the state variable $A(t)$ by:

$$p(t) = \frac{e^{-\frac{\pi}{\beta}t}}{(-\gamma - \kappa_2)^\theta \beta^{\theta-1} (A_0 + B_0 \frac{\kappa_1}{\kappa_2})^\theta} \quad (3.23)$$

The Hamiltonian associated to this optimal control problem can be defined by:

$$\mathcal{H}(A, C, p, t) = e^{-\rho t} u(C(t)) + p(t) \left(\frac{wL}{\beta} - \frac{C(t)}{\beta} + \frac{\pi}{\beta}(A(t) - B_0) \right)$$

Now we check that each of the four conditions of the theorem is satisfied. The * below indicates that the derivative is being evaluated at $(A^*(t), C^*(t), p(t), t)$. Firstly, we shall check if the Hamiltonian is maximized by the control variable trajectory (3.22). Since the Hamiltonian is strictly concave in C , its maximum is given by the first-order condition:

$$\frac{e^{-\rho t}}{(C^*(t))^\theta} = \frac{p(t)}{\beta}$$

Plugging (3.22) and (3.23) in the above expression yields an identity. In what regards the co-state variable movement equation, the usual condition is required:

$$-\frac{\partial \mathcal{H}^*}{\partial A} = \dot{p}(t)$$

Since $\partial \mathcal{H}^* / \partial A = p(t)(\pi/\beta)$ and $\dot{p}(t)/p(t) = -\pi/\beta$, from (3.23), the above condition follows.

The third condition is the 'transversality condition' $\liminf(p(t)(A(t) - A^*(t))) \geq 0$, which must hold for every admissible $A(t)$. Note first that $\lim_{t \rightarrow \infty} p(t)A^*(t) = 0$, since, by multiplying the expressions for $p(t)$ and $A^*(t)$, one obtains an exponential function with growth rate $-\pi/\beta(1 - 1/\theta) - \rho/\theta < 0$. Thus, it suffices to prove that $\liminf(p(t)A(t)) \geq 0$, for every admissible $A(t)$. This follows from Lemma 3.1 and the facts that every admissible state is non-negative and $\kappa_1/\kappa_2 > 0$.

Last but not least, the Hamiltonian is concave in (A, C) , since it is linear in A and strictly concave in C .

Thus, (A^*, B^*, C^*) solves the unconstrained optimal control problem. Now note that $\dot{A} \geq 0$ for every t since $\gamma > 0$ and $\dot{A}(t) = \gamma(A_0 + \frac{\kappa_1}{\kappa_2}B_0)e^{\gamma t}$. Thus (A^*, B^*, C^*) , also solves (3.2)-(3.8). $\square$

Now we turn to the more interesting case of a positive m . It will prove useful, in order to address constraint (3.7), to define, given $\phi = (\alpha, \rho, L, \beta, \theta, A_0, B_0, m) \in \Phi$, $\Lambda : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Lambda(g'_0, \gamma') = A_0 \frac{m(1 - g'_0)(m + \gamma' + \lambda) - g'_0 \gamma' \lambda}{m}$$

Now we prove a very useful Lemma:

Lemma 3.2. Given $\phi \in \Phi_2$, $\Lambda(g_0(\phi), \gamma(\phi)) \geq 0$

Proof. Let $\phi \in \Phi_2$. Note that $\Lambda(g_0, \gamma)$ is affine and therefore monotonic in g_0 . Suppose now $\gamma(\phi) \geq -m - \lambda$. Note that $\Lambda(0, \gamma(\phi)) \geq 0$ and $\Lambda(1, \gamma(\phi)) \geq 0$. Thus, since $\Lambda(g_0, \gamma)$ is monotonic in g_0 , $\Lambda(g_0(\phi), \gamma(\phi)) \geq 0$. Now suppose $0 < \gamma(\phi) < -m - \lambda$. Necessarily $g_0(\phi) \geq \bar{g}_0(\phi)$. Note that $\Lambda(\bar{g}_0(\phi), \gamma(\phi)) = 0$ and $\Lambda(1, \gamma(\phi)) \geq 0$, so that $\Lambda(g_0(\phi), \gamma(\phi)) \geq 0$, by monotonicity. $\square$

We show, as it will be useful in the discussion that follows the proof of the next proposition, that given $\phi \in \Phi$, if $\Lambda(g_0(\phi), \gamma(\phi)) \geq 0$, then $\phi \in \text{Phi}_2$ or $\gamma(\phi) = 0$. Since $0 \leq g_0 \leq 1$, given γ , $\Lambda(g_0, \gamma) \geq 0, \forall g_0$ if and only if $\Lambda(0, \gamma) = A_0(m + \gamma + \lambda) \geq 0$ and $\Lambda(1, \gamma) = -A_0(g_0 \lambda \gamma)/m \geq 0$, or equivalently, if and only if $\gamma \geq -m - \lambda$. On the other hand, if $0 \leq \gamma < -m - \lambda$, see that for g_0^* , $\Lambda(g_0^*, \gamma) = 0$. Since $\Lambda(0, \gamma) < 0$, for $0 \leq \gamma < -m - \lambda$ and $\Lambda(\cdot, \gamma)$ is monotone, for $g_0 \geq g_0^*$ and $0 < \gamma \leq -m - \lambda$, $\Lambda(g_0, \gamma) > 0$. We discuss later why we chose to let the $\gamma = 0$ case under type-1 equilibria.

Proposition 3.3: Given an economy described by $\phi = (\alpha, \rho, L, \beta, \theta, A_0, B_0, m) \in \Phi$ such that $m > 0$ and $\phi \in \Phi_2$, the equilibrium trajectories are given by:

$$A^*(t) = \frac{mA_0 - (m + \lambda)B_0}{m(\gamma - \lambda)}(m + \gamma)e^{\gamma t} + \frac{-mA_0 + (m + \gamma)B_0}{m(\gamma - \lambda)}(m + \lambda)e^{\lambda t} \quad (3.24)$$

$$B^*(t) = \frac{mA_0 - (m + \lambda)B_0}{m(\gamma - \lambda)}me^{\gamma t} + \frac{-mA_0 + (m + \gamma)B_0}{m(\gamma - \lambda)}me^{\lambda t} \quad (3.25)$$

$$C^*(t) = \frac{mA_0 - (m + \lambda)B_0}{m}(-\beta)(m + \gamma + \lambda + \kappa_2)e^{\gamma t} \quad (3.26)$$

Proof. Firstly, note that for $\phi \in \Phi_2$ and $m > 0$, since $m + \lambda(\phi) < 0$, $\gamma(\phi) > 0$. we check that $C^*(t) > 0, \forall t$. This is true since $m + \lambda < 0$, $\gamma + \kappa_2 < 0$ (by Lemma 3.1) and $mA_0 - (m + \lambda)B_0 = m(A_0 - B_0) - \lambda B_0 > 0$, since this expression is non-negative once $m > 0$ and $\lambda < 0$ and it would be null if and only if $A_0 = B_0 = 0$, which violates the definition of Φ . Now, it must be checked if the trajectories given above in fact satisfy (3.3) - (3.8). In what regards (3.3), we check (3.1) instead, since they are equivalent for w and π from (2.4) and (2.7), respectively:

$$\dot{A}^*(t) = \frac{mA_0 - (m + \lambda)B_0}{m(\gamma - \lambda)}(m + \gamma)\gamma e^{\gamma t} + \frac{-mA_0 + (m + \gamma)B_0}{m(\gamma - \lambda)}(m + \lambda)\lambda e^{\lambda t} \quad (3.27)$$

Plugging equations (3.24), (3.25), (3.26) and (3.27) in (3.1) yields an identity. In what regards restriction (3.4), note that:

$$\dot{B}^*(t) = \frac{mA_0 - (m + \lambda)B_0}{m(\gamma - \lambda)}m\gamma e^{\gamma t} + \frac{-mA_0 + (m + \gamma)B_0}{m(\gamma - \lambda)}m\lambda e^{\lambda t} \quad (3.28)$$

Plugging equations (3.24), (3.25), (3.26) and (3.28) in (3.4) yields an identity. In what concerns the initial conditions (3.5) and (3.6), note that:

$$A^*(0) = \frac{mA_0(m + \gamma - m - \lambda) - (m + \lambda)(m + \gamma)B_0 + (m + \lambda)(m + \gamma)B_0}{m(\gamma - \lambda)} = A_0$$

$$B^*(0) = \frac{mA_0 - mA_0 + B_0(m + \gamma - m - \lambda)}{\gamma - \lambda} = B_0$$

In order to show that physical constraint (3.7) is satisfied, using a little trick is convenient. Firstly, note that the time derivative of $A^*(t)$, given in equation (3.27) has the same sign as:

$$\frac{mA_0 - (m + \lambda)B_0}{m(\gamma - \lambda)}(m + \gamma)\gamma + \frac{-mA_0 + (m + \gamma)B_0}{m(\gamma - \lambda)}(m + \lambda)\lambda e^{(-\gamma + \lambda)t}$$

Since the last function is monotonic, $\dot{A}^*(t) \geq 0, \forall t$ if and only if the above expression is non-negative at $t = 0$ and at infinity. In the limit, since $\gamma > 0 > \lambda$:

$$\lim_{t \rightarrow \infty} \dot{A}^*(t) = \frac{mA_0 - (m + \lambda)B_0}{m(\gamma - \lambda)}(m + \gamma)\gamma > 0,$$

since every term of the above fraction is positive. In what regards $t = 0$, it is enough bear in mind Lemma 3.2 and to check that $\dot{A}^*(0) = \Lambda(g_0(\phi), \gamma(\phi))$:

$$\begin{aligned} \dot{A}^*(0) &= \frac{mA_0 - (m + \lambda)B_0}{m(\gamma - \lambda)}(m + \gamma)\gamma + \frac{-mA_0 + (m + \gamma)B_0}{m(\gamma - \lambda)}(m + \lambda)\lambda \\ &= A_0 \frac{m(1 - g_0)(m + \gamma + \lambda) - g_0\gamma\lambda}{m} = \Lambda(g_0(\phi), \gamma(\phi)) \end{aligned}$$

In what regards restriction (3.8), notice that it is satisfied once (3.7) is satisfied.

Now, we check that the trajectories are in fact optimal. As we did in the no-research case, we shall use Sufficiency Theorem 6.11 of Seierstad e Sydsaeter (1986). Using the Hamiltonian and the Lagrangian defined in (3.13) and (3.14), let the co-state variable and Lagrange multiplier be:

$$p(t) = \frac{m^\theta}{(mA_0 - (m + \lambda)B_0)^\theta (-m - \gamma - \kappa_2 - \lambda)^\theta \beta^{\theta-1}} e^{(-\frac{\pi}{\beta} + m)t} \quad (3.29)$$

$$q(t) = 0 \quad (3.30)$$

Now we show that each condition of the Theorem is satisfied. Below, the $*$ indicates that the derivatives are being evaluated along $(D^*(t), C^*(t), p(t), q(t), t)$. Firstly we have to check that, $\forall C > 0$, it holds that:

$$\frac{\partial \mathcal{L}^*}{\partial C}(C - C^*(t)) \leq 0, \forall t$$

From (3.14), we see that:

$$\frac{\partial \mathcal{L}^*}{\partial C} = \frac{e^{-\rho t}}{C^{*\theta}} - \frac{p(t) + q(t)}{\beta}$$

By plugging equations (3.26), (3.29) and (3.30) in the above expression, we see that $\partial \mathcal{L}^* / \partial C = 0$, and the condition holds.

Now we look at the co-state variable movement equation:

$$-\dot{p}(t) = \frac{\partial \mathcal{L}^*}{\partial D}$$

First note that $\partial \mathcal{L}^* / \partial D = (p(t) + q(t))(\pi/\beta) - p(t)m = p(t)(\pi/\beta - m)$. Since $\dot{p}(t)/p(t) = m - \pi/\beta$, the condition holds.

Now we turn our attention to the 'transversality condition'

$$\liminf(p(t)(D(t) - D^*(t))) \geq 0$$

which must hold for all admissible $D(t)$ ($D(t) \geq 0, \forall t$). Using (3.24) and (3.25), one has:

$$D^*(t) = \frac{mA_0 - (m + \lambda)B_0}{m(\gamma - \lambda)}\gamma e^{\gamma t} + \frac{-mA_0 + (m + \gamma)B_0}{m(\gamma - \lambda)}\lambda e^{\lambda t}$$

Now see that $\lambda + (m - \pi/\beta) < \gamma + (m - \pi/\beta) \leq \pi/\beta - m - \rho + (m - \pi/\beta) = -\rho < 0$, since γ decreases in θ and $\theta \geq 1$, and therefore $\lim D^*(t)p(t) = 0$. Moreover, since $\gamma > 0$, $\pi/\beta > m + \rho > m$ and $\lim p(t) = 0$. Thus, it is enough to prove that $\liminf p(t)D(t) \geq 0$ for all admissible $D(t)$. In order to do that, first note that $p(t)$ is positive for all t since $-m - \lambda - \gamma - \kappa_2 > 0$, as seen above. Since $D(t) \geq 0$, for every admissible trajectory of the state variables, the result follows.

Validity of the complementary slackness condition for the Lagrange multiplier is trivial, since q is identically null. Finally, we check that the left-hand side of restriction (3.12) is quasi-concave in (D, C) , since it is linear in these arguments, and that the Hamiltonian (3.13) is concave in (D, C) , since it is linear in D and strictly concave in C .

□

Note that the asymptotic growth rates of $A(t)$, $B(t)$ and $C(t)$ all equal γ , since $\lambda < 0$, whence this model presents a balanced growth path.

We had to treat the cases $m = 0$ and $m > 0$ separately in Propositions 3.2 and 3.3 on the grounds that for $m = 0$, the expressions (3.24) and (3.26) are not well defined. In this fashion, it is natural to ask whether as m tends to zero, (3.26) boils down to (3.22). Indeed:

$$\lim_{m \rightarrow 0_+} \frac{mA_0 - (m + \lambda)B_0}{m} (-\beta)(m + \lambda + \gamma + \kappa_2) e^{\gamma t} = \beta \left(\frac{\frac{\pi}{\beta} - \rho}{\theta} + \kappa_2 \right) e^{\frac{\pi}{\beta} - \rho} \lim_{m \rightarrow 0_+} \frac{mA_0 - (m + \lambda)B_0}{m}$$

if the last limit exists, and it does. In order to evaluate it, note that:

$$\frac{d\lambda}{dm} = -\frac{1}{2} \left(1 + \frac{m - \kappa_2 - 2\kappa_1}{\sqrt{(m - \kappa_2)^2 - 4m\kappa_1}} \right) \quad (3.31)$$

$$\lim_{m \rightarrow 0_+} \frac{d\lambda}{dm} = \lim_{m \rightarrow 0_+} \left(-\frac{1}{2} \left(1 + \frac{m - \kappa_2 - 2\kappa_1}{\sqrt{(m - \kappa_2)^2 - 4m\kappa_1}} \right) \right) = -1 - \frac{\kappa_1}{\kappa_2} \quad (3.32)$$

Applying the L'Hopital Rule, and then (3.31) and (3.32) to the limit:

$$\lim_{m \rightarrow 0_+} \frac{mA_0 - (m + \lambda)B_0}{m} = \lim_{m \rightarrow 0_+} \frac{A_0 - B_0(1 + \frac{d\lambda}{dm})}{1} =$$

$$A_0 - B_0 \left(1 + \lim_{m \rightarrow 0_+} \frac{d\lambda}{dm}\right) = A_0 + B_0 \frac{\kappa_1}{\kappa_2},$$

The restrictions placed upon γ are necessary particularly in light of the fact that for $\phi \notin \Phi_2$, $\dot{A}^*(0) < 0$. Recalling the function $\Lambda(g_0, \gamma)$ defined previously, if $\gamma < 0$, $\Lambda(1, \gamma) < 0$ and $\Lambda(0, \gamma) < 0$, and the physical restriction at $t = 0$ is not satisfied for any g_0 . If $0 < \gamma \leq -m - \lambda$, see that for g_0^* , $\Lambda(g_0^*, \gamma) = 0$. Since $\Lambda(0, \gamma) < 0$, for $0 < \gamma < -m - \lambda$ and $\Lambda(\cdot)$ is monotone in its first argument, for $g_0 < g_0^*$ and $0 < \gamma \leq -m - \lambda$, $\Lambda(g_0, \gamma) < 0$. In particular, all of this implies, that if $g_0 = 0$, we are in a type-2 equilibrium if $\gamma \geq -m - \lambda$; on the other hand, if $g_0 = 1$, we are in a type-2 equilibrium if $\gamma > 0$.

Considerations regarding $\Lambda(g_0, \gamma)$ help us understand how the parameters γ and g_0 interact in determining whether at $t = 0$ agents will get involved in R&D activities or not. Since if $\gamma \leq 0$, $\Lambda(g_0, \gamma) < 0$, $\forall g_0$, in this subspace of Φ , agents never innovate in $t = 0$, irrespective of the values of g_0 , as stressed in the previous subsection. On the other hand, if $\gamma \geq -m - \lambda$, $\Lambda(g_0, \gamma) > 0$, $\forall g_0$, and irrespective of g_0 , agents will choose to innovate at $t = 0$. Thus, for the subspace of Φ where $\gamma \leq 0$ or $\gamma \geq -m - \lambda$, g_0 exerts no influence whatsoever in the R&D investment decision in $t = 0$ and in the type of equilibrium the economy yields, so that 0 and $-m - \lambda$ can be understood as threshold values regarding R&D profitability.

In fact, the value of $\Lambda(g_0, \gamma)$ presents a more intimate relation with the type of equilibrium attained by the economy. This is the case since the Lagrange multipliers $q(t)$ and $\mu(t)$ are identically null in type-2 equilibria, and, as a result, the solutions in Propositions 3.2 and 3.3 are the solution of the unconstrained - with respect to the physical constraints - household problem (indeed, coinciding with the solutions found in Cysne e Turchick (2012)). Thus, when such restrictions are not binding, households adopt the consumption plan given by equation (3.26), or its limit if $m = 0$. This is particularly important, since, as calculations show, if $\dot{A}(t) \geq 0$ is not binding for $t = 0$, for trajectories given by equations (3.24) to (3.26), then $\dot{A}(t) \geq 0$ and $\dot{B}(t) \geq 0$ will not be binding for any other t . Thus, we are in a type-2 equilibrium if and only if $\Lambda(g_0, \gamma) \geq 0$.

On the other hand, if $0 < \gamma < -m - \gamma$, the sign of $\Lambda(g_0, \gamma)$ is determined jointly by g_0 and γ . In this respect, this range of values of γ might be deemed as "transition" values, where considerations about γ alone - regardless of g_0 - are not enough to determine if agents will innovate at $t = 0$, and consequently the kind of equilibrium attained. In particular, for $\gamma > -m$, $\partial\Lambda(g_0, \gamma)/\partial g_0 > 0$, and a bigger stock of copied goods initially, A_0 kept constant, creates an upward shift in the incentives to invest in R&D in such subspace.

In fact, since $\Lambda(g_0, \gamma) < 0$, $\forall g_0$ if $\gamma < 0$, they can never transition from this boundary consumption state, since it would imply $\dot{A}(t^*_+) < 0$. For the switching case, studied in the

next section, consumers start consuming at the boundary but then transition to an unconstrained consumption trajectory at some finite t^* , specifically for t^* satisfying, $g(t^*) = \bar{g}_0$, in trajectories of Proposition 3.1.

3.4 Switching case

In this subsection we study the solution to (3.2) - (3.8) when $\phi \in \Phi_3$. It turns out that in this subspace of Φ initially the trajectories present no research and as a result consumption at the boundary. As $g(t) = B(t)/A(t)$ grows, eventually it reaches $\bar{g}_0$, moment when R&D begins. This is formalized in the next proposition.

Proposition 3.4: Define:

$$t^* := \frac{\ln\left(\frac{(m+\lambda)(m+\gamma)(1-g_0)}{\gamma\lambda}\right)}{m}$$

Given an economy described by $\phi \in \Phi$ such that and $\phi \in \Phi_3$, the equilibrium is given by:

$$A^*(t) = \begin{cases} A_0, & \text{for } t < t^* \\ A_0 \frac{-\lambda e^{\gamma(t-t^*)} + \gamma e^{\lambda(t-t^*)}}{\gamma - \lambda}, & \text{for } t \geq t^* \end{cases} \quad (3.33)$$

$$B^*(t) = \begin{cases} A_0 + (B_0 - A_0)e^{-mt}, & \text{for } t < t^* \\ \frac{mA_0}{\gamma - \lambda} \left(\frac{\lambda}{m + \gamma} e^{\gamma(t-t^*)} + \frac{\gamma}{m + \lambda} e^{\lambda(t-t^*)} \right), & \text{for } t \geq t^* \end{cases} \quad (3.34)$$

$$C^*(t) = \begin{cases} \beta(A_0(-\kappa_2 + \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt}), & \text{for } t < t^* \\ A_0\beta\frac{\lambda}{m + \gamma}(m + \gamma + \lambda + \kappa_2)e^{\gamma(t-t^*)}, & \text{for } t \geq t^* \end{cases} \quad (3.35)$$

Proof. Firstly, we check that t^* defined above is in fact positive, which happens if and only if

$$\frac{(m + \lambda)(m + \gamma)(1 - g_0)}{\gamma\lambda} > 1$$

Since $\gamma > 0$ and $\lambda < 0$, this holds, for positive γ , if and only if

$$(m + \lambda)(m + \gamma)(1 - g_0) - \gamma\lambda < 0 \iff g_0(m + \lambda)(m + \gamma) > m(m + \gamma + \lambda) \iff g_0 < \frac{(m)(m + \gamma + \lambda)}{(m + \gamma)(m + \lambda)} = \bar{g}_0,$$

since $m + \lambda < 0$ and $m + \gamma > 0$. Now it is necessary to check if restrictions (3.3) to (3.8) are indeed satisfied. That the initial conditions (3.5) and (3.6) are satisfied stems from the fact that for $t < t^*$, equations (3.15) to (3.17) coincide with (3.33) to (3.35), and there the initial

conditions are indeed observed. Analogously, in what concerns conditions (3.3), (3.4) and (3.7) for $t < t^*$, they are satisfied, since they are satisfied in Proposition 3.1. For $t \geq t^*$, in what regards the physical restriction (3.7), note that:

$$\dot{A}^*(t) = A_0 \frac{-\lambda\gamma}{\gamma - \lambda} (e^{\gamma(t-t^*)} - e^{\lambda(t-t^*)}) > 0$$

In what regards restriction (3.8), notice that it is satisfied once (3.7) is satisfied. The movement equations (3.3) and (3.4) hold since the trajectories embodied in equations (3.33) to (3.35) coincide (with a lag of t^*), for $t \geq t^*$, with (3.24) to (3.26) and taking $B_0 = A_0 \bar{g}_0$.

It is also necessary to check that the state variables trajectories are in fact continuous. Firstly, note that $A^*(t^*) = A_0$. Furthermore

$$\lim_{t \rightarrow t^*_+} B(t) = B(t^*) = \frac{mA_0}{\gamma - \lambda} \left(\frac{-\lambda(m + \lambda) + \gamma(m + \gamma)}{(m + \lambda)(m + \gamma)} \right) = \frac{mA_0}{\gamma - \lambda} \frac{(\gamma - \lambda)(m + \gamma + \lambda)}{(m + \lambda)(m + \gamma)} = A_0 g_0^*$$

and

$$\begin{aligned} \lim_{t \rightarrow t^*_-} B(t) &= A_0 + (B_0 - A_0)e^{-mt^*} = A_0 + (B_0 - A_0)e^{\ln(\frac{\gamma\lambda}{(1-g_0)(m+\gamma)(m+\lambda)})} = \\ &= A_0 \left(1 - \frac{\gamma\lambda}{(m + \gamma)(m + \lambda)} \right) = A_0 \frac{(m)(m + \gamma + \lambda)}{(m + \lambda)(m + \gamma)} = A_0 g_0^* \end{aligned}$$

Now we show that the consumption trajectory is in fact optimal. Using the Hamiltonian and the Lagrangian defined in (3.13) and (3.14), respectively, let the co-state variable and Lagrange Multiplier be:

$$p(t) = \begin{cases} \frac{\pi}{\beta^\theta} e^{mt} \int_t^{t^*} \frac{e^{-(\rho+m)\tau} d\tau}{(A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta}, & \text{for } t < t^* \\ \frac{(m+\gamma)^\theta}{A_0^\theta (\lambda(m+\gamma+\lambda+\kappa_2))^\theta \beta^{\theta-1}} e^{(m-\frac{\pi}{\beta})(t-t^*)}, & \text{for } t \geq t^* \end{cases} \quad (3.36)$$

$$q(t) = \begin{cases} \frac{e^{-\rho t}}{\beta^{\theta-1}(A_0(-\kappa_2 - \kappa_1) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta} - \frac{\pi}{\beta^\theta} e^{mt} \int_t^{t^*} \frac{e^{-(\rho+m)\tau} d\tau}{(A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta}, & \text{for } t < t^* \\ 0, & \text{for } t \geq t^* \end{cases} \quad (3.37)$$

In order to prove optimality, we utilize, once again, Theorem 6.11 in Seierstad e Sydsaeter (1986). Note that the co-state variable is continuous, since $\lim_{t \rightarrow t^*_-} p(t) = \lim_{t \rightarrow t^*_+} p(t) = 0$ and therefore, we check the validity of the usual conditions. The * below indicates that the derivative is beeing evaluated at $(D^*(t), C^*(t), p(t), q(t), t)$, where the co-state variables and Lagrange multipliers are as defined in equations (3.36) and (3.37) and trajectories are as of equations (3.33) - (3.35). Firstly we have to check that, $\forall C > 0$, it holds that:

$$\frac{\partial \mathcal{L}^*}{\partial C} (C - C^*(t)) \leq 0, \forall t$$

Now we check the co-state variable movement equation is satisfied:

$$-\dot{p}(t) = \frac{\partial \mathcal{L}^*}{\partial D}$$

Firstly, take $t < t^*$. Evaluating $\partial \mathcal{L}^*/\partial D$:

$$\begin{aligned} \frac{\partial \mathcal{L}^*}{\partial D} &= (p(t) + q(t)) \frac{\pi}{\beta} - p(t)m = \\ &= \frac{\pi}{\beta^\theta} \frac{e^{-\rho t}}{(A_0(-\kappa_2 - \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt})^\theta} - m \frac{\pi}{\beta^\theta} e^{mt} \int_t^{t^*} \frac{e^{-(\rho+m)\tau} d\tau}{(A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta} \end{aligned}$$

Using Leibniz Rule and the Product Rule, it is easy to see that $-p(t)$ equals the last expression above for $t < t^*$. For $t \geq t^*$, first note that $\partial \mathcal{L}^*/\partial D = (p(t) + q(t))(\pi/\beta) + -p(t)m = p(t)(\pi/\beta - m)$. Note also that $\dot{p}(t)/p(t) = m - \frac{\pi}{\beta}$, which leads to the desired result.

In what concerns the complementary slackness of the Lagrange Multiplier, note that, since $q(t) = 0$ for $t \geq t^*$, there is nothing to check in this interval. The proof that $q(t) \geq 0$ for $t < t^*$ is very intricate and left to the Appendix.

The 'transversality condition' $\liminf(p(t)(D(t) - D^*(t))) \geq 0$, which must hold for all admissible $D(t)$, follows from the discussion carried out in proving Proposition 3.3, since the co-state and state variables defined in Propositions 3.3. and 3.4 coincide asymptotically.

Finally, left-hand side of restriction (3.12) is quasi-concave in (D, C) , since it is linear in these arguments, and that the Hamiltonian (3.13) is concave in (D, C) , since it is linear in D and strictly concave in C .

□

As outlined above, for $t < t^*$, the trajectories (3.33) to (3.35) coincide with the trajectories (3.15) to (3.17) in Proposition 1. The associated co-state variables and Lagrange multipliers presented in Propositions 3.1 and 3.4 also coincide, up to a constant. For $t \geq t^*$, the trajectories (3.33) to (3.35), up to a lag of t^* , coincide with trajectories (3.24) to (3.26), as do co-state variables and Lagrange multipliers defined in Propositions (3.3) and (3.4)

In connection with the discussion carried out in the previous section, that initially the consumers are not able to stick to an unconstrained consumption trajectory, since it would entail $\dot{A}(0) < 0$, as shown. Nonetheless, in opposition to what happens when $\gamma < 0$, there exists $0 \leq g_0 \leq 1$ such that $\Lambda(g_0, \gamma) = 0$, if $0 \leq \gamma \leq -m - \lambda$, insomuch as $\Lambda(0, \gamma) \leq 0$ and $\Lambda(1, \gamma) \geq 0$ and Λ is affine in g_0 . In fact g_0^* is such that $\Lambda(g_0^*, \gamma) = 0$. Therefore, consumers wait until the ratio between the measure of copied goods and all intermediate goods reaches g_0^* , so they can engage in R&D, given that the physical restriction concerning $A(t)$ will not be violated. This transition happens in t^* ; in fact, t^* is such that $\lim_{t \rightarrow t^*} g(t) = g_0^*$. Since $g(0) < g_0^*$, $g(t)$ is

increasing when there is no research and $\lim_{t \rightarrow \infty} g(t) = 1$, if there were no transition, then such t^* exists.

It is interesting to note that if $\gamma(\phi) = 0$, then there will be neither R&D nor violation of (3.7). In fact, if $\phi \in \Phi$ is such that $\gamma(\phi) = 0$, such ϕ is in the boundary of Φ_1 , Φ_2 and Φ_3 (in the $g_0 - \gamma$ plane). However, if $\gamma(\phi) = 0$ and $g_0(\phi) < 1$, and we chose $\phi \in \Phi_3$, $t^* \rightarrow \infty$, what would correspond to exactly no R&D ever, so that it is more sensible to let this case under type-1 equilibria. For $g_0(\phi) = 1$, type-2 and type-1 equilibria trajectories coincide. Then we could, at principle, choose either, but this would complicate a bit the formulation of the sets. Thus, we chose to treat the case $\gamma(\phi) = 0$, for any $g_0(\phi)$, as a type-1 equilibrium.

4 Optimal Policy

Having found the equilibrium trajectories in the parametric space Φ , we are now in position to finally deal with the aim of this work, calculating the optimal patent length parameter m^* . In contrast to Cysne e Turchick (2012), we find analytically that, under logarithmic utility, it will be optimal to provide either full or no patent protection, i.e., given $\omega \in \Omega$, it is possible that the policymaker prefers to induce no research, as such trajectories yield higher discounted utility values than BGP trajectories. Moreover, as we deal with sufficiently high values of θ , numerical simulations indicate that it is even possible to find interior values for optimal patent duration, which strikingly well reflect utility models duration in OECD countries.

For logarithmic utility, we apply the following strategy for solving the problem. Firstly, we present the concept of implementability. Taking into account that m is endogenous to the kind of equilibrium that arises in the economy, a type- i equilibrium is implementable ($i \in \{1, 2, 3\}$), given $\omega \in \Omega$, if there exists a policy choice m so that $(\omega, m) \in \Phi_i$. If a type-1 equilibrium is implementable, we optimize with respect to m , and do the same for type-2 equilibria. We then show that the policymaker will never find it optimal to implement a type-3 equilibrium. Lastly, we compare the value functions attached to type-1 and type-2 equilibria when both types of equilibrium are implementable.

4.1 Optimal policy for type-1 and type-2 equilibria

4.1.1 Implementability

The policymaker, when faced with $\omega \in \Omega$, must pick m so that the intertemporal utility is maximized. Nevertheless, m is endogenous to the determination of the kind of equilibrium that arises, since γ , $\bar{g}_0$ and λ are functions of m .

The following alternative characterizations, for $m > 0$ of Φ_2 and Φ_3 will be useful in what follows:

$$\Phi_2 = \{\phi \in \Phi : \gamma(\phi) \geq -\frac{m'(m' + \lambda(\phi))(1 - g_0(\phi))}{m' - (m' + \lambda(\phi))g_0(\phi)}\}$$

$$\Phi_3 = \{\phi \in \Phi : 0 < \gamma(\phi) < -\frac{m'(m' + \lambda(\phi))(1 - g_0(\phi))}{m' - (m' + \lambda(\phi))g_0(\phi)}\}$$

We check equivalence in the Appendix. Denote, for convenience, given $\phi \in \Phi$:

$$z(\phi) = -\lim_{m' \rightarrow m} \frac{m'(m' + \lambda(\phi))(1 - g_0(\phi))}{m' - (m' + \lambda(\phi))g_0(\phi)}$$

We now list some properties of the function z :

1. $z(\omega, 0) = 0, \forall \omega \in \Omega$:

Proof. If $m = 0$, using the L'Hopital Rule (it is clear that the denominator and numerator assume the value zero when $m = 0$, since $\lambda = 0$):

$$-\lim_{m' \rightarrow 0} \frac{m'(m' + \lambda(\phi))(1 - g_0(\phi))}{m' - (m' + \lambda(\phi))g_0(\phi)} = -\lim_{m' \rightarrow 0} (1 - g_0(\phi)) \frac{(m' + \lambda(\phi)) + m'(1 + \frac{d\lambda(\phi)}{dm})}{1 - (1 + \frac{d\lambda(\phi)}{dm})g_0(\phi)} = 0$$

The numerator of the last fraction is null since $m + \lambda = 0$ when $m = 0$ and $\lim_{m' \rightarrow 0} d\lambda(m')/dm$ does not explode, by (3.32). The denominator of the last fraction is never null, since it is monotone in g_0 and assumes positive values for $g_0 = 0$ and $g_0 = 1$ $\square$

2. $z(\phi) \geq 0, \forall \phi \in \Phi$

Proof. If $m = 0$, we already know that $z(\phi) = 0$. Now, for $m > 0$:

$$-\lim_{m' \rightarrow m} \frac{m'(m' + \lambda(\phi))(1 - g_0(\phi))}{m' - (m' + \lambda(\phi))g_0(\phi)} = -\frac{m(m + \lambda(\phi))(1 - g_0(\phi))}{m - (m + \lambda(\phi))g_0(\phi)}$$

We already know that $m + \lambda < 0$ and $m - (m + \lambda)g_0 > 0$, so that the results follows. Equality happens if and only if $g_0 = 1$ $\square$

3. Given $\phi \in \Phi$, if $m > 0$, $\partial z(\phi)/\partial g_0(\phi) < 0$

Proof. Note that $\lambda(\phi)$ is constant in both A_0 and B_0 , so that, by the Chain Rule, when $m > 0$:

$$\frac{\partial z(\phi)}{\partial g_0(\phi)} = \frac{m(m + \lambda)(m - (m + \lambda)g_0 + (1 - g_0)(1 - (1 + \frac{\partial \lambda}{\partial m})g_0))}{(m - (m + \lambda)g_0)^2}$$

Now, from (3.31), $\partial \lambda / \partial m < -1$, since $m - 2\kappa_1 - \kappa_2 > \sqrt{(m - \kappa_2)^2 - 4m\kappa_1} > 0$. Since $m + \lambda < 0$ and $m - (m + \lambda)g_0 > 0$, the denominator of the above function is strictly negative and the result follows $\square$

Now we introduce the concept of implementability.

Definition 4.1. Let $\omega \in \Omega$. A type- i equilibrium is **implementable** for ω ($i \in \{1, 2, 3\}$) if there exists $m \in [0, +\infty]$ such that $(\omega, m) \in \Phi_i$.

Now we prove three useful lemmas:

Lemma 4.1. Given $\omega \in \Omega$, a type-1 equilibrium is implementable for ω

Proof. Let $\omega = (\alpha, \rho, L, \beta, \theta, A_0, B_0) \in \Omega$, pick $m = \max(\pi/\beta - \rho, 0)$ in which case $\gamma \leq 0$ and $(\omega, m) \in \Phi_1$. $\square$

In particular, for any $\omega \in \Omega$, $(\omega, \infty) \in \Phi_1$.

Lemma 4.2. Let $\omega = (\alpha, \rho, L, \beta, \theta, A_0, B_0) \in \Omega$. A type-2 equilibrium is implementable for ω if and only if:

$$\frac{\pi}{\beta} = \frac{L}{\beta}(1 - \alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} > \rho \quad (4.1)$$

Proof. Let $\omega \in \Omega$. Suppose (4.1) holds. Then, $\gamma(\omega, 0) > 0$. The result follows from the fact that whenever $m = 0$, $(\omega, 0) \in \Phi_2 \iff \gamma(\omega, 0) > 0$

Now, let $\omega \in \Omega$, suppose a type-2 equilibrium is implementable for ω . For any $m > 0$, $z(\omega, m) \geq z(\omega, 0)$, from the facts (1) and (2) regarding z . Moreover, since $\gamma(\omega, m)$ is strictly decreasing in m , $\gamma(\omega, 0) > \gamma(\omega, m)$. Then, $\gamma(\omega, 0) - z(\omega, 0) > \gamma(\omega, m) - z(\omega, m)$, for any $m > 0$. Since, by hypothesis, a type-2 equilibrium is implementable, there exists $m \geq 0$ such that $\gamma(\omega, m) - z(\omega, m) \geq 0$ and necessarily $\gamma(\omega, 0) - z(\omega, 0) > 0$. As a result, (4.1) holds. $\square$

Interpreting γ as the returns over investment, the feasibility constraint for type-2 equilibria in equation (4.1) is just telling us that when returns to investment, given the most "investment-friendly" choice of m , i.e., zero, are negative - and in consequence, it can never be non-negative for any choice of m - research can not be implemented, since the agents, given the physical restrictions, will prefer to stand by and enjoy the cash flowing from the already carried out innovations.

Lemma 4.3. Given $\omega \in \Omega$, Type-3 equilibria are implementable for ω only if type-2 equilibria are implementable for ω

Proof. Let $\omega \in \Omega$ and suppose type-2 equilibria are not implementable for ω . By Lemma (4.2), $\pi/\beta \leq \rho$. Then, for any $m \geq 0$, $(\pi/\beta - m - \rho) \leq 0$. Then, type-3 equilibria are not implementable. $\square$

Corollary 4.3.1. If $\pi/\beta \leq \rho$, only type-1 equilibria are implementable.

From a purely mathematical viewpoint, the fact that only type-1 equilibria are always implementable stems from the facts that there is no upper bound, but an infimum of 0, to the value of m , γ is decreasing in m and that type-2 and type-3 equilibria involve γ above a certain

threshold, while type-1 equilibria appears when γ is below a threshold of zero. From an economic perspective, it shows that conditions for a R&D friendly environment must be furnished; otherwise, agents will prefer to live without investing in research.

4.1.2 Optimal policy for type-1 equilibrium

This result, that holds for all values of θ , shows that, whenever in a type-1 equilibrium, it is optimal to do totally away with innovation, by excluding patent protection from the legal framework.

Proposition 4.1: Given $\omega \in \Omega$ and the consumption trajectory of type-1 equilibrium, for any $\theta \geq 1$, intertemporal utility is maximized by providing no patent protection.

Proof. Let $\omega = (\alpha, \rho, L, \beta, \theta, A_0, B_0) \in \Omega$. We divide our analysis in two cases:

Case I: $\theta = 1$

In this case, consumers value consumption as given by $U_0 = \int_0^\infty e^{-\rho t} \ln C(t) dt$; plugging in the expression the equation (3.17), we find the utility obtained by consumers in type-1 equilibria:

$$U_0 = \int_0^\infty e^{-\rho t} \ln(\beta(A_0(-\kappa_2 + \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt})) dt$$

Since $e^{-\rho t}$ does not depend on m and the natural logarithmic function is strictly increasing, m must be chosen in order to maximize $Q(m) := A_0(-\kappa_2 + \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt}$. Notice that $dQ/dm = -m\kappa_1(A_0 - B_0)e^{-mt} > 0, \forall m \in [0, \infty)$. Thus, in order to maximize utility the policymaker should pick $m = \infty$ or analogously, the elimination of intellectual property legislation in the economy. Note that, by Lemma (4.1), by picking such m , a type-1 equilibrium arises in the economy. The resulting utility level is:

$$V_1 = \frac{\ln(\beta A_0(-\kappa_2 - \kappa_1))}{\rho} \quad (4.2)$$

Case II: $\theta > 1$

In this case, consumers value consumption as given by:

$$U_0 = \int_0^\infty e^{-\rho t} \frac{(\beta(A_0(-\kappa_2 - \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt}))^{1-\theta} - 1}{1-\theta} dt$$

Since $e^{-\rho t}$ does not depend on m and, for $\theta > 1$, the transformation $(\cdot)^{1-\theta}/(1-\theta)$ is strictly increasing, we shall choose m to maximize $Q(m)$ defined above, i.e., set $m = \infty$, which, by Lemma (4.1) yields a type-1 equilibrium. The value function in this case is given by (we suppress the subtraction for 1 in the numerator for algebraic simplicity with no loss,

since we are not concerned with limits as $\theta = 1$, and that configures an strictly increasing transformation):

$$W_1 = \frac{(\beta A_0(-\kappa_2 - \kappa_1))^{1-\theta}}{(1-\theta)\rho} \quad (4.3)$$

□

We provide the following intuition for the above result: if a type-1 equilibrium is implemented, the individuals will simply not invest in R&D, and the aforementioned trade-off cuts only in one direction. Thus, by eliminating intellectual property rights from the economy in $t = 0$, the policymaker brings the price of intermediate goods to its competitive level, what raises consumption to its maximal level, without hindering innovation.

4.1.3 Optimal policy for type-2 equilibrium and logarithmic utility

In this section, we prove that for logarithmic utility, given we are in a type-2 equilibrium, it is optimal to set $m = 0$.

Before proceeding, we prove three useful lemmas:

Lemma 4.4. $\frac{d\lambda(m)}{dm} \geq \frac{\lambda}{m}$

Proof. Using equation (3.31) and the definition of λ :

$$\begin{aligned} \frac{d\lambda(m)}{dm} \geq \frac{\lambda}{m} &\iff \frac{\kappa_2 + \sqrt{(m - \kappa_2)^2 - 4m\kappa_1}}{m} \geq \frac{m - \kappa_2 - 2\kappa_1}{\sqrt{(m - \kappa_2)^2 - 4m\kappa_1}} \iff \\ &\quad -\kappa_2 \leq -\frac{m(m - \kappa_2 - 2\kappa_1)}{\sqrt{(m - \kappa_2)^2 - 4m\kappa_1}} + \sqrt{(m - \kappa_2)^2 - 4m\kappa_1} \iff \\ -\kappa_2 \sqrt{(m - \kappa_2)^2 - 4m\kappa_1} &\leq -m(m - \kappa_2 - 2\kappa_1) + (m - \kappa_2)^2 - 4m\kappa_1 = \kappa_2^2 - m(\kappa_2 + 2\kappa_1) \end{aligned}$$

Since both members of the above inequality are positive, it is true that

$$\begin{aligned} -\kappa_2 \sqrt{(m - \kappa_2)^2 - 4m\kappa_1} &\leq \kappa_2^2 - m(\kappa_2 + 2\kappa_1) \iff \\ \kappa_2^2(\kappa_2^2 - 2m\kappa_2 + m^2 - 4m\kappa_1) &\leq \kappa_2^4 - 2m\kappa_2^3 - 4m\kappa_2^2\kappa_1 + m^2\kappa_2^2 + 4m^2\kappa_1\kappa_2 + 4m^2\kappa_1^2 \iff \\ 0 &\leq 4m^2\kappa_1(\kappa_2 + \kappa_1) \end{aligned}$$

The result holds since both κ_1 and κ_2 are strictly negative

□

Lemma 4.5. $\frac{d^2\lambda(m)}{dm^2} > 0$

Proof. Differentiating $d\lambda/dm$:

$$\begin{aligned} \frac{d^2\lambda(m)}{dm^2} &= -\frac{1}{2} \frac{\sqrt{(m-\kappa_2)^2 - 4m\kappa_1} - \frac{(m-\kappa_2-2\kappa_1)^2}{\sqrt{(m-\kappa_2)^2 - 4m\kappa_1}}}{(m-\kappa_2)^2 - 4m\kappa_1} = \\ &= -\frac{1}{2} \frac{(m-\kappa_2)^2 - 4m\kappa_1 - (m-\kappa_2-2\kappa_1)^2}{((m-\kappa_2)^2 - 4m\kappa_1)^{1.5}} = -\frac{1}{2} \frac{-4\kappa_1^2 - 4\kappa_1\kappa_2}{((m-\kappa_2)^2 - 4m\kappa_1)^{1.5}} > 0 \end{aligned}$$

□

Lemma 4.6. If $\gamma > 0$, then $(\pi/\beta)(1/\alpha) > \rho(\kappa_1/\kappa_2)$.

Proof. Since $\gamma > 0$, $\pi/\beta > m + \rho \geq \rho$. Therefore, it is enough to prove that $1/\alpha > \kappa_1/\kappa_2$. Firstly note, by the definition of κ_1 and κ_2 :

$$\frac{\kappa_1}{\kappa_2} = \frac{-1 - \alpha + \alpha^{\frac{\alpha}{\alpha-1}}}{1 + \alpha},$$

Thus, we have to show:

$$\frac{1}{\alpha} > \frac{-1 - \alpha + \alpha^{\frac{\alpha}{\alpha-1}}}{1 + \alpha} \iff 2 + \alpha + \frac{1}{\alpha} > \alpha^{\frac{\alpha}{\alpha-1}} \iff 3 + \alpha > \alpha^{\frac{\alpha}{\alpha-1}}$$

We shall prove that the last inequality is correct. Firstly, by taking the auxiliary variable $x = \alpha - 1$, it is easy to see that $\lim_{\alpha \rightarrow 1} \alpha^{\frac{\alpha}{\alpha-1}} = e < 3 + \alpha$. Besides, using logarithmic differentiation:

$$\frac{d}{d\alpha} (\alpha^{\frac{\alpha}{\alpha-1}}) = \frac{\alpha^{\frac{\alpha}{\alpha-1}} (\alpha - \ln\alpha - 1)}{(1 - \alpha)^2}$$

The last expression has the same sign as $(\alpha - \ln\alpha - 1)$, which we claim to be positive in the relevant range of α . At $\alpha = 1$ this expression assumes value zero. Its derivative is given by $1 - 1/\alpha$, which is negative for $0 < \alpha < 1$. Thus $(\alpha - \ln\alpha - 1)$ is strictly decreasing in $\alpha \in (0, 1)$, and since at $\alpha = 1$ this expression assumes value zero, it is positive for $0 < \alpha < 1$. Therefore, $\alpha^{\frac{\alpha}{\alpha-1}}$ is increasing in $(0, 1)$ and cannot be greater than e , which completes our point.

□

Proposition 4.2: Given $\omega = (\alpha, \rho, L, \beta, \theta, A_0, B_0) \in \Omega$ such that $\theta = 1$ and $h(\omega, 0) = 2$ and the consumption trajectory of type-2 equilibrium, intertemporal utility is maximized by providing full patent protection. ($m = 0$).

Proof. First of all, note that, from Lemma (4.2), type-II equilibria are implementable, so that the restriction set is not empty.

Firstly, by substituting equations (3.26) and (3.35) in the expression of the intertemporal utility of consumers and performing some calculations, we can find the levels of utility attained by consumers in a type-2 equilibrium with $m = 0$ and $m > 0$, respectively:

$$\begin{aligned}
 U_0 &= \int_0^\infty e^{-\rho t} \ln((\beta)(-\gamma - \kappa_2)(A_0 + \frac{\kappa_1}{\kappa_2}B_0)e^{\gamma t}) dt = \\
 &\int_0^\infty e^{-\rho t} (\gamma t + \ln((\beta)(-\gamma - \kappa_2)(A_0 + \frac{\kappa_1}{\kappa_2}B_0))) dt = \\
 &\frac{1}{\rho} \left(\frac{\gamma}{\rho} + \ln((\beta)(A_0)(-\gamma - \kappa_2) \left(1 + \frac{\kappa_1}{\kappa_2}g_0\right)) \right) \\
 U_0 &= \int_0^\infty e^{-\rho t} \ln\left(\frac{mA_0 - (m + \lambda)B_0}{m}(-\beta)(m + \gamma + \lambda + \kappa_2)e^{\gamma t}\right) dt = \\
 &\int_0^\infty e^{-\rho t} (\gamma t + \ln\left(\frac{mA_0 - (m + \lambda)B_0}{m}(-\beta)(m + \gamma + \lambda + \kappa_2)\right)) dt = \\
 &\frac{1}{\rho} \left(\frac{\gamma}{\rho} + \ln\left(\frac{m - (m + \lambda)g_0}{m}(-\beta)(A_0)(m + \gamma + \lambda + \kappa_2)\right) \right)
 \end{aligned}$$

Now define a function $J : (0, \infty) \times [0, 1] \rightarrow (-\infty, \infty)$, whose law is given by:

$$J(m, g_0) = \frac{\gamma}{\rho} + \ln\left(\frac{m - (m + \lambda)g_0}{m}(-\beta)(A_0)(m + \gamma + \lambda + \kappa_2)\right)$$

Note that $J(m, g_0)$ represents the intertemporal utility attained by consumers when in a type-2 equilibrium, multiplied by ρ . Notice that $J(\cdot)$ could not be defined for $m = 0$; nevertheless, note that the intertemporal utility for $m > 0$ converges to the one for $m = 0$ when $m \rightarrow 0$, as discussed above. Thus, we could continuously extend J to $m = 0$ by assigning $J(0, g_0) = \gamma/\rho + \ln[(\beta)(A_0)(-\gamma - \kappa_2)(1 + (\kappa_1/\kappa_2)g_0)]$. We shall not do so, since it is better to work with the limits of the partial derivatives when $m = 0$, rather than working with the derivatives of the limit. Notwithstanding this, by proving that $\partial J(m, g_0)/\partial m < 0, \forall (m, g_0) \in (0, \infty) \times [0, 1]$, which we shall do next, and taking into consideration that we could extend J to $m = 0$ assigning the level of utility effectively attained, we show that when in a type-2 equilibrium the optimal policy is $m^* = 0$

Firstly note that:

$$\frac{\partial J(m, g_0)}{\partial m} = -\frac{1}{\rho} - \frac{1}{m} + \frac{\frac{d\lambda}{dm}}{m + \gamma + \lambda + \kappa_2} + \frac{1 - g_0 - g_0 \frac{d\lambda}{dm}}{m - (m + \lambda)g_0} \quad (4.4)$$

$$\frac{\partial J(m, 0)}{\partial m} = -\frac{1}{\rho} + \frac{\frac{d\lambda}{dm}}{m + \gamma + \lambda + \kappa_2} \quad (4.5)$$

$$\lim_{m \rightarrow 0} \frac{\partial J(m, 0)}{\partial m} = -\frac{1}{\rho} + \frac{1 + \frac{\kappa_1}{\kappa_2}}{\rho - \frac{\pi}{\beta} - \kappa_2} \quad (4.6)$$

We shall prove $\partial J(m, g_0)/\partial m < 0, \forall (m, g_0) \in (0, \infty) \times [0, 1]$ in three steps.

Step i): $\partial J(m, g_0)/\partial m \leq \partial J(m, 0)/\partial m$

From (4.4) and (4.5) :

$$\frac{\partial J(m, g_0)}{\partial m} \leq \frac{\partial J(m, 0)}{\partial m} \iff \frac{1}{m} \geq \frac{1 - g_0 - g_0 \frac{d\lambda}{dm}}{m - (m + \lambda)g_0} \iff \frac{\lambda}{m} \leq \frac{d\lambda}{dm}$$

which was proven in Lemma 4.4.

Step ii): $\partial J(m, 0)/\partial m \leq \lim_{m \rightarrow 0} \partial J(m, 0)/\partial m$

Differentiating $\partial J(m, 0)/\partial m$ with respect to m and using Lemma 4.5:

$$\frac{\partial^2 J(m, 0)}{\partial m^2} = \frac{\frac{d^2 \lambda}{dm^2}(m + \gamma + \lambda + \kappa_2) - (\frac{d\lambda}{dm})^2}{(m + \gamma + \lambda + \kappa_2)^2} < 0$$

Thus $\partial J(m, 0)/\partial m$ is decreasing in m , and in particular $\lim_{m \rightarrow 0} \partial J(m, 0)/\partial m$ is an upper bound to $\partial J(m, 0)/\partial m$

Step iii): $\lim_{m \rightarrow 0} \partial J(m, 0)/\partial m < 0$

Remember, from the proof of Lemma 3.1 , that $\pi/\beta + \kappa_2 = -(\pi/\beta)(1/\alpha) < 0$. Thus, it is the case that:

$$\frac{1}{\rho} > \frac{1 + \frac{\kappa_1}{\kappa_2}}{\rho - \kappa_2 - \frac{\pi}{\beta}} \iff -\kappa_2 - \frac{\pi}{\beta} = \frac{\pi}{\beta} \frac{1}{\alpha} > \rho \frac{\kappa_1}{\kappa_2}$$

which was proven to be true in Lemma 4.6.

Finally yet importantly, since, by hypothesis, $h(\omega, 0) = 2$, picking $m = 0$ yields a type-2 equilibrium.

□

From an economic perspective this indicates that when in a research equilibrium, and under logarithmic utility - where substitution and income effects outweigh each other - long-run consumption gains due to investment in R&D completely outweigh short-run gains that would accrue from a lower level of investment. As $m^* = 0$, the value function V_2 attached to a type-2 equilibrium is given by

$$V_2 = \frac{1}{\rho} \left(\frac{\gamma}{\rho} + \ln \left((\beta)(A_0)(-\gamma - \kappa_2)(1 + \frac{\kappa_1}{\kappa_2} g_0) \right) \right) \quad (4.7)$$

4.2 Optimal policy in general for logarithmic utility

The next Proposition provides a useful reduction of the problem being tackled:

Proposition 4.3: Given $\omega = (\alpha, \rho, L, \beta, \theta, A_0, B_0) \in \Omega$, the policymaker never finds it optimal to implement a type-3 equilibrium

Proof. Firstly, consider the optimization problem (3.2) with restrictions given by equations (3.3) - (3.6) and (3.8). Note that we do not include the $\dot{A}(t) \geq 0$ restriction. For a fixed tuple of parameters for which $\gamma > 0$, the solution of this problem is given by trajectories (3.24) to (3.26). This stems from the fact that the Lagrange multiplier associated with the $\dot{A}(t) \geq 0$ restriction is identically null, and all other optimality conditions are not altered. Now, as we add the the $\dot{A}(t) \geq 0$ restriction, for $0 < \gamma < -m(m + \lambda)(1 - g_0)/(m - (m + \lambda)g_0)$, the solution is given by trajectories (3.33) to (3.35). The solution for $\gamma \geq -m(m + \lambda)(1 - g_0)/(m - (m + \lambda)g_0)$ is not changed. This implies in particular that, for every element of Φ such that $\gamma > 0$, the utility levels attained by trajectories (3.24) to (3.26) (type-2 equilibrium) are greater than or equal to those attained under trajectories (3.33) to (3.35) (type-3 equilibria) since the former arise in an optimization problem with less constraints.

In order to complete the argument, now consider the policymaker is faced with a tuple of parameters other than m such that type-2 and type-3 equilibria are implementable (this is the only material case for the proposition being proved). By Lemma (4.3), it is not possible that only type-3, but not type-2, equilibria are implementable. If he chooses m such that the economy falls in a type-3 equilibrium, he would prefer to be, with the same m , in a type-2 equilibria (of course, that would only be feasible if restriction (3.7) were not a concern; here, we are just comparing utility levels, i.e., numbers). Nevertheless, this option is never preferred, in the sense of yielding strictly higher utility levels, than implementing a type-2 equilibrium with optimal m ($m = 0$ if $\theta = 1$, from Proposition 4.2). $\square$

A straightforward consequence of Proposition 4.3. consists in that, on order to find m^* - for $\theta = 1$ - we shall now just compare levels of utility V_1 and V_2 defined in (4.2) and (4.7), when a type-2 equilibrium is feasible, i.e. when $(L/\beta)(1 - \alpha)(\alpha^{\frac{1+\alpha}{1-\alpha}}) > \rho$. On the other hand, if $(L/\beta)(1 - \alpha)(\alpha^{\frac{1+\alpha}{1-\alpha}}) \leq \rho$, every choice of m by the policymaker generates a type-1 equilibrium, by Corollary (4.3.1), in which case, by Proposition 4.1., $m = \infty$ should be chosen.

The condition $V_1 \geq V_2$, using equations (4.2) and (4.7), is equivalent to

$$\frac{\gamma}{\rho} \leq \ln \left(\frac{-\kappa_1 - \kappa_2}{(-\gamma - \kappa_2)(1 + \frac{\kappa_1}{\kappa_2}g_0)} \right) \quad (4.8)$$

which can be rewritten, in terms of fundamental parameters of the model, as

$$\frac{L}{\beta\rho}(1 - \alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} - 1 \leq \ln \left(\frac{\frac{L}{\beta}(1 - \alpha)(\alpha^{\frac{\alpha}{1-\alpha}})}{(\frac{L}{\beta}(1 - \alpha)\alpha^{\frac{2\alpha}{1-\alpha}} + \rho)(1 + \frac{-1-\alpha+\alpha^{\frac{\alpha}{1-\alpha}}}{1+\alpha}g_0)} \right) \quad (4.9)$$

Proposition 4.4: For $\theta = 1$, if condition (4.1) holds and condition (4.9) does not hold, it is optimal to pick $m = 0$. Otherwise, it is optimal to pick $m^* = \infty$.

Figure 2 – Values of m^* for $\rho = 0.05$ and $L/\beta = 1$

Proof. Follows from Propositions 4.1, 4.2 and 4.3 and equations (4.2) and (4.7). $\square$

Firstly, let us see heuristically that the condition (4.9) is not innocuous, since for small positive γ and small g_0 it will be satisfied. In order to see this, first note that for $g_0 = 0$ the right-hand side of (4.8) boils down to

$$\ln\left(\frac{-\kappa_1 - \kappa_2}{-\gamma - \kappa_2}\right) > 0$$

since $-\gamma - \kappa_2 > 0$, $-\kappa_1 - \kappa_2 > 0$ and $\gamma > 0 > \kappa_1$. If we rise ρ so that γ approaches zero, the above expression approaches $\ln(1 + \kappa_1/\kappa_2) > 0$, and being continuous in γ , there exists $\gamma^* > 0$ such that $\gamma^*/\rho < \ln((-\kappa_1 - \kappa_2)/(-\gamma^* - \kappa_2))$.

While we defer an analytic comparative statics exercise to Section 5, we proceed here with a little numerical exercise in order to illustrate the implications of Proposition 4.4. The results of such an exercise are depicted in Figures 2 and 3. In the first one, we plot g_0 against α , keeping the values of L/β fixed at 1 and of ρ at 0.05.

In figure 3, we plot α against L/β , setting $g_0 = 0.5$ and $\rho = 0.05$. Those values are chosen in order to fit the growth rates of the economy - if research is on - to real world values. Then, for the grid of Figure 2, the growth rates hits a maximum of 2.5% - and interest rates range between 5.5% and 7.5%, and no research cannot be optimal if γ is higher than 1.8%. In figure 3, by its turn, the growth (interest) rates hit a maximum of 9% (14%), but no research can be optimal only if $\gamma < 1.5\%$. The yellow region refers to $m^* = 0$, while the blue one to $m = \infty$; if heavy, due to the feasibility constraint, and if light, due to direct comparison of the value functions expressed in Equation (4.8).

Some interesting conclusions can be taken from the analysis of figures 2 and 3. Firstly, note, from both figures, that increasing α tends to favour optimality of type-1 equilibria. Analogously, as we lower L/β and g_0 , no research gets more desirable vis-à-vis investing in R&D.

Figure 3 – Values of m^* for $\rho = 0.05$ and $g_0 = 0.5$

These facts, with some qualification, specially regarding the behaviour of α , will be established analytically in the section 5.

The result presented in Proposition 4.4 is quite striking, since it entails that, even when a BGP with research can be implemented, it may be optimal to just extinguish patent law, removing every incentive for investment and implementing an environment without technological innovation. This might go counter to common sense, since innovation is often linked to greater welfare. Nevertheless, in our view, it tells a reasonable story. When agents discount future heavily (high ρ) or when the cost of innovation is high enough, for example, what brings about low returns accruing from innovation, which is captured by a low positive or negative γ , it may be optimal to "sit over" the already existing resources of the economy and enjoy solely the income of salaried work in final good firms. Once again, if the physical restrictions were not a concern, a negative growth rate would not influence consumers decisions, as long as they set consumption at $t = 0$ at a high enough level, which fatally violates the $\dot{A}(0) \geq 0$ restriction. In the next section we discuss mathematically and economically the influence of each fundamental of the model in this trade-off, captured in the comparison of the values of V_1 and V_2 .

4.3 Optimal patent duration for non-logarithmic utility

In this section we study the optimal patent duration for the case $\theta > 1$. Since we were unable to find analytically the optimal m in type-2 equilibria for $\theta > 1$, the analysis carried out in this subsection is purely numerical. We apply the same *modus operandi* of precedent subsections. For completion, we also present a numerical analysis, for the optimal patent duration in type-2 equilibria. Both tasks are intertwined, since to study optimality in general we must compute the value function attached to type-2 equilibria. Then we compare this value function with the value function attached to a type-1 equilibrium, which, as studied in Proposition 4.1., involves picking $m = \infty$. type-3 equilibria will not be considered in such analysis, by

Proposition 4.3.

4.3.1 Optimal patent duration for type-2 equilibria

In this section we study the optimal values of patent duration, given the economy is in a type-2 equilibrium. Firstly notice that, given an element of Φ such that $\theta > 1$ and that we are in a type-II equilibrium, the attained utility U_0 is given by (we suppress the subtraction for 1 in the numerator for algebraic simplicity with no loss, since we are not concerned with limits as $\theta = 1$, and that configures a strictly increasing transformation):

$$U_0 = \frac{\left((-m - \gamma - \lambda - \kappa_2)(m - (m + \lambda)g_0)\frac{\beta A_0}{m}\right)^{1-\theta}}{(1-\theta)(\rho + \gamma(\theta - 1))} \quad (4.10)$$

Denote by m^{**} the optimal value of the parameter m - the value of m which maximizes U_0 defined in (4.10) - for a type-2 equilibrium. Then, the value function W_2 of a type-2 equilibrium is given by - where $\gamma(\cdot)$ and $\lambda(\cdot)$ express the dependence of the values of these parameters to m :

$$W_2 = \frac{\left((-m^{**} - \gamma(m^{**}) - \lambda(m^{**}) - \kappa_2)(m^{**} - (m^{**} + \lambda(m^{**}))g_0)\frac{\beta A_0}{m^{**}}\right)^{1-\theta}}{(1-\theta)(\rho + \gamma^{**}(\theta - 1))} \quad (4.11)$$

Unlike the logarithmic utility case, we were unable to find an analytic expression for m^{**} . Nonetheless, numerical simulations indicate that, for, $\theta < 4$, it is indeed the case that $m^{**} = 0$.

However, if we rise θ , we can find interior values of m^{**} . In this respect, observe Figures 4, 5 and 6, which depict the best policy choice for a tuple $(\alpha, \theta, g_0, L/\beta, \rho)$. We grid θ against α , leaving the values of g_0 , ρ and L/β fixed. We kept α within the interval $[0, 35, 0, 80)$, in accordance with the numerical analysis of patent duration optimality in Kwan e Lai (2003), corresponding to realistic values of α either in its capital share or inverse of markup ratio interpretation. The white region is relative to $m^{**} = 0$, whilst the coloured one refers to an interior value of m^{**} , whose respective value T of patent duration, in years, is given by the side colour palette.

Some comments regarding the figures are in order. Firstly, it is possible to see that for higher values of θ and/or α , finding an interior m^{**} is more common. Also, the optimal term of patents ranges, in figures 4 and 5 from 15 to 30 years, which is quite close to patents terms in jurisdictions around the globe, specially the ones which adhered to the TRIPS agreement, which stipulates 20 years as the baseline patent duration. The consumption growth rates associated to the regions in which $m^{**} > 0$ holds, range from 0.5% to 2.5% in figure 4 and from 0,2 % to 1,8% in figure 4. It is also noteworthy that as we rise g_0 , it gets more difficult to find an interior solution for m^{**} , and in order to get it, it is necessary to rise the value of θ .

Figure 4 – Optimal patent term - type-2 eq. - for $g_0 = 0$, $\rho = 0.05$ and $L/\beta = 4$

Figure 5 – Optimal patent term - type-2 eq. - for $g_0 = 0.2$, $\rho = 0.08$ and $L/\beta = 4$

Figure 6 – Optimal patent term - type-2 eq. - for $g_0 = 0.6$, $\rho = 0.06$ and $L/\beta = 10$

Figure 7 – Optimal values of m for $\alpha = 0.4$, $\rho = 0.05$ and $L/\beta = 2$

Notwithstanding this, for the values of parameters studied in the current exercise, the solution to the general optimal policy problem - when the policymaker is not restricted to implement a type-2 equilibrium - still involves picking boundary values of m^* , i.e., the value functions attached to the interior values of m^{**} are dominated by the corresponding value function of type-1 equilibria. In the next section we tackle this properly, showing some regions for which we can find interior m^* .

4.3.2 Optimal patent duration in general

Now, we turn to the question of optimality of patent length in general.

Firstly, we concentrate our analysis on a region of values of $(\alpha, \theta, g_0, L/\beta, \rho)$ for which m^* assumes only boundary values, as in the logarithmic utility case. The region chosen comprises $\theta \in [1, 9]$, $\rho = 0.05$ and $L/\beta = 2$. We pick also three different values of α , 0.4, 0.625 and 0.75, in order to assess the impact of this variable upon the optimal policy choice, keeping its value reasonably well fit to both the markup and the capital share interpretation. The value of L/β was chosen so that the growth rate of the economy - in case there is research - ranges from 0,1 % to 9,0 % a.a. We plot θ against g_0 ; the yellow region is the one where $m = \infty$ is optimal. The blue one corresponds to the region of optimality of $m = 0$:

Some interesting conclusions can be drawn from the analysis of figures 7 to 9. First and foremost, our results depart considerably from those obtained in Cysne e Turchick (2012), inasmuch they find that for such manifold, $m^* = 0$. The discrepancy in results emanates, as expected, from duly taking the physical restrictions into account, which brings about a solution, for a subset of Φ , with no research, which happens to be associated with a higher value function than the one associated with a type-2 equilibrium.

In what concerns comparative statics, firstly note that, alike the logarithmic utility case, lower g_0 favours the optimality of $m = \infty$. This is mathematically expected, since, for any

Figure 8 – Optimal values of m for $\alpha = 0.625$, $\rho = 0.05$ and $L/\beta = 2$ Figure 9 – Optimal values of m for $\alpha = 0.75$, $\rho = 0.05$ and $L/\beta = 2$

value of θ in the relevant range, the value function of a type-1 equilibrium is not altered by the value of g_0 , since $g(0_+) = 1$ when m is set at infinity in $t = 0$, and there subsists a direct relation between intertemporal utility levels attained and g_0 in type-2 equilibria, as equation (4.11) reveals. Economically, this indicates that for economies in which IPR are deteriorated, it may be optimal to introduce a 'shock' of patent protection.

Notice also, from Figure 9, that for $\alpha = 0.75$, and $g_0 < 0.25$, $T = 0$ is the optimal patent term, irrespective of the value of θ . However, for the three values of α utilized, for high enough g_0 , it is optimal to set $T = \infty$.

It is also interesting to mention that, in the spirit of equation (4.8), lower values of γ favour the optimality of $m = 0$. The explanation of this phenomenon for the logarithmic utility case carries on to the context now analysed: γ is a measure of returns to investment in R&D. In fact, for $\alpha = 0.4$, the values of γ attached to the grid depicted in Figure 7 range from 1% to 9%, but $m = \infty$ is optimal only if $\gamma < 2.5\%$; for $\alpha = 0.625$, γ ranges from 0.5% to 4.7%, but

Figure 10 – Optimal patent term for $g_0 = 0.6$, $\rho = 0.12$ and $L/\beta = 20$

$m = \infty$ is optimal only if $\gamma < 1.5\%$.

In what regards the coefficient of relative risk aversion, as we lower the value of θ , the optimality of $m = 0$ is favoured. This phenomenon is related to the fact that in a type-2 equilibrium the consumption path is more smooth, since the consumption growth rate – yet positive – is decreasing and reaches zero asymptotically. In a type-2 equilibrium, in opposition thereto, it is ever constant and positive. Notice also that for $\alpha = 0.4$ and $\alpha = 0.625$, for low enough values of θ , it is optimal to set $m = 0$ irrespective of the value of g_0 .

In what concerns α , for the range analysed above, as we rise α , the optimality of $m = \infty$ is favoured. Notice that in our model, α represents both the inverse of the markup as the capital share in production.¹ Roughly, higher values of α lower the attractiveness of R&D if taken into account its mark-up effects; the capital share effect points in the opposite direction. Thus, for the range of α analysed in the figures, the mark-up effect dominates. However, as simulations showed, if α is taken to (unrealistically) low values (just below 0.03), the capital share effect dominates, and it would always be optimal, given the other parameters, to set $m = \infty$.

If we take yet higher values of θ , it may be the case that an interior value of m is optimal, as depicted in figures 10 to 12. In these figures, the white regions refer to $m^* = \infty$, and the blue area to $m^* = 0$. The area painted in other colors represents the region for which m^* is interior, and the values of the optimal patent term, in years, are given by the side color palette. We now grid θ against α , keeping the values of ρ , L/β and g_0 fixed.

The growth rates of the economy for figure 12 range from 3% to 5%, consistent with real-world values. These values for Figure 10 range from 7 % to 9 %, a bit higher. Interestingly enough, the optimal patent terms in interior solutions, in figures 10 and 12 range from 3 years to 10 years, in line with the IPR terms of the so-called utility models in some countries. In fact,

¹ Ideally, the inverse of the markup and the capital share should be represented by different parameters. See Alvarez-Pelaez e Groth (2005), for an explanation of the consequences and a taxonomy of such simplification for semi-endogenous growth models

Figure 11 – Optimal patent term for $g_0 = 0$, $\rho = 0.06$ and $L/\beta = 20$

Figure 12 – Optimal patent term for $g_0 = 0.50$, $\rho = 0.08$ and $L/\beta = 12$

in France, the maximal term for an utility model is 6 years. In other countries such as Germany, Denmark, China, Italy and Ireland, it is 10 years. For figure 11, the optimal terms are a little lower, but not too far apart from such values.

5 Discussion

In this section we discuss thoroughly some points outlined in the text body

5.1 General discussion

Besides being intuitive, the physical constraints in our setting - specially $\dot{A} \geq 0$ play a role of guaranteeing 'fairness'. This is the case since if it were allowed for the stock of $A(t)$ to decrease, agents would be receiving money, which would be spent in consumption, from the reversion of already conceived ideas, which, after all, can be deemed as a cheat to the savings scheme in our framework.

Barro e Martin (1995) acknowledge this fact, stressing that for $\gamma < 0$ the solution to the innovator's problem would be to spend - if possible - a negative amount in research. Since they rule out this possibility, they mention the emergence of the corner solution of Proposition 3.1, where nothing is spent in R&D activities, without providing a formal proof to this fact. In Cysne e Turchick (2012), no restriction for the values of parameters other than m is mentioned in the moment of resolution of the dynamic system. Notwithstanding that, they impose, when solving for the optimal patent duration parameter, that m should be restricted to the interval $[0, \pi/\beta - \rho]$ in order to guarantee a non-negative growth rate. Such *modus operandi* is not very fruitful for their scope, however. Firstly, it may be the case that $\pi/\beta < \rho$, and then the aforementioned 'interval' makes no mathematical sense. This corresponds, in our framework, to the case where $m = \infty$ is vacuously optimal. Secondly, for m close enough to the supremum of this interval, the economy falls in a type-3 equilibria, and the calculated utility levels refer to a non-admissible trajectory.

The reader might wonder why we spent so much energy in the case where $\pi/\beta < \rho$, in opposition to the status quo of the literature. We believe that ruling out such cases brings a loss of economic intuition to horizontal innovation models. Allowing for certain tuples of preference and production parameters to fill this condition allows the analyst to examine which economic variables underlie the choice to either engage in R&D activity, or to live by the rents of the already conceived innovations, inherited at time 0. The framework presented in the present work is suited to studying formally such questions, which is attained through the introduction of the physical constraints directly to the consumers' optimization problem.¹

Moreover, even if $\pi/\beta > \rho$, considering 'uninteresting' type-1 equilibria (in the sense of not encompassing endogenous growth) is more than not innocuous, as shown extensively in Section 4, given that it may be optimal for the policymaker to induce a equilibrium with no

¹ In Barro e Martin (1995) this is justifiable insofar they are concerned with the characterization of steady-state balanced growth models, abstracting away from policy action

research. If the physical constraints were not introduced, a formal analysis of type-1 equilibria would not be possible, and we would have to content ourselves with schemes such as the restriction of the values of m introduced in Cysne e Turchick (2012), which rule out $m = \infty$. In this respect, notice that, even if the two issues appointed above concerning such restriction of admissible values of m were not present, it might as well be the case that $m^* = \infty$, and the type of trajectories induced by such choice dominate all choices of m in $[0, \pi/\beta - \rho]$. The analysis of R&D policy, in this case, would remain impoverished. In this sense, the introduction of the physical constraints, rather than just ruling out implausible equilibria, also permits the analysis to be extended to any plausible parameter values.

Moreover, the models *sub examine* fail to account for the possibility that, for $\gamma > 0$, the presented trajectories would present the same kind of issues of the case $\gamma < 0$, as shown by Proposition 3.4. We repute that such failure to account for this possibility, although stemming ultimately from the option of not introducing explicitly the physical constraints, is induced by the fact that only steady state trajectories are considered in those frameworks. In fact, as we derived the whole equilibrium trajectories, we were able to perceive the influence of the values of g_0 in the type of equilibrium arising in the economy, given our explicit consideration of physical constraints.

This point might require a little further explanation. As we can see from Proposition 3.4, asymptotically, for the subspace of the parametric space under examination, trajectories encompassing R&D do not violate physical constraints. Thus, in the one hand, the mentioned works are correct in not differentiating the asymptotic trajectories in Propositions 3.3 and 3.4, placing both under the same $\gamma > 0$ tag. In the long run, the physical restrictions are not violated once it is assumed that the economy is initially in a steady state, and, as a result, the value of $B(t)/A(t)$ is the one given by the steady-state conditions. Hence, the value of this ratio, $m/(m + \gamma) > \bar{g}_0$, since $m + \gamma + \lambda < 0$, assumes the role of g_0 in their frameworks, and in this parametric region, we are in a type-2 equilibrium .

5.2 Comparative Statics

Now we discuss how the values assumed by the fundamental parameters of the model, given $\theta = 1$, influence the optimal choice of the patent duration m , in a comparative statics exercise. In line with the discussion carried out in Section 4.2, there are two channels through which this influence can be exerted, i.e., through the feasibility constraint in condition (4.1) and through condition (4.9). Before proceeding, note that specific values of parameters A_0 and B_0 do not matter for dominance of one type of equilibrium over another or feasibility, but rather their ratio g_0 ; the same can be said of L and β . Thus, our analysis will encompass the influence of $(g_0, L/\beta, \rho, \alpha)$ over the satisfaction of conditions (4.9) and (4.1).

In what concerns the feasibility constraint of type-2 equilibria, note that g_0 exerts no

influence upon it. Clearly, the constraint is more easily satisfied for bigger L/β and lower ρ . Such a result is intuitive insofar more labor input and lower invention costs enlarges profits, which magnifies the returns from investment. On the other hand, with lower discount factor ρ , future consumption flows are valued more heavily, and interpreting blueprints as capital goods transferring utility from the present to the future, their attractiveness increases.

To the contrary, we observe a non-monotone dependence between α and π/β , depicted in Figure 13, bringing about a dubious effect of the value of this parameter to the satisfaction of condition (4.1), which reads $\pi/\beta > \rho$. To understand this, firstly note that π , as given by equation (2.7), is a product between unitary profit $1/\alpha - 1$ - which happens to be mark-up - and the demanded quantity given by the right-hand side of equation (2.6). The non-monotonicity stems from the facts that the mark-up is increasing in α , which we shall coin the mark-up effect, while the quantity is decreasing in this parameter. To understand this last phenomenon, note that as we rise α , the capital share of production increases, and final good firms demand more capital, what increases the quantity demanded to monopolized intermediate goods firms, which we shall coin the capital share effect. Algebraically, if we differentiate the expression in the right-hand side of equation (2.6) with respect to α :

$$L\alpha^{\frac{2}{1-\alpha}} \frac{2}{1-\alpha} \left(\frac{1}{\alpha} + \frac{\ln \alpha}{1-\alpha} \right) > 0$$

The parameter α enters both the price and quantity terms of the unitary profit since, in our model, it is related both to the mark-up enjoyed by intermediate good firms, as well as to the capital share. Had we used a specification in which the mark-up and elasticity of substitution parameters do not coincide, we would expect a monotonous relation between profits and the values of such parameters.

From the figure, we see that for lower values of α , the capital share effect dominates, and thus profits are increasing in α . Nevertheless, from $\alpha = 0,68$ onwards, the markup effect dominates, and then profits are decreasing in α . Thus, given $(g_0, L/\beta, \rho)$, if for $\alpha < 0,68$ a type-2 equilibrium is feasible, then it will be feasible for α^* such that $\alpha \leq \alpha^* \leq 0,68$. Conversely, if it is not feasible for such α , it will be not be feasible for α^* such that $\alpha^* \leq \alpha$. Analogously, if, given $(g_0, L/\beta, \rho)$, a type-2 equilibrium is not feasible for $\alpha > 0,68$, then it will not be feasible for α^* such that $\alpha^* \geq \alpha$.

In what regards the comparison between utility levels V_1 and V_2 and satisfaction of condition (4.9), only the parameter g_0 has an indubitable effect over it. Note that g_0 appears only in the right-side of this inequality, multiplied by a positive constant - which happens to be, in function of α , the ratio $\kappa_2/\kappa_1 > 0$ - and in the denominator of the argument of a strictly increasing function and therefore the right-hand side of (4.9) is decreasing in g_0 . Thus, as we increase g_0 , the attractiveness of type-1 equilibria in terms of maximal levels of utility attained decreases. This phenomenon stems from the fact, that, if the policymaker sets $m = \infty$ at $t = 0$,

Figure 13 – Values of $(1 - \alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$

$g(0_+) = 1$, no matter the value of g_0 . Thus, the value of this parameter is immaterial to the level of utility attained when m is set at its maximal possible value, what can be seen in equation (4.2), which does not contain any g_0 term. To the contrary, the value of g_0 has a direct relation with the level of utility attained in a type-2 equilibrium when $m = 0$, as can be checked in equation (4.7). The influence of g_0 over the level of utility attained in a type-2 equilibrium shall be understood from the following mechanism: the rise of the stock of copies lowers the average price of intermediate goods, raising marginal productivity of labor and increasing wages.

In order to analyse the effect of L/β and ρ over the satisfaction of condition (4.9), it is convenient to define a function $\varpi : (0, 1) \times [0, 1] \times (0, +\infty)^2 \rightarrow (-\infty, +\infty)$ whose law is given by

$$\varpi(\alpha, g_0, L/\beta, \rho) = \ln \left(\frac{\frac{L}{\beta}(1 - \alpha)(\alpha^{\frac{\alpha}{1-\alpha}})}{(\frac{L}{\beta}(1 - \alpha)\alpha^{\frac{2\alpha}{1-\alpha}} + \rho)(1 + \frac{-1 - \alpha + \alpha^{\frac{\alpha}{\alpha-1}}}{1 + \alpha}g_0)} \right) - \frac{L}{\beta\rho}(1 - \alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} - 1 \quad (5.1)$$

The function $\varpi(\alpha, g_0, L/\beta, \rho)$ is numerically equal to $V_1 - V_2$ as given by equations (4.2) and (4.7). Hence, if $\varpi(\alpha, g_0, L/\beta, \rho) > 0$, an optimal type-1 equilibrium (with $m = +\infty$, from Proposition 4.1) is preferred; otherwise, an optimal type-2 equilibrium (with $m = 0$, from Proposition 4.2) is preferred.

Our analysis for ρ is summarized in Proposition 5.1. Firstly define:

$$\Omega_1 = \left((\alpha, g_0, L/\beta) \in (0, 1) \times [0, 1] \times (0, +\infty) : (1 + \alpha)\alpha^{\frac{\alpha}{1-\alpha}} \left(1 + \frac{-1 - \alpha + \alpha^{\frac{\alpha}{\alpha-1}}}{1 + \alpha}g_0 \right) < 1 \right)$$

Proposition 5.1: Given $(\alpha, g_0, L/\beta) \in \Omega_1$, there exists a unique $\rho^* \in (0, \pi/\beta)$ such that $\varpi(\alpha, g_0, L/\beta, \rho^*) = 0$. Furthermore, for any $\rho > \rho^*$, it is optimal to pick $m = \infty$; otherwise, the policymaker should pick $m = 0$.

Proof. We begin with a lemma:

Lemma 5.1. Given $(\alpha, g_0, L/\beta) \in \Omega_1$, $\partial\varpi(\alpha, g_0, L/\beta, \rho)/\partial\rho > 0$ if $\rho \in (0, \pi/\beta)$

Proof. Let $(\alpha, g_0, L/\beta) \in \Omega_1$. Computing $\partial\varpi(\alpha, g_0, L/\beta, \rho)/\partial\rho$:

$$\frac{\partial\varpi}{\partial\rho} = \frac{1}{\rho^2} \frac{L}{\beta} (1 - \alpha) \alpha^{\frac{1+\alpha}{1-\alpha}} - \frac{1}{\frac{L}{\beta} (1 - \alpha) \alpha^{\frac{2\alpha}{1-\alpha}} + \rho}$$

Therefore, if $\partial\varpi/\partial\rho \leq 0$ then:

$$\rho^2 - \frac{L}{\beta} (1 - \alpha) \alpha^{\frac{1+\alpha}{1-\alpha}} \rho - \left(\frac{L}{\beta} (1 - \alpha) \alpha^{\frac{1+\alpha}{1-\alpha}} \right)^2 \alpha^{-1} \geq 0 \quad (5.2)$$

Note that this is a second-degree inequation in ρ ; since the independent term is negative, it possesses two real roots with distinct signs. The positive root $\bar{\rho}$ is given by:

$$\bar{\rho} = \left(\frac{L}{\beta} (1 - \alpha) \alpha^{\frac{1+\alpha}{1-\alpha}} \right) \left(\frac{1 + \sqrt{1 + 4\alpha^{-1}}}{2} \right)$$

Thus:

$$\rho \geq \bar{\rho} = \frac{\pi}{\beta} \frac{1 + \sqrt{1 + 4\alpha^{-1}}}{2} > \frac{\pi}{\beta} \frac{1 + \sqrt{5}}{2} > \frac{\pi}{\beta}$$

□

Now we show existence and uniqueness of ρ^* . Let $(\alpha, g_0, L/\beta) \in \Omega_1$. Since $\partial\varpi/\partial\rho > 0$ for $\rho \in (0, \pi/\beta)$, by the above discussion, there exists at most one value of $\rho \in (0, \pi/\beta)$ such that $\varpi(\alpha, g_0, L/\beta, \rho) = 0$. As ρ approaches 0 from the right, the left-hand side of (4.9) gets arbitrarily large, and the right-hand side is a real number, so that $\lim_{\rho \rightarrow 0+} \varpi(\alpha, g_0, L/\beta, \rho) < 0$. In opposition, by using (5.1) we can see that $\varpi(\alpha, g_0, L/\beta, \pi/\beta) > 0$. Thus, by the Intermediate Value Theorem, there exists $\rho^* \in (0, \pi/\beta)$ such that $\varpi(\alpha, g_0, L/\beta, \rho^*) = 0$.

Finally, if $\rho > \rho^*$, it may be the case that $\rho \geq \pi/\beta$, and only type-1 equilibria are implementable, and $m^* = \infty$. It may also be the case that $\rho < \pi/\beta$, and $\varpi(\alpha, g_0, L/\beta, \rho) > 0$, from the above discussion, so that $m^* = \infty$. If $\rho < \rho^*$, $\varpi(\alpha, g_0, L/\beta, \rho) < 0$, so that $m^* = 0$

□

For L/β , we provide an analogous result. Firstly define:

$$\Omega_2 = \left((\alpha, g_0, \rho) \in (0, 1) \times [0, 1] \times (0, +\infty) : (1 + \alpha) \alpha^{\frac{1}{1-\alpha}} \left(1 + \frac{-1 - \alpha + \alpha^{\frac{\alpha}{p\beta a - 1}}}{1 + \alpha} g_0 \right) < 1 \right)$$

Proposition 5.2: Given $(\alpha, g_0, \rho) \in \Omega_2$, there exists a unique $(L/\beta)^* \in (\rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1), +\infty)$ such that $\varpi(\alpha, g_0, (L/\beta)^*, \rho) = 0$. Moreover, for any $L/\beta < (L/\beta)^*$, it is optimal to pick $m = \infty$; otherwise, the policymaker should pick $m = 0$.

Proof. We begin by proving a Lemma:

Lemma 5.2. Given $(\alpha, g_0, \rho) \in \Omega_2$, $\partial\varpi(\alpha, g_0, L/\beta, \rho)/\partial(L/\beta) < 0$ if $L/\beta \in (\rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1), +\infty)$

Proof. Let $(\alpha, g_0, \rho) \in \Omega_2$. Computing $\partial\varpi(\alpha, g_0, L/\beta, \rho)/\partial(L/\beta)$:

$$\frac{\partial\varpi}{\partial\frac{L}{\beta}} = \frac{\beta}{L} - \frac{(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\rho} - \frac{(1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}}}{\frac{L}{\beta}(1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} + \rho}$$

After some algebraic manipulation, and creating the auxiliary variable $z = L/\beta\rho$, $\partial\varpi/\partial(L/\beta) < 0$ if:

$$z^2 + z\frac{\alpha^{\frac{2\alpha}{\alpha-1}}}{1-\alpha} - \frac{\alpha^{\frac{3\alpha+1}{\alpha-1}}}{(1-\alpha)^2} > 0$$

Notice then that this second-degree inequality has two real roots with different signs, since the independent term is negative. Its positive root is:

$$\bar{z} := \frac{\alpha^{\frac{2\alpha}{\alpha-1}}}{1-\alpha} \left(\frac{\sqrt{1+4\alpha^{-1}} - 1}{2} \right)$$

Then, for $L/\beta \in (\rho\bar{z}, \infty)$, $\partial\varpi(\alpha, g_0, L/\beta, \rho)/\partial(L/\beta) < 0$. It is the case that $\rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1) > \rho\bar{z}$. To see this:

$$\frac{\alpha^{\frac{1+\alpha}{\alpha-1}}}{1-\alpha} > \bar{z} = \frac{\alpha^{\frac{2\alpha}{\alpha-1}}}{1-\alpha} \left(\frac{\sqrt{1+4\alpha^{-1}} - 1}{2} \right) \iff 2\alpha^{-1} + 1 > \sqrt{1+4\alpha^{-1}} \iff 4\alpha^{-2} > 0$$

□

Now we show existence and uniqueness of $(L/\beta)^*$. Let $(\alpha, g_0, \rho) \in \Omega_2$. Since $\partial\varpi/\partial(L/\beta) > 0$ for $(L/\beta) \in (\rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1), +\infty)$, by the Lemma, there exists at most one value of $(L/\beta) \in (\rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1), +\infty)$ such that $\varpi(\alpha, g_0, L/\beta, \rho) = 0$. As (L/β) approaches $+\infty$, the left-hand side of (4.9) gets arbitrarily large, and the right-hand side is a real number, by the L'Hopital Rule, so that $\lim_{(L/\beta) \rightarrow +\infty} \varpi(\alpha, g_0, L/\beta, \rho) > 0$. In opposition, by (5.1), $\varpi(\alpha, g_0, \rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1), \rho) < 0$. Thus, by the Intermediate Value Theorem, there exists $(L/\beta)^* \in (\rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1), +\infty)$ such that $\varpi(\alpha, g_0, (L/\beta)^*, \rho) = 0$.

Finally, if $L/\beta < (L/\beta)^*$, it may be the case that $L/\beta \leq \rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1)$, type-2 equilibria are not implementable and $m^* = \infty$. If $L/\beta > \rho\alpha^{\frac{1+\alpha}{\alpha-1}}/(\alpha+1)$, then $\varpi(\alpha, g_0, L/\beta, \rho) > 0$, from the above discussion and $m^* = \infty$. If $L/\beta > (L/\beta)^*$, then $\varpi(\alpha, g_0, L/\beta, \rho) < 0$ and $m^* = 0$.

□

In our model, as aforementioned, the parameter β may be interpreted as the pecuniary cost attached to putting up an invention, which might enclose, for example, education, research, equipment and facilities costs. The parameter also encompasses the administrative costs linked to making the IPR related to the intermediate good binding, including the costs of hiring a lawyer, administrative and judiciary costs in general and the opportunity costs of waiting until a decision is taken. In this respect, although we do not model β as choice variable, inasmuch a State agent is not able to alter its value to any desired level with a flick of his wrist, actions from the government, such as putting up a sound and (almost) free of contradictions legal framework, as well as effective courts and agencies are apt to lower the value of β . We show next that if β is taken to a sufficiently low value, holding (α, g_0, L, ρ) fixed, it is optimal to set $m = 0$.

Proposition 5.3: Given (α, g_0, L, ρ) and $\theta = 1$, there exists $\beta^* > 0$ such that, for every $\beta \leq \beta^*$, it is optimal to implement $m = 0$.

Proof. We just have to show that for sufficiently low values of β , the condition in equation (4.1) holds and that condition (4.9) does not hold. For the first condition to be valid, it is enough that

$$\beta < \beta_1 := \frac{L}{\rho}(1 - \alpha)(\alpha^{\frac{1+\alpha}{1-\alpha}})$$

In what concerns condition (4.9) not holding, we can see that the left-hand expression assumes arbitrarily high values as β approaches zero. On the other hand, if β is taken to zero, the right-hand side of (4.9) shows a 0/0 indeterminacy. Applying the L'Hopital Rule, and since the logarithmic function is continuous, we see that this limit is

$$\ln\left(\frac{(\alpha^{\frac{-\alpha}{1-\alpha}})}{(1 + \frac{-1-\alpha+\alpha^{\frac{\alpha}{\alpha-1}}}{1+\alpha}g_0)}\right)$$

which is clearly finite. Hence, by continuity, there exists $\beta_2 > 0$ such that for $\beta < \beta_2$, equation (4.9) is not satisfied. Now we just set $\beta^* = \min(\beta_1/2, \beta_2)$. □

6 Conclusion

We deployed the lab-equipment specification of the R&D based growth model of Rivera-Batiz e Romer (1991) in order to calculate optimal patent lengths. Due to the duly consideration of physical constraints assuring expanding product-variety, our results depart from those obtained in the literature thus far.

In particular, for logarithmic utility, we found out, in sharp contrast with Cysne e Turchick (2012) and Kwan e Lai (2003), that the optimal patent length is either null or infinite. Those results, although not corresponding in general to patent terms observed in jurisdictions around the globe, may be informative of how the values of underlying economic parameters may exert influence over the desirability of R& D investments. For instance, as proposition 5.1 shows, for low enough costs attached to the R&D activity, which can be lowered through an adequate governmental action, it will be optimal to provide infinite patent duration.

As we rise the value of the coefficient of relative risk aversion, we are able to find an interior solution for optimal patent duration problem which, for relevant parametric values, resemble utility models terms in OECD countries.

A possible extension could involve a parametric specification where the elasticity of substitution and the capital share do not coincide, in order to dissociate both effects over the optimal patent length. Another line of research may involve utilizing a two-dimensional policy space, in which not only the patent duration, but also its scope may be picked by a benevolent policymaker, as initiated by Ribeiro e Turchick (2014)

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A Equivalence between partitions of Φ

For convenience, as in the text, denote:

$$z(\phi) = - \lim_{m' \rightarrow m} \frac{m'(m' + \lambda(\phi))(1 - g_0(\phi))}{m' - (m' + \lambda(\phi))g_0(\phi)}$$

We shall bother only proving the equivalence for Φ_2 , since $\{\Phi_1, \Phi_2, \Phi_3\}$ is a partition of Φ and the condition for Φ_1 is not altered. We remind here, for the convenience of the reader, the three relevant properties of z :

1. $z(\omega, 0) = 0, \forall \omega \in \Omega$.
2. $z(\phi) \geq 0, \forall \phi \in \Phi$.
3. Given $\phi \in \Phi$, if $m > 0$, $\partial z(\phi)/\partial g_0(\phi) < 0$.

Let $\phi \in \Phi_2$ in the main formulation. Since $z(\phi)/\partial g_0(\phi) < 0$, given $\phi' \in \Phi$ the maximal possible value $z(\phi')$ happens when $g_0(\phi') = 0$, so that $z(\phi') \leq -m - \lambda(\phi')$. Thus, if $\gamma(\phi) > -m - \lambda(\phi)$, $\gamma(\phi) > z(\phi)$, from transitivity. If $0 < \gamma(\phi) < -m - \lambda(\phi)$ and $g_0(\phi) > \bar{g}_0(\phi)$, we have:

$$\begin{aligned} g_0(\phi) > \bar{g}_0(\phi) &\implies -m(m + \lambda(\phi))(1 - g_0(\phi)) < \\ &\gamma(\phi)(m - (m + \lambda(\phi))g_0(\phi)) \implies \gamma(\phi) > z(\phi) \end{aligned}$$

Let $\phi \notin \Phi_2$ in the main formulation. Then, $\gamma(\phi) \leq 0$, or $0 < \gamma(\phi) < -m - \lambda(\phi)$ and $g_0(\phi) \leq g_1^*(\phi)$. Since $0 \leq z(\phi)$, in the former case, it is clear that $\phi \notin \Phi_2$ in the alternative formulation. In the latter case, $\gamma(\phi) \leq z(\phi)$, and $\phi \in \Phi_3$, so that in neither case $\phi \in \Phi_2$ in the alternative formulation.

B Complementary slackness for Proposition 3.4

In this appendix we prove the complementary slackness condition in Proposition 3.4 for $t < t^*$, i.e., that, for all $t \in [0, t^*)$, given $0 < \gamma < -m - \lambda$ and $g_0 < g_0^*$, it holds that

$$\frac{e^{-\rho t}}{\beta^{\theta-1}(A_0(-\kappa_2 - \kappa_1) + \kappa_1(A_0 - B_0)e^{-mt})^\theta} - \frac{\pi}{\beta^\theta} e^{mt} \int_t^{t^*} \frac{e^{-(\rho+m)\tau} d\tau}{(A_0(-\kappa_1 - \kappa_2) + \kappa_1(A_0 - B_0)e^{-m\tau})^\theta} \geq 0$$

This condition can be rewritten as

$$\frac{\frac{e^{-(\rho+m)t}}{((- \kappa_2 - \kappa_1) + \kappa_1(1 - g_0)e^{-mt})^\theta}}{\int_t^{t^*} \frac{e^{-(\rho+m)\tau} d\tau}{((- \kappa_1 - \kappa_2) + \kappa_1(1 - g_0)e^{-m\tau})^\theta}} \geq \frac{\pi}{\beta}, \forall t \in [0, t^*)$$

For simplicity, we denote

$$f(t) := \frac{e^{-(\rho+m)t}}{((- \kappa_2 - \kappa_1) + \kappa_1(1 - g_0)e^{-mt})^\theta}$$

$$F(t) := \frac{f(t)}{\int_t^{t^*} f(\tau) d\tau}$$

We divide the proof in some steps. Firstly, we prove some lemmas:

Lemma B.1. $\lambda \geq \frac{m(-\kappa_1 - \kappa_2)}{\kappa_2}$

Proof. Remind that λ is the negative root of $g(x) = x^2 + (m + \kappa_2)x + m(\kappa_1 + \kappa_2) = 0$. This equation has two real roots with different signs, since the independent term is negative. Since $-m(\kappa_2 + \kappa_1)/\kappa_2 < 0$, the proposition holds if and only if $g(-m(\kappa_2 + \kappa_1)/\kappa_2) \geq 0$, which is the case since:

$$g\left(\frac{m(-\kappa_1 - \kappa_2)}{\kappa_2}\right) = \frac{m^2 \kappa_1 (\kappa_2 + \kappa_1)}{\kappa_2^2} \geq 0$$

□

Lemma B.2. There is at most one critical point for $F(t)$ in $[0, t^*)$, which must necessarily be a local minimum

Proof. Taking the second time derivative of $F(t)$, we see that

$$F''(t) = \frac{f''(t)(\int_t^{t^*} f(\tau)d\tau)^2 + 3f'(t)f(t)\int_t^{t^*} f(\tau)d\tau + 2(f(t))^3}{(\int_t^{t^*} f(\tau)d\tau)^3}$$

We could cut off a term $\int_t^{t^*} f(\tau)d\tau$ since in the relevant range it is always strictly positive. In a critical point $t_c \in [0, t^*)$, it is the case that

$$f'(t_c) \int_{t_c}^{t^*} f(\tau)d\tau = -(f(t_c))^2 \quad (\text{B.1})$$

By plugging the last expression in the expression for $F''(t)$, in the numerator term $2(f(t))^3$, we conclude that $F''(t_c)$ has the same sign as

$$f''(t_c) \left(\int_{t_c}^{t^*} f(\tau)d\tau \right)^2 - (f(t_c))^3 = \frac{(f(t_c))^3}{(f'(t_c))^2} (f''(t_c)f(t_c) - (f'(t_c))^2)$$

where the last equality was obtained by isolating the integral in Equation (B.1). Since $f(t) > 0, \forall t$, the last expression has the same sign as $f''(t_c)f(t_c) - (f'(t_c))^2$, which, by its turn, has the same sign of the first-derivative of $f'(t_c)/f(t_c)$. Now, see that

$$f'(t) = \frac{-(\rho + m)e^{-(\rho+m)t}(-\kappa_2 - \kappa_1 + \kappa_1(1 - g_0)e^{-mt})^\theta + \theta m \kappa_1(1 - g_0)e^{-mt}e^{-(\rho+m)t}}{(-\kappa_2 - \kappa_1 + \kappa_1(1 - g_0)e^{-mt})^{\theta+1}}$$

So that:

$$\frac{f'(t_c)}{f(t_c)} = -(\rho + m) + \frac{\theta m \kappa_1(1 - g_0)e^{-mt_c}}{-\kappa_2 - \kappa_1 + \kappa_1(1 - g_0)e^{-mt_c}}$$

We have:

$$\left(\frac{f'(t_c)}{f(t_c)} \right)' = \frac{m^2 \theta (1 - g_0) e^{-mt_c} \kappa_1 (\kappa_2 + \kappa_1)}{(-\kappa_2 - \kappa_1 + \kappa_1(1 - g_0)e^{-mt_c})^2} > 0$$

The above expression cannot be zero since $g_0 < 1$ in a type- II equilibrium. Thus, we see that the second derivative of $F(t)$ is positive in a critical point. Hence, every critical point is a local minimum. On the other hand, since $F(t)$ is C-2 class, and there are no inflection points, there cannot exist two consecutive local minima or local maxima, by the Intermediate Value Theorem. Since there are no maxima, there exists at most one local minimum, and since every critical point is a local minimum, there exists at most one critical point, which is a local minimum. $\square$

Lemma B.3. If $0 < \gamma < -m - \lambda$ and $g_0 < g_0^*$, then $\gamma(-\kappa_2 - \kappa_1 g_0) \leq -m \kappa_1(1 - g_0)$

Proof. Notice that the right-hand side of the above expression is decreasing in g_0 and the left-hand side increasing. Thus, the above expression holds for every g_0 in the relevant range, if and only if it holds in its supremum, g_0^* . Since $g_0^* = m(m + \gamma + \lambda)/((m + \gamma)(m + \lambda))$, we have, $\forall g_0 < g_0^*$:

$$\begin{aligned} \gamma(-\kappa_2 - \kappa_1 g_0) \leq -m\kappa_1(1 - g_0) &\iff \gamma\left(-\kappa_2 - \frac{\kappa_1 m(m + \gamma + \lambda)}{(m + \gamma)(m + \lambda)}\right) \leq -\frac{m\kappa_1 \lambda \gamma}{(m + \gamma)(m + \lambda)} \\ &\iff (\kappa_1 m - \kappa_2 m - \kappa_2 \lambda)(m + \gamma) \geq 0 \end{aligned}$$

Since $(m + \gamma) \geq 0$ the above fact holds if $\lambda \geq \frac{m(-\kappa_1 - \kappa_2)}{\kappa_2}$, which is proven to be true in Lemma B.1. $\square$

Now, with the lemmas proven, we must divide the problem in two cases, depending on whether there is a critical point for $F(t)$ in $[0, t^*)$. If there is no critical point, since the function must be increasing when close enough to t^* , it will be increasing in all the relevant range, and the lowest value of $F(t)$ is attained in $t = 0$. Thus, it will be enough to prove the proposition for $t = 0$. Otherwise, since the only critical point, if existent, is a local minimum, it will also be the global minimum.

Case i) There is no critical point

In this case, it is enough to prove that $F(0) \geq \pi/\beta$. We have that $F'(0) \geq 0$, which is equivalent to (since $f'(t) < 0$):

$$\int_0^{t^*} f(\tau) d\tau \leq -\frac{f^2(0)}{f'(0)}$$

Thus, it is enough to prove that

$$-\frac{f^2(0)}{f'(0)} \leq \frac{f(0)}{\frac{\pi}{\beta}} \quad (\text{B.2})$$

From the definition of $f(t)$ and the expression of its derivative, we have:

$$\begin{aligned} f(0) &= \frac{1}{(-\kappa_2 - \kappa_1 g_0)^\theta} \\ (f(0))^2 &= \frac{1}{(-\kappa_2 - \kappa_1 g_0)^{2\theta}} \\ f'(0) &= \frac{-(\rho + m)(-\kappa_2 - \kappa_1 g_0) + \theta m \kappa_1 (1 - g_0)}{(-\kappa_2 - \kappa_1 g_0)^{\theta+1}} \end{aligned}$$

Then, by substituting the above expression in Equation (B.2), one can conclude that this expression is equivalent to $\gamma(-\kappa_2 - \kappa_1 g_0) \leq -m\kappa_1(1 - g_0)$, which was proven to be true in Lemma B.3.

Case ii) There is one critical point $t_c \in [0, t^*)$

In this case, in order to prove that the proposition holds for all t in $[0, t^*)$, it suffices to show that it holds in t_c , since it is a global minimum for $F(t)$ in the relevant range. Since $dF(t)/dt$ evaluated at t_c is zero, from Equation (B.1), $F(t_c)$ can be rewritten as

$$-\frac{f'(t_c)}{f(t_c)} = \rho + m - \frac{\theta m \kappa_1 (1 - g_0) e^{-m t_c}}{-\kappa_2 - \kappa_1 + \kappa_1 (1 - g_0) e^{-m t_c}}$$

Thus, $F(t_c) \geq \pi/\beta$ if

$$\gamma \leq -\frac{m \kappa_1 (1 - g_0) e^{-m t_c}}{-\kappa_2 - \kappa_1 + \kappa_1 (1 - g_0) e^{-m t_c}}$$

Since the right-hand side is decreasing in t_c , for the above expression to hold for any t_c in the relevant range, it is enough that it holds for $t_c = t^*$, an upper bound for t_c . Substituting $t_c = t^*$ in the above expression and manipulating, it is enough to prove that (remember $\gamma > 0$):

$$\begin{aligned} \gamma \geq \frac{-m \kappa_1 \lambda \gamma}{(-\kappa_2 - \kappa_1)(m + \gamma)(m + \lambda) + \kappa_1 \gamma \lambda} &\iff (-\kappa_2 - \kappa_1)(m + \gamma)(m + \lambda) + \kappa_1 \lambda \gamma \geq -m \kappa_1 \gamma \\ &\iff (\kappa_1 m - \kappa_2 m - \kappa_2 \lambda)(m + \gamma) \geq 0 \end{aligned}$$

Since $(m + \gamma) \geq 0$ the above fact holds if $\lambda \geq \frac{m(-\kappa_1 - \kappa_2)}{\kappa_2}$, which is proven to be true in Lemma B.1.