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Optimal Legislative Funding

Financiamento Ótimo do Legislativo

Beatriz Silva Garcia

Orientador: Prof. Dr. David Turchick

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Prof. Dr. Marco Antonio Zago
Reitor da Universidade de São Paulo

Prof. Dr. Adalberto Américo Fischmann
Diretor da Faculdade de Economia, Administração e Contabilidade

Prof. Dr. Hélio Nogueira da Cruz
Chefe do Departamento de Economia

Prof. Dr. Márcio Issao Nakane
Coordenador do Programa de Pós-Graduação em Economia

BEATRIZ SILVA GARCIA

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Resumo

Propõe-se um modelo teórico para estudar a formação de governo por um corpo legislativo, composto por partidos eleitos em representação proporcional. Uma vez que a configuração do Legislativo é definida, os partidos eleitos devem formar um governo, escolher uma política de governo e uma distribuição de recursos e benefícios legislativos entre os partidos presentes na casa através de um jogo de barganha. Uma massa de eleitores é assumida capaz de votar estrategicamente. Nosso objetivo é estudar como uma limitação nos recursos disponíveis entre os legisladores pode afetar o comportamento dos agentes envolvidos no sistema, tanto eleitores quanto legisladores. Mostra-se que uma queda da distância relativa entre as ideologias aumenta o bem-estar social e diminui a quantia necessária de recursos que mantem o acordo legislativo ótimo. Ademais, há um limite superior para essa quantia quando a distância ideológica aumenta.

Palavras-chaves: formação de governo, barganha legislativa, financiamento do legislativo, rent-seeking, votação estratégica.

Abstract

We propose a model to study government formation by a legislative body composed by parties elected with a proportional representation rule. Once the legislative configuration is determined, the elected parties must form a government, choose a government policy and a distribution of legislative resources and benefits among the elected parties through a bargaining game. A mass of voters is assumed capable of voting strategically. Our goal is to study how a limitation on the available resources among legislators may affect the behavior of the agents involved in this system, both voters and legislators. We show that a decrease in the relative distance between ideologies increases the social welfare and decreases the necessary amount of resources to keep the optimal agreement. Moreover, there is an upper limit to this amount when the ideological distance increases.

Keywords: government formation, legislative bargaining, legislative funding, rent-seeking, strategic voting.

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1 Introduction

Life in society requires articulation among its members to define rules and policies. In this organization, the government can be seen as the highest body in a decision hierarchy. The study of government formation is then essential to understand how policies are defined. In a dictatorial system, where an individual or just an ideologically homogeneous group is the top of the decision chain, an explicit articulation is not necessary, since everybody is subject to the choices of the homogeneous group in power.

In such a system, ideologies of individuals are completely ignored for policy determination, therefore, unless the dictator is a representative agent of society or a benevolent central planner, social welfare will not be at its optimum.

In a democratic system, however, the definition of an election rule in which members of society choose the participants in the highest decision body is required. When this body is composed of a legislative body, the lowest heterogeneity in preferences of the participants makes coalition formation a necessary and fundamental factor of system stability. In the parliamentary, the government is defined by the coalition that has the support of the majority of parties in the legislature.

The coalition formation process is carried out through a bargaining game in which the parties negotiate and make agreements on the conditions necessary for a majority group to support a common government policy.

The parties and voters' behavior in a representative system is the subject of a wide range of studies. However, the impact of restricting the resources that parties use for political bargaining is not directly considered in those studies. This work proposes a theoretical investigation on the impact of this restriction, since the resources' magnitude is an important element of the coalition formation that may affect the behavior of both agents in the electoral process, voters and parties.

Different studies analyze the formation of coalitions and factors possibly relevant to the decision making of the parties. For a comprehensive review of this literature see

Laver (1998), which summarizes differences in approach and methodology.

Laver (1998) considers basically two hypotheses about the motivations of parties: the office-seeking and the policy-seeking assumptions. In the first case, parties base their decisions on the desire to get into office and derive some benefit from taking part in the government. In the second case, the parties are concerned with the government policies and their main interest is getting into office in order to influence these policies. It is common to consider a mix of these two motivations for a better representation of reality.

At the bargaining process, each party has the chance of making a proposal to other parties by suggesting a government policy and a portfolio distribution. If it has the support of the majority of the house, this coalition succeeds and the government is formed. This portfolio is a fixed amount of resources and benefits that can be distributed among parties in the legislature. We assume parties are interested both in supporting a government policy ideologically close to its ideal policy and also in receiving as many benefits as possible.

Bandyopadhyay & Chatterjee (2006) survey several specifications for this bargaining process, and Martin & Stevenson (2001) test several assumptions usually used in theoretical approaches. Some of their main results are that in fact both policy seeking and office seeking have a significant role for government formation indeed and that the government coalition usually includes the largest legislative party

However, it is important to stress that voters have a great role in democratic parliamentary systems, since they are responsible for determining the composition of the legislature. Despite of the fact that the individuals do not take part into the government, they also have preferences on the government policy. Therefore, voters will be concerned with the government policy that is supported by the legislative body.

Assuming perfect information among agents, i.e. everyone is fully aware of the preferences of all involved, voters are able to anticipate the behavior of parties in the legislature, and they will thus vote in such a way that the political government arising from party negotiations is ideologically closer to their ideal policy. This behavior is what

we call strategic voting. When it is not possible, voters show sincere voting, in which they vote for the party that represents their ideology, with no need to consider the agreements that the parties can reach in the legislature.

This relation between voters' behavior and coalition formation in the legislature has been the concern of several empirical and theoretical studies. For instance, Debus & Müller (2013) show how voters' preferences could also impact the coalition formation and the government outcomes. Herrmann (2014) presents empirical evidence on strategical voting. He shows how election polls can influence voters' decisions by changing their expectations about coalition formation.

To the same end, Armstrong & Duch (2010) analyze the historical patterns in coalition formation as an indicative of the possibility of strategic behavior in voters' decisions. Hobolt & Karp (2010) also study the conditions that lead to voters to adopt strategic behaviors under elections on plurality and proportional systems. Duch, May & Armstrong (2010) empirically show that this behavior is common in systems with coalition government formation.

Once determined how voters and parties behave, we are able to describe the system equilibrium from the interaction of agents in each stage: electoral and legislative.

The research developed by Austen-Smith & Banks (1988) proposes a general model that includes three stages of government formation. In the first stage (pre-election), parties announce the ideology they seek to defend at the legislature. Then, voters decide for which party to vote. The shares of votes obtained by each party determines the number of seats they will occupy at the legislature. In the last stage, the elected parties participate in a bargaining game to form a government and decide on a government policy and a division of the legislative resources.

Analyzing the whole system, we can identify factors and incentives that lead to objective and verifiable behavior of agents. The available legislative resources have an important role in the coalition formation. This is one way for parties convince others

with different ideology to join them in a voting on policy. Thus, it is intuitive that the magnitude of this legislative resources has an important role on the coalitions formations. A larger amount can increase the advantage of the party that decides how it will be divided. If a larger amount we suppose that it will be able to convince a bigger set of parties, and if the resources decrease, its set of possibilities may reduce. We can thus predict changes in the behavior of parties in the coalition formation and therefore in the voting behavior in the elections process.

In view of this, we intend to study how limitations on legislative resources, which would impact directly the representatives' payoffs, thus changing the pattern of their behavior could be welfare optimizing.

Our main goal is to identify the impact of restricting the bargaining power of the parties in the legislature on government outcomes and social welfare. In fact, we assume within the legislative body structure the formation of coalitions and party connections through the exchange of benefits and resources. Hence, a decrease in the size of these benefits may act as a restrainer on parties' bargaining power.

Since the size of this portfolio can influence the outcome of the bargaining game, we shall employ a model similar to the one proposed by Austen-Smith & Banks (1988) in order to analyze how the resource constraint impacts government outcome.

In the following chapter, we will describe our model and hypotheses. Chapter 3 and 4 are devoted to the analysis of the effects on social welfare and the optimum magnitude of legislative resources. Chapter 5 concludes, while the proofs to the aforementioned propositions are detailed in the Appendix.

2 The model

We propose a model to study a government formed by a legislative body, in which the council is composed by parties elected with a closed-list proportional representation rule. Each party announces a list of candidates to take part in the legislature and the seats are allocated according to the proportion of votes obtained by each party. Once the legislative configuration is defined, they will choose the government policy and the distribution of the portfolio of legislative resources among the elected parties through a bargaining game.

In fact, we observed in the legislative policy structure the formation of coalitions and party connections made through exchange of benefits and resources between the parties in order to form up a majority group. Since the size of this portfolio can influence the outcomes of this bargaining game, the central aim of this model is to analyze how the resource constraint can impact the government policy.

An expected result would be that unlimited resources could ensure the formation of coalitions among ideologically distant parties, leading to a distant equilibrium of government policy that would be socially optimal. In return, the presence of resources that allow partisan negotiations bargaining is necessary for the legislative between consensus to preserve the governability. This trade-off arises the need of defining the optimal size of these resources available in the house in such a way that maximizes social welfare.

To this end, we adopt a simplification of the three periods model proposed by Austen-Smith & Banks (1988). Besides those mentioned above, they also propose a first period when each party chooses its policy position, so they can maximize its vote share at the elections. In our model, we consider that party ideologies are exogenously defined. Each party is interested in representing one group of voters.

Let $\Omega = \{\alpha, \beta, \gamma\}$ be a discrete space of ideologies. The ideological distribution of voters is given by the vector $\sigma = \{\sigma_\alpha, \sigma_\beta, \sigma_\gamma\}$. By a correspondence between the ideological space and a policy space $P \subset \mathbb{R}$, we can define for each group of voters in Ω their ideal

policy, $(P_\alpha, P_\beta, P_\gamma)$.

Unlike the model of Austen-Smith & Banks (1988), the parties electoral strategy and how they choose which ideological position they will defend is not our concern. We only assume that there will be a party representing each group of voters. Therefore, we define three parties $\Phi = \{L, M, R\}$ with the following ideal policy respectively:

$$P_L = \min\{P_\alpha, P_\beta, P_\gamma\}$$

$$P_M = \text{mid}\{P_\alpha, P_\beta, P_\gamma\}$$

$$P_R = \max\{P_\alpha, P_\beta, P_\gamma\}$$

These parties are elected to compose the legislative body by a closed-list proportional representation rule. Each party announces a list of candidates in the pre-electoral period and voters choose a party to vote for. Then, the seats of the legislature are allocated among the parties, according to the proportion of votes obtained by each one, and the candidates that get the seats are defined by the party's list.

Voters can show two types of behavior: sincere voting and strategic voting. The first one consists of choosing the party that represents their ideal policy, regardless of the other voters' choice. Alternatively, voters may be more concerned about the government policy arising from the legislative process. Therefore, they decide their vote based on the expected outcome of the bargaining game and not just the policy position of the party. If voters are aware that their vote can influence the government policy and are able to anticipate the equilibrium of the legislative stage, they can attempt to vote strategically rather than sincerely.

We assume that only voters with the ideal policy P_M vote strategically. Thus, voters with extreme ideologies, represented by parties R and L , will necessarily vote for the party that represents them, i.e. will vote sincerely. We also assume that the strategic group of voters behaves in a coordinated way. In other words, they can organize and define the portion of its members that will vote for each party, ensuring that the legislature configuration will lead to a government policy favorable to their group's interests.

Following the election, the participation of parties in the house is represented by the vector $\mathbf{w} = (w_L, w_M, w_R)$. This distribution of seats is exactly equal to the proportion of votes obtained, i.e. there is no requirement of a minimum number of votes for the party to be elected.

Once the configuration of seats in the house is defined, the elected parties should choose the government policy, $y \in P$, by a simple majority poll. It is also up to the legislature the decision of how the funds and benefits of the house, represented by the exogenous variable G , should be divided among the elected parties – that is, to choose a point $\mathbf{g}$ from $\Delta(G) = \{(g_L, g_M, g_R) \in \mathbb{R}^3 : g_k \geq 0, \forall k \in \Phi, \sum_{k \in \Phi} g_k = G\}$.

When a party has numerical majority in the house, it has enough power to win the poll alone, so it can define government policy according to its ideal and hold all the resources of the house. Otherwise, a bargaining game between the parties begins and there is the formation of a minimum coalition that supports an agreed policy. As we shall see in the next chapter, in equilibrium every party gets a set in the house. Since we have three parties, the bargaining game holds in up to three periods.

In the first period, $t = 1$, the largest party makes a proposal for a winning coalition $C \subseteq \Phi$ – that is, one such that $w_C > 1/2$, where $w_C = \sum_{k \in C} w_k$. Since there will be three parties in the house, unless the largest party has more than 50% of the seats, any subset of Φ of size 2 will be a winning coalition. The proposal consists of a government policy y_1 and a distribution of the resources $\mathbf{g}_1 = (g_{1L}, g_{1M}, g_{1R})$. The party invited for the coalition accepts or rejects the proposal. If it is accepted, the government is formed and the game ends, otherwise, the bargaining continues to the next period, in which the second largest party tries to form a winning coalition.

If it is not successful in this task, the smallest party gets to make the offers at $t = 3$. However, if no agreement is met in any of the periods, since the parties did not reach any consensus, no policy will be established and no benefits will be distributed among them. In this case, the utility of each party is defined as constant and equal to zero.

In the case of a tie between any parties at the elections, the order in which they will make their coalition proposals will be defined by a coin toss before the beginning of the bargaining game ($t = 0$).

As such, once party k is elected, its preferences are represented by the utility function:

$$u_k(y, g_k) = \begin{cases} g_k - (y - P_k)^2 & \text{if } k \text{ is elected and a government is formed} \\ 0 & \text{if } k \text{ is elected and no government is formed} \\ -c & \text{if } k \text{ is not elected} \end{cases} \quad (2.1)$$

where c is a constant such that $c \geq (P_R - P_L)^2$.

For the voters' preferences, we assume that their utility depends on lump sum tax, paid by every voter and is intended for the legislature resources, and the ideological distance between the government policy and their ideal policy. Thus, the utility for voter i is $s_i(y) = -G - (y - P_i)^2$. However, if the legislative body cannot achieve an agreement through this bargaining game, there is no consensus and no policy will be adopted. In this case, we define that all voters will obtain the worst possible utility: $s_i = -G - c$.

Thereby, the model proposed is based on a sequential game with two stages: elections are held for the composition of the house ($t = 0$), and then elected parties decide the government policy from a bargaining game and the government is formed ($t = 1, 2, 3$). Since we assume a game with perfect information among the agents, i.e. the action of each period can be anticipated, the game can be solved by backward induction. First, we analyze the legislative stage and find the equilibrium policy for a given distribution of parties in the house. Then, we specify voters' behavior.

3 Legislative Equilibrium

As mentioned above, once the legislative body is elected, the government policy should be chosen by the house with a majority poll and the outcomes depend on the proportion of each party in the house, $\mathbf{w} = (w_L, w_M, w_R)$. If there is a party with the numerical majority, it can support its own ideal policy and hold all the house resources. Otherwise a bargaining game begins, so that a winning coalition is formed in order to elect a government policy.

In the following analysis, we describe in detail the bargaining process in the case in which the parties have distinct shares in the legislature. We assume that, in case of a tie between the parties in the election, the order by which they make the proposals of coalitions is defined tossing a coin before the bargaining game starts. The results of these cases are also described in the analysis presented below.

When a party is deciding which proposal it will make, it is necessary to ensure that the other party invited for the coalition is willing to accept it. Thus, the proposal must give the other party at least the same utility that it would obtain if it rejects the proposal. At $t = 3$, the outside option of accepting a proposal is a situation with no agreement, in which every party obtains zero utility. However, in the other periods, the opportunity cost of accepting a proposal is the utility obtained with the agreement of the following period. This reservation utility for party k at period t , v_{tk} , can be expressed as:

$$\begin{aligned} v_{3k} &= 0 \\ v_{2k} &= u_k(y_3, \mathbf{g}_3) = g_{3k} - (y_3 - P_k)^2 \\ v_{1k} &= u_k(y_2, \mathbf{g}_2) = g_{2k} - (y_2 - P_k)^2 \end{aligned}$$

where y_t and g_{tk} are the government policy and the benefits obtained by party k through the proposal agreed upon at period t .

We assume a game with perfect information, so that parties are able to anticipate the proposals of the following periods to determine their best course of action. If a single

party possesses the absolute majority of the seats, a singleton winning coalition emerges. Otherwise, an agreement between two parties is sufficient to form a winning coalition C with $w_C > 1/2$. To decide which will be the optimum proposal, we have to compare the utility obtained in every possible coalition. If party k makes the proposal at period t , it will choose the coalition with the agreement that gives itself the greatest possible utility. Suppose that party k proposes a coalition with party j at period t . The optimum proposal must maximize the following problem:

$$\begin{aligned} \max_{\mathbf{g}, y} \quad & g_k - (y - P_k)^2 \\ \text{s.t.} \quad & g_k - (y - P_k)^2 \geq v_{tk} \end{aligned} \tag{3.1}$$

$$g_j - (y - P_j)^2 \geq v_{tj} \tag{3.2}$$

$$g_L + g_M + g_R = G \tag{3.3}$$

$$g_L, g_M, g_R \geq 0 \tag{3.4}$$

Note that (3.1) makes the proposal worthwhile for party k , otherwise it will prefer the agreement of the following period; (3.2) ensures the minimum utility that makes party j , invited to the coalition, willing to accept the proposal (we assume that when a party is indifferent between accepting or not a proposal, it chooses to take on the agreement); (3.3) and (3.4) are the requirements of the distribution of benefits among the parties.

Additionally, for $v_{tj} \geq 0$, restriction 3.2 will hold with equality. Otherwise, if $g_j - (y - P_j)^2 > v_{tj} \geq 0$, we would have $g_j > v_{tj} + (y - P_j)^2 \geq 0$. Thus, for any value of y , k 's utility could be increased by decreasing g_j until the constraint is satisfied with equality. Therefore, all problems below are simplified using the equality on this restriction, if $v_{tj} \geq 0$.

Let $\Gamma_t = (y_t, \mathbf{g}_t, C_t)$ be the optimal proposal for period t , where y_t is the optimum government policy at t ; $\mathbf{g}_t$ is the optimum distribution at t ; and C_t is the minimum coalition formed at t . As we assume perfect information and the bargaining game is solved by backward induction, the legislative equilibrium will be a subgame perfect equilibrium.

Anticipating some relevant results on the impact of the magnitude of G on the

government policy of the equilibrium at the legislative stage, we can mention:

1. When party M is the first one to make a coalition proposal, it will be able to form a coalition with the smallest party that supports M 's ideal policy as the government policy even with party M holding all the resources G . The magnitude of G will define the conditions in which this agreement will be possible. If it is not sufficiently large, the equilibrium will be a no-deal situation on the house.
2. When party M is the second one to make a coalition proposal, the party with the most seats in the legislature will choose to make a coalition with the smallest party, regardless of the ideological distances. A decrease in the magnitude of G can lead to a government policy closer to the ideal policy of the largest party. It can benefit this group, but the impact on social welfare is ambiguous.
3. When party M is the last one to make a proposal, the decrease of G 's magnitude can sometimes improve the utility of the majority of voters by leading to a government policy closer to their ideal policy. However, this impact of G on the government policy depends on the ideological distance setting and will be analyzed after we include another important element of the model: voters' behavior.

We shall now describe the equilibrium achieved at each period for all possible legislative configurations, which summarizes the equilibrium for each scenario. Followed by Proposition 1, detailed proofs of each step can be found in the Mathematical Appendix.

First, we analyze the case in which party M holds the greatest participation in the legislature, for instance $\mathbf{w}_M > \mathbf{w}_R > \mathbf{w}_L$.¹ The party with the lowest weight in the house, L , makes the proposal at $t = 3$. Since all parties have the same opportunity cost at $t = 3$, party L will choose the closest party to form a coalition, M . The optimum proposal in

¹ The analysis of the case where $\mathbf{w}_M > \mathbf{w}_L > \mathbf{w}_R$ is similar by symmetry.

this case will be the solution of the problem:

$$\begin{aligned}
& \max_{\mathbf{g}, y} \quad g_L - (y - P_L)^2 \\
& \text{s.t.} \quad g_L - (y - P_L)^2 \geq 0 \\
& \quad \quad g_M - (y - P_M)^2 = 0 \\
& \quad \quad g_L + g_M = G \\
& \quad \quad g_L, g_M \geq 0
\end{aligned}$$

In order to simplify the notation, for any $j, h \in \Phi$, let P_{jh} be the mid-distance policy between P_j and P_h : $P_{jh} = (P_j + P_h)/2$.

If $G \geq 2(P_M - P_{ML})^2$, then the solution for the problem above will be:

$$\begin{aligned}
y_3^* &= P_{ML} \\
g_{3L} &= G - (P_M - P_{ML})^2 \\
g_{3M} &= (P_M - P_{ML})^2 \\
g_{3R} &= 0 \\
C_3 &= \{M, L\}
\end{aligned}$$

This solution is the same as in Austen-Smith & Banks (1988), since they consider that the benefit G is sufficiently large so that any coalition can be formed. However, if $G < 2(P_M - P_{ML})^2$, no agreement will be met.

Actually, this reasoning holds for every configuration in the legislature. Since the problem at $t = 3$ is similar regardless of which party is the smallest, whoever is making the proposal will choose the ideologically closest party to form the coalition and will propose a similar government policy and benefit distribution as shown above.²

Lemma 1. *For any given distribution of parties in the legislature, let j be the party that makes the proposal at $t = 3$ and $|P_j - P_h| < |P_j - P_k|$. If $G \geq 2(P_j - P_{jh})^2$, then the*

² The proof can be verified in the Mathematical Appendix.

optimal proposal at this period will be:

$$\Gamma_3 = \begin{cases} y_3 = P_{jh} \\ g_{3j} = G - (P_j - P_{jh})^2 \\ g_{3h} = (P_j - P_{jh})^2 \\ C_3 = \{j, h\} \end{cases}$$

If $G < 2(P_j - P_{jh})^2$, then no agreement will be met.

In order to simplify our analysis, we make three assumptions. If a party is indifferent between making a proposal to one or another party, it will choose the ideologically closest one. If it is ideologically equidistant to the two other parties, it will choose the party with the lowest participation in the house. In the case of indifference between making an agreement or not, we also assume that parties will always prefer to do so.

Let $d_R \equiv P_R - P_M$ and $d_L \equiv P_M - P_L$. Still with the $w_M > w_R > w_L$ case in mind, the relevant threshold for G becomes $2(P_L - P_{ML})^2 = 2(d_L/2)^2 = (d_L)^2/2$.

The opportunity costs of each party at the previous period, $t = 2$, are:

$$\begin{aligned} \text{If } G \geq \frac{(d_L)^2}{2}, \quad & v_{2R} = u_R(P_{ML}, 0) = \frac{-(2d_R + d_L)^2}{4} \\ & v_{2M} = u_M(P_{ML}, (P_M - P_{ML})^2) = 0 \\ & v_{2L} = u_L(P_{ML}, G - (P_M - P_{ML})^2) = G - \frac{(d_L)^2}{2} \\ \text{If } G < \frac{(d_L)^2}{2}, \quad & v_{2R} = v_{2M} = v_{2L} = 0 \end{aligned}$$

Since party M has the closest ideology to party R and it also has the lowest opportunity cost, it will be optimal for party R to propose a coalition to party M at $t = 2$.³ The optimal proposal will be a function of the ideological distances between the parties, d_L and d_R , and the size of the legislative resources, G . As G increases, the government policy proposed for the coalition will get closer to the ideal policy of party R . The ideological distances will determine the size of G necessary to support each policy.

³ We compare the utility obtained with each possible coalition at the proof of Proposition 1 in the mathematical appendix.

At $t = 1$, party M makes a proposal anticipating all the following periods. Since the outside option of its proposal is an agreement with the coalition $C_2 = \{M, R\}$ and a government policy $y_2 \in [P_M, P_R]$, we have $v_{1L} = u_L(y_2, 0) = -(y_2 - P_L)^2$, whence party M can make a proposal to party L with its own ideal policy as the government policy, since $u_L(P_M, g_{1L}) \geq -(P_M - P_L)^2 \geq -(y_2 - P_L)^2, \forall y_2 \in [P_M, P_R]$. Thus, it is possible for M to ensure a higher utility to party L and to increase its own utility, even by holding all the resource benefits G .

However, there is a lower bound for the size of G that ensures an agreement in each period of the bargaining game. If such a minimum is not met, that rationale above would be invalid, since the opportunity costs would be $v_{tj} = 0, \forall t \in 1, 2, 3, j \in \Phi$. Thus, the maximization problem faced at each period would be similar to that described in Lemma 1.⁴

Now suppose that party M is the second largest party in terms of legislative participation, for instance $\mathbf{w}_R > \mathbf{w}_M > \mathbf{w}_L$.⁵ At $t = 3$, the optimal proposal is determined by party L as set out in Lemma 1. Thus, for $G \geq (d_L)^2/2$ the optimal coalition is $C_3 = \{L, M\}$ and the government policy is $y_3 = P_{ML}$. Otherwise, no agreement is met.

At $t = 2$, for $G \geq 1/2 (d_L)^2$, party M should choose between a coalition with party R or L . In the appendix, we show that an agreement with party R would provide a higher utility for party M , regardless of the ideological distances between parties. Since the outside option for its proposal is the government policy $y_3 = P_{ML}$ with a coalition between M and L , party M can propose to party R an agreement with its own ideal political position, P_M , as the government policy and thus can also give a higher utility to party R , without sharing the legislative resources, G .

However, if $G < 1/2 (d_L)^2$ and party M is closer to party R , $d_R < d_L$, the outside option for this case is a no agreement situation where all parties obtain zero utility. Therefore, for G sufficiently large ($\geq (d_R)^2/2$), it is possible to make a coalition with the

⁴ Details can be found in the mathematical appendix.

⁵ $\mathbf{w}_L > \mathbf{w}_M > \mathbf{w}_R$ is a symmetric case.

closest party R , as proved in Lemma 1. Otherwise, a no deal situation is even better for the proponent.

At $t = 1$, when party R makes the proposal, if it chooses a coalition with party M , the only feasible agreement is the same one made at $t = 2$. However, if it chooses a coalition with party L , it is possible for party R to obtain a greater utility. For $G \geq (d_L)^2/2$, the optimum proposal is $y^* = P_M$ and $g_R^* = G$, regardless of the ideological distances. For $(d_L)^2/2 > G \geq (d_R)^2/2$ and $d_L > d_R$, the optimum proposal is $y^* = P_{RM}$ and $g_R^* = G$. Otherwise, it will prefer a no deal situation where everyone obtains zero utility. Therefore, a lower amount of resources G can lead to a government policy closer to the ideal policy of the major party.

This result indicates the importance of analyzing the impact of G on social welfare, which we leave to chapter 5. On one hand, a reduction of G increases the utility of all voters, since this resource comes from a lump-sum tax paid by all of them. On the other hand, note that as it leads to a government policy closer to the ideal policy of the group with the greatest presence in the legislature. At least one other group (but possibly both, which may account for the majority of voters) becomes worse off.

Finally, suppose that party M holds the lowest participation in the legislature, for instance $\mathbf{w}_R > \mathbf{w}_L > \mathbf{w}_M$.⁶ At $t = 3$, party M will make a coalition with the closest party, as prescribed by Lemma 1. At $t = 2$, if G is sufficiently large to compensate for the ideological distance, party L will prefer a coalition with the party with the lowest opportunity cost at that period, R . However, as the value of G decreases, it may become better for party L to choose the closest party, M , even with its higher opportunity cost. The proposal will also differ by value of G and the ideological distances setting. The optimum solutions for this period are summarized in the following table:

⁶ Again, the equilibrium of the legislative configuration $\mathbf{w}_L > \mathbf{w}_R > \mathbf{w}_M$ is a symmetric case.

Table 1 – Optimum settlements at period $t = 2$ ($w_R > w_L > w_M$)

$d_L < d_R$	$d_R \leq d_L \leq \sqrt{2}d_R$	$\sqrt{2}d_R < d_L$
$G > \frac{(d_L)^2}{2} :$ $C_2 = \{R, L\}$ $y_2 = P_{ML}$ $g_{2L} = G$	$G > (d_R)^2 + d_R d_L :$ $C_2 = \{R, L\}$ $y_2 = P_{RL}$ $g_{2L} = G - \frac{(d_R + d_L)^2}{4}$ $g_{2R} = \frac{(d_R + d_L)^2}{4}$	$G > d_R d_L (1 + \sqrt{2}) - \frac{(d_L)^2}{2} :$ $C_2 = \{R, L\}$ $y_2 = P_{RL}$ $g_{2L} = G - \frac{(d_R + d_L)^2}{4}$ $g_{2R} = \frac{(d_R + d_L)^2}{4}$
$\frac{(d_L)^2}{2} = G :$ $C_2 = \{M, L\}$ $y_2 = P_{ML}$ $g_{2L} = \frac{(d_L)^2}{4}$ $g_{2M} = \frac{(d_L)^2}{4}$	$(d_R)^2 + d_R d_L \geq G \geq \frac{(d_R)^2}{2} :$ $C_2 = \{M, L\}$ $y_2 = P_{ML}$ $g_{2L} = \frac{(d_R)^2}{2} - \frac{(d_L)^2}{4}$ $g_{2M} = G - g_{2L}$	$d_R d_L (1 + \sqrt{2}) - \frac{(d_L)^2}{2} \geq G \geq \frac{(d_R)^2}{2} :$ $C_2 = \{M, L\}$ $y_2 = P_M - \frac{\sqrt{2}d_R}{2}$ $g_{2M} = G$
$\frac{(d_L)^2}{2} > G :$ No agreement.	$\frac{(d_R)^2}{2} > G :$ No agreement.	$\frac{(d_R)^2}{2} > G :$ No agreement.

At $t = 1$, since party M will have the lowest opportunity cost and it is also the closest party to R in terms of ideological distance, there will not be a trade-off between establishing an agreement with party L or M . Therefore, the optimum coalition will be between parties R and M , regardless of the ideological distances setting and the value of G . However, those elements will impact which proposal will be optimal:

Table 2 – Optimum settlements at period $t = 1$ ($w_R > w_L > w_M$)

$d_L < d_R \leq \sqrt{2}d_L$	$\sqrt{2}d_L < d_R$	$d_R \leq d_L \leq \sqrt{2}d_R$
$G > \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = G - g_M^*$ $g_M^* = \frac{(d_R)^2 - (d_L)^2}{4}$	$G > \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = G - g_M^*$ $g_M^* = \frac{(d_R)^2 - (d_L)^2}{4}$	$G \geq (d_R)^2 + d_R d_L :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = G - g_M^*$ $g_M^* = \frac{(d_R)^2 - (d_L)^2}{4}$
$G = \frac{(d_L)^2}{2} :$ $C^* = \{M, L\}$ $y^* = P_{RM}$ $g_R^* = \frac{2(d_L)^2 - (d_R)^2}{4}$ $g_M^* = G - g_R^*$	$G = \frac{(d_L)^2}{2} :$ $C^* = \{M, L\}$ $y^* = P_M + \frac{(d_L)}{\sqrt{2}}$ $g_M^* = G$	$(d_R)^2 + d_R d_L \geq G \geq \frac{(d_R)^2}{2} :$ $C^* = \{M, L\}$ $y^* = P_{RM}$ $g_R^* = \frac{(d_R)^2}{4}$ $g_M^* = G - g_R^*$
$\frac{(d_L)^2}{2} > G :$ No agreement.	$\frac{(d_L)^2}{2} > G :$ No agreement.	$\frac{(d_R)^2}{2} > G :$ No agreement.
$\sqrt{2}d_R < d_L \leq 2d_R$	$2d_R < d_L < 3d_R$	$3d_R \leq d_L$
$G > (1 + \sqrt{2})d_L d_R - \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = G - g_M^*$ $g_M^* = \frac{(d_R)^2 - (d_L)^2}{4}$	$G > (1 + \sqrt{2})d_L d_R - \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = 2P_M - P_{RL}$ $g_R^* = G$	$G > (1 + \sqrt{2})d_L d_R - \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_R$ $g_R^* = G$
$(1 + \sqrt{2})d_L d_R - \frac{(d_L)^2}{2} \geq G \geq \frac{(d_R)^2}{2} :$ $C^* = \{M, L\}$ $y^* = P_{RM}$ $g_R^* = \frac{(d_R)^2}{4}$ $g_M^* = G - g_R^*$		
$\frac{(d_R)^2}{2} > G :$ No agreement.		

Again, we can observe from this equilibrium an effect of the magnitude of G on the government policy. Depending on the ideological distance setting, the decrease of G can sometimes benefit the majority of voters and sometimes distance the government policy from their ideal point. A detailed analysis is presented in chapter 5 after including voters' behavior in the elections.

The equilibrium for any legislature configuration is described by the following theorem.

Proposition 1. *Let k makes the proposal at $t = 1$, h , at $t = 2$ and j , at $t = 3$.*

1. If party k has the majority in the house, $y^* = P_k$ and $g_k^* = G$, regardless of the size of G ;

2. If no party has the majority, we have the following possible cases:

a) **Party M proposes at $t = 1$:** If $G \geq (d_i)^2/2$, where $d_i = \min\{d_L, d_R\}$, the equilibrium will be $C^* = \{M, j\}$, where $w_j = \min\{w_L, w_R\}$, $y^* = P_M$ and $g_M^* = G$. Otherwise, no agreement is met.

b) **Party M proposes at $t = 2$:** The optimum coalition will be $C^* = \{R, L\}$ and the best proposal will depend on the value of G and the ideological distance setting:

- If $G \geq (d_j)^2/2$, $y^* = P_M$ and $g_k^* = G$.
- If $(d_j)^2/2 > G \geq (d_k)^2/2$, $y^* = (P_M + P_k)/2$ and $g_k^* = G$.

Otherwise, no agreement is met.

c) **Party M proposes at $t = 3$:** The optimum coalition will be $C^* = \{M, k\}$, where $w_k = \max\{w_L, w_R\}$ and the best proposal will depend on the value of G and the ideological distance setting:

- For $d_h \geq 3d_k$:
 - If $G > (1 + \sqrt{2})d_h d_k - (d_h)^2/2$, the optimum will be $y^* = P_k$ and $g_k^* = G$.
 - If $(1 + \sqrt{2})d_h d_k - (d_h)^2/2 \geq G \geq (d_k)^2/2$, the optimum will be $y^* = P_{kM}$, $g_k^* = (d_k)^2/4$ and $g_M^* = G - g_k^*$.
- For $2d_k < d_h < 3d_k$:
 - If $G > (1 + \sqrt{2})d_h d_k - (d_h)^2/2$, the optimum will be $y^* = 2P_M - P_{kh}$ and $g_k^* = G$.
 - If $(1 + \sqrt{2})d_h d_k - (d_h)^2/2 \geq G \geq (d_k)^2/2$, the optimum will be $y^* = P_{kM}$, $g_k^* = (d_k)^2/4$ and $g_M^* = G - g_k^*$.
- For $\sqrt{2}d_k < d_h \leq 2d_k$:
 - If $G > (1 + \sqrt{2})d_h d_k - (d_h)^2/2$, the optimum will be $y^* = P_{kM}$, $g_k^* = G - g_M^*$ and $g_M^* = (d_k)^2/4 - (d_h)^2/4$.

- If $(1 + \sqrt{2})d_h d_k - (d_h)^2/2 \geq G \geq (d_k)^2/2$, the optimum will be $y^* = P_{kM}$, $g_k^* = (d_k)^2/4$ and $g_M^* = G - g_k^*$.
- For $d_k \leq d_h \leq \sqrt{2}d_k$:
 - If $G > (d_k)^2 + d_k d_h$, the optimum will be $y^* = P_{kM}$, $g_k^* = G - g_M^*$ and $g_M^* = (d_k)^2/4 - (d_h)^2/4$.
 - If $(d_k)^2 + d_k d_h \geq G \geq (d_k)^2/2$, the optimum will be $y^* = P_{kM}$, $g_k^* = (d_k)^2/4$ and $g_M^* = G - g_k^*$.
- For $\sqrt{2}d_h < d_k < d_h$:
 - If $G > (d_h)^2/2$, the optimum will be $y^* = P_{kM}$, $g_k^* = G - g_M^*$ and $g_M^* = (d_k)^2/4 - (d_h)^2/4$.
 - If $G = (d_h)^2/2$, the optimum will be $y^* = P_M + d_h/\sqrt{2}$ and $g_M^* = G$.
- For $d_h < d_k \leq \sqrt{2}d_h$:
 - If $G > (d_h)^2/2$, the optimum will be $y^* = P_{kM}$, $g_k^* = G - g_M^*$ and $g_M^* = (d_k)^2/4 - (d_h)^2/4$.
 - If $G = (d_h)^2/2$, the optimum will be $y^* = P_{kM}$, $g_k^* = (d_h)^2/2 - (d_k)^2/4$ and $g_M^* = G - g_k^*$.

Otherwise, no agreement is met.

Note that the intervals of G above are defined by quadratic functions of the ideological distances, and that the government policy y is defined by a linear function of P_M and the ideological distances. Thus, considering the following transformations, we can reduce the number of variables in the model, in order to make the graphical display of such relations easier. We consider from now on the following variables:

- $d = d_L/d_R$;
- $H = G/(d_R)^2$;
- $h_i = g_i/(d_R)^2 \ \forall i \in \{L, M, R\}$; and
- $z = (y - P_M)/d_R$.

The next chapter focuses on the electoral behavior, considering that voters are able to anticipate the equilibrium of this legislative bargaining process.

4 Electoral Equilibrium

Considering that voters are discretely distributed across three ideologies (L , M and R), we can illustrate any possible distribution as a point of a 2-simplex. Each vertex of the triangle below represents an ideology and its corresponding height represents the proportion of this group within the population of voters.

If all groups of voters choose a sincere behavior in the elections, the share of votes obtained by a party will be equal to the proportion of its ideology in the population, i.e. $w_k = \sigma_k$, $\forall k \in \{L, M, R\}$. Hence, the simplex below can illustrate any configuration of the legislature in the case of sincere voting:

Figure 1 – Configurations of the legislature.

The regions in which $w_k \geq 1/2$ for some $k \in \{L, M, R\}$ illustrate the cases in which some party has the majority of the house. The other areas are the cases where parties have to form a coalition through a bargaining process.

We suppose that only groups of voters with the extreme ideologies, L and R , show a sincere behavior during elections. Voters with ideal policy P_M are able not only to anticipate the bargaining processes equilibrium for each legislative configuration, but also

to coordinate their votes and determine the house composition from a set of possible regions.

Consider for instance the case in which the proportion of ideologies between voters is represented by the red point in Figure 2. If all voters choose a sincere behavior at the elections, this point would also represent the legislature configuration, and the government policy would be the equilibrium of the bargaining game given society's ideological distribution. However, voters with P_M as the ideal policy can strategically modify the house configuration to any other region below the red lines, since $w_L \geq \sigma_L$ and $w_R \geq \sigma_R$ from our assumptions.

Figure 2 – Voters' strategies.

Since voters' utility depends on the government policy, and not on the policy defended by the party for which they choose to vote, the group that behaves strategically will decide their vote by choosing the region for which the optimum government policy is closer to their ideal.

The set of possible choices depends on the voters' distribution of ideologies:

- If ideology M is the majority in the population (region VII), they can support any legislative configuration;

- If some extreme ideology has the majority in the population (region VIII or IX), they cannot change the legislative configuration, since they cannot decrease the share of voters obtained by the party that represents the extreme ideology;
- If ideology M has the lowest representation in the population (region IV or V), this group of voters can only modify the legislative configuration to some other region where they remain the minority (IV, V, VIII and IX);
- If M is the intermediate group of voters (region III or VI), besides the same options available in the previous case, they can also choose their original region (for instance, by voting sincerely);
- If ideology M has the largest representation in the population (region I or II), there will be two possible set of choices depending on the relation between the proportions:
 - If voters' distribution is such that the presence of the second greatest ideology is bounded between $1/4$ and $1/3$, they can change to any region of the simplex;
 - If the proportion is such that the presence of the second greatest ideology is bounded between $1/3$ and σ_M , they can give the majority to an extreme party (R or L), change to some region where they are minority (IV and V), strengthen the presence of a extreme party (L or R , the one that has higher participation), but remain as the intermediate (VI and III), or decide for a sincere behavior.

The analysis should consider the size of G and the ideological distances, since those elements impact the equilibrium of the government policy in each region and the group with ideology M will choose the configuration that yields the government policy that maximizes their utility.

Considering each possible ideological distance setting with $1 < d$, the following tables summarize the information for the elections equilibrium analysis that will be relevant

for the welfare analysis in the next chapter. The optimum strategies of voters with ideal policy $z = 0$ are also described in the last column of those tables.¹

Table 3 – Strategy of voters with ideal policy $z = 0$ for $\mathbf{3} \leq \mathbf{d}$.

Region	Interval of H			Voting Strategy
	$H < 0.5$	$0.5 \leq H < d^2/2$	$d^2/2 \leq H$	
I	no deal	$z^* = 0$		sincere
II				
III		$z^* = 0.5$	$z^* = 0$	
IV		$z^* = 0.5$	$z^* = 1$	
V ^(a)		$z^* = -d/2$		change to IV
VI		$z^* = 0$		sincere

(a) In the specific case where $H = 0.5$, the optimum government policy will be $z^* = -d/\sqrt{2}$. However, the voting strategy is still to move to region IV.

Table 4 – Strategy of voters with ideal policy $z = 0$ for $\mathbf{2} < \mathbf{d} < \mathbf{3}$.

Region	Interval of H			Voting Strategy
	$H < 0.5$	$0.5 \leq H < d^2/2$	$d^2/2 \leq H$	
I	no deal	$z^* = 0$		sincere
II				
III		$z^* = 0.5$	$z^* = 0$	
IV		$z^* = 0.5$	$z^* = (d-1)/2$	change to IV
V ^(a)		$z^* = -d/2$		
VI		$z^* = 0$		

(a) In the specific case where $H = 0.5$, the optimum government policy will be $z^* = -d/\sqrt{2}$. However, the voting strategy is still to move to region IV.

¹ The cases where $1 > d$ are analogous, one would only need to exchange regions I and II, III and VI, and IV and V in these tables.

Table 5 – Strategy of voters with ideal policy $z = 0$ for $\sqrt{2} < d \leq 2$.

Region	Interval of H			Voting Strategy
	$H < 0.5$	$0.5 \leq H < d^2/2$	$d^2/2 \leq H$	
I	no deal	$z^* = 0$		sincere
II				
III		$z^* = 0.5$	$z^* = 0$	
IV		$z^* = 0.5$		change to IV
V ^(a)		$z^* = -d/2$		
VI		$z^* = 0$		sincere

(a) In the specific case where $H = 0.5$, the optimum government policy will be $z^* = -d/\sqrt{2}$. However, the voting strategy is still to move to region IV.

Table 6 – Strategy of voters with ideal policy $z = 0$ for $1 \leq d \leq \sqrt{2}$.

Region	Interval of H			Voting Strategy
	$H < 0.5$	$0.5 \leq H < d^2/2$	$d^2/2 \leq H$	
I	no deal	$z^* = 0$		sincere
II				
III		$z^* = 0.5$	$z^* = 0$	
IV		$z^* = 0.5$		change to IV
V		$z^* = -d/2$		
VI		$z^* = 0$		sincere

The voters' strategies are defined by choosing the region in which the government policy is the closest to their ideal, given the set of possible choices and considering this group of voters' utility, $s_M = -G - (y - P_M)^2$. Applying a variable transformation, we can rewrite this utility as: $s_M/(d_R)^2 = q = -H - z^2$.

The main conclusions of this stage are:

1. If voters with the ideal policy $z = 0$ have the greatest presence in the population (region I or II), this group will vote sincerely. In this case, the party that represents their ideology has enough bargaining power to support its ideal policy as the government policy.
2. If voters with the ideal policy $z = 0$ have the second greatest presence in the population and the greatest group is ideologically more distant from them (region VI

for the distance settings considered for the tables above), this group will also prefer a sincere behavior in the elections. In spite of the coalition being formed between the two extreme parties, the party that represents their ideology $z = 0$ at the legislature will have a great impact on the optimum proposal. Due to its advantage of being the second proposer and the greatest party having an extreme ideology, more distant from the other parties, the party with ideal policy $z = 0$ can lead the equilibrium to its ideal.

3. If voters with ideal policy $z = 0$ have the second greatest presence in the population and the largest group is ideologically closer to them (region III for the distance settings considered for the tables above), they will have the same advantage described in the previous item. However, it is important to highlight that a decrease in the bargaining resources H can lead to a government policy more distant from their ideal policy, since the bargaining power of the party that represents them will also decrease. The greater the distance from the lowest party to the other two, the greater the magnitude of H that is needed to ensure the advantage of their party.
4. Suppose now that voters with ideal policy $z = 0$ have the lowest presence in population. They will always choose a strategy that increases the participation of the ideologically closest party at the legislature and give them the greatest participation in the house. Hence, they can lead the optimum proposal to a government policy that is closer to their ideal policy.

Considering now all the possibilities of values for d , the figures below illustrate how the utility of voters with ideal policy $z = 0$ varies according to the ideological distance and the magnitude of H .

Figure 3 – Utility function over regions I, II and VI of the distribution of ideologies within the population.

Figure 4 – Utility function over region III of the distribution of ideologies within the population.

Figure 5 – Utility function over regions IV and V of the distribution of ideologies within the population.

We can identify in these figures the intervals of H and ideological distance ratio d and their effect on the utility of voters with intermediate ideology. An increase in H can improve this group's utility since it increases the bargaining power of the party that represents their ideal policy in the legislature and then the government policy can get closer to their ideal point. However, when this group has the lowest presence in society (Figure 5), there is an interval for $d \in [2, 3]$ in which the magnitude of H that supports the optimum government policy depends on d . For this interval of the ideological distance ratio, we note that the lower d , the lower the magnitude of H necessary to support that government policy.

Notice that for $d \in (0, 1]$ the magnitude of H that supports an agreement between parties will depends on the relative distance between ideologies regardless the distribution of ideology on the population. Therefore, the magnitude of H decrease as the relative distance of the left ideology d get smaller.

In the next chapter, we develop the analysis of the optimum size of H that maximizes the social welfare and its relation to the ideological distance ratio.

5 Social Welfare

In this chapter, we perform a welfare analysis for each region of the ideological distribution of voters. Previously, we observed different effects of the magnitude of resources present in the legislature (H) on the behavior of parties in the coalition formation process and on the utility of voters that behave strategically by anticipating post-election results.

The magnitude of legislative resources directly favors the party with the largest participation in the house. However, it also increases the bargaining power of the other parties present in the house, since all upcoming optimum proposals are anticipated by the proposer.

Regarding the voters' behavior, it is considered that they are able to anticipate the government policy from the legislative stage equilibrium. However, the strategic behavior of voters with ideal policy $z = 0$ increases their utility only in cases where this group represents a minority of voters. The magnitude of H does not affect the electoral behavior of the strategic group, but interferes with their utility by its effect on the equilibrium government policy.

Considering the aggregate of utilities of all voters, the social welfare function is defined as:

$$W = (\sigma_M) \left(-H - (z)^2 \right) + (\sigma_L) \left(-H - (z + d)^2 \right) + (1 - \sigma_M - \sigma_L) \left(-H - (z - 1)^2 \right)$$

where σ_M e σ_L are the proportions of voters with ideology M and L in the population.

Figure 6 shows the optimum social welfare obtained for different ideological distance setting and ideological distribution of voters.

Figure 6 – Optimum Social Welfare (W^*) depending on ideological distance (d), considering different ideological distribution of voters

(a) Ideology M prevails.

(b) Ideology R prevails.

(c) Ideology L prevails.

We can observe from this figure an intuitive result. A higher relative distance between the two most dominant ideologies in population gives a lower social welfare in cases that the intermediate ideology is one of the dominant ideologies. However, if it has the lowest

presence in population, this result will depend on its proportion.

Regarding now the value of H that leads to those optimum social welfare, it will depend on the ideological distance setting. This result is indicated at the Figure 7 below. The pattern of maker also indicates which ideology is prevalent in population.

We can see that the optimal legislative resources H^* is the same for all ideological distribution in population. It thus depends only on the relative distance of voters' ideology. However, in all cases, it is only necessary to set a minimum value of H that supports an agreement between parties. When the relative distance between parties' ideologies decreases, i.e. parties get closer in terms of ideology, the magnitude of H necessary to support an agreement between these parties is smaller, regardless of the ideological distribution of voters. These results are summarized in terms of the original variables at the following proposition.

Proposition 2. *Setting a value of d_R to 1, the optimal legislative resources G^* will not depend on the ideology distribution in population. It will be depends only on the magnitude of d_L , such that:*

1. *for $0 < d_L < 1$, $G^* = (d_L)^2/2$; and*
2. *for $1 \leq d_L$, G^* is limited by 0.5.*

Therefore, the optimal legislative resources will be positively related to the ideological distance d_L .

Figure 7 – Optimum Financial Funding (H^*) depending on ideological distance (d), considering different ideological distribution of voters

(a) Ideology M prevails.

(b) Ideology R prevails.

(c) Ideology L prevails.

6 Conclusion

Our main conclusions can be classified in three categories. We summarize the impact of a resource restriction on the legislative outcome:

1. When the party with intermediate ideology is the largest in the house, it will be able to form, together with the smallest party, a coalition that supports its ideal policy as the government policy. The magnitude of the resources of the legislature will impact only the conditions in which the agreement will be possible. Thus, there will be an amount necessary for the coalition formation. If this amount is not achieved, no agreement will be set.
2. When the party with intermediate ideology is the second largest, the largest in the legislature will prefer to make a coalition with the smallest one, regardless of the ideological distances between them. A decrease in the magnitude of legislative resources can lead to a government policy closer to the largest party's ideology. It can benefit this group, but the impact on social welfare is ambiguous. Actually, it may depend on the ideological distances setting.
3. When the party with intermediate ideology is the smallest, a decrease in the magnitude of resources can sometimes improve the utility of the majority of voters by leading to a government policy closer to their ideal policy. However, this impact of the legislative resources on the government policy depends on the ideological distances.

Regarding the voters' behavior, strategical voting is adopted only for cases in which the group of voters with intermediate ideology has the lowest presence in the population. We have also identified how magnitude of resources in proportion to the ideological distance setting affect the utility of the strategical voters.

An increase in legislative resources proportion can improve this group's utility since it raises the bargaining power of the party that represents their ideology in the legislature

and then the government policy approaches their ideal point. However, when this group has the lowest presence in the population, there is an interval for the ideological distance ratio on which the magnitude of resources proportion that supports the agreement depends on this ration. For this interval, we note that the closest the center ideology is to the right, the lower is the magnitude of resources in proportion of distance between the center and the right ideology necessary to support that government policy (considering that the distance between center and left is constant).

Finally, the interesting finding on the optimum legislative funding is that a low relative distance between ideologies in population leads to a low amount of resources necessary to support the agreement in the legislature. However, if this relative distance between ideologies increases, there will be a limit for the optimum amount of resource necessary to maintain the social equilibrium. Indeed, the situation in which there are a low ideological distance and a low size of legislative resources seems to be the best situation for the social welfare.

Since we assume for this study that voters with an extreme policy as ideal will only vote sincerely, an interesting step for future research would be considering that every voter can adopt strategic voting. Furthermore, if one accepts that voters and parties can choose mixed strategies when they are indifferent between the possible options, the optimum solution could be different.

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Mathematical Appendix

Proof of Proposition 1. Let party j make the proposal at $t = 3$ and $|P_j - P_h| < |P_j - P_k|$, i.e. party h is closer to party j than to party k . Party j should choose which party it will make the proposal for the coalition. If it chooses party k , the optimal proposal maximizes the following problem:

$$\begin{aligned} \max_{\mathbf{g}, y} \quad & g_j - (y - P_j)^2 \\ \text{s.t.} \quad & g_j - (y - P_j)^2 \geq 0 \\ & g_k - (y - P_k)^2 = 0 \\ & g_k + g_j = G \\ & g_k, g_j \geq 0 \end{aligned}$$

that can be rewritten as:

$$\begin{aligned} \max_y \quad & G - (y - P_k)^2 - (y - P_j)^2 \\ \text{s.t.} \quad & G - (y - P_k)^2 \geq 0 \end{aligned} \tag{6.1}$$

where $g_j = G - (y - P_k)^2$ and $g_k = (y - P_k)^2$.

The solution is $y = (P_j + P_k)/2 \equiv P_{jk}$, $g_k = (P_{jk} - P_k)^2$, $g_j = G - (P_{jk} - P_k)^2$. However, this is possible only if $G \geq 2(P_{jk} - P_j)^2$, otherwise, constraint (6.1) will not be satisfied. If $G < 2(P_{jk} - P_j)^2$, even the global optimum will provide a negative utility to party j . It will then prefer a no deal situation, in which all parties obtain an utility equal to zero.

The optimal proposal if party j chooses party h will be similar to the solution described above. The utility obtained in each situation is reported in the following table.

Table 7 – Utility obtained by party j at $t = 3$.

Interval of G	Coalition for party j	
	party h	party k
$G \geq 2(P_{jk} - P_j)^2$	$u_j = G - 2(P_{jh} - P_j)^2$	$u_j = G - 2(P_{jk} - P_j)^2$
$2(P_{jk} - P_j)^2 > G \geq 2(P_{jh} - P_j)^2$	$u_j = G - 2(P_{jh} - P_j)^2$	no deal ($u_j = 0$)
$2(P_{jh} - P_j)^2 > G$	no deal ($u_j = 0$)	

We can see in Table 7 that party j will always obtain a greater utility if it proposes a coalition to the party with the smallest ideological distance, party h as we are assuming here.

Remember that if party j is ideologically equidistant from the other two parties, we assume by an ad hoc hypothesis that it will choose a coalition with the lowest participation in the house. Thus, the equilibrium analysis in this case is similar to the problem described above. ■

Proof of Proposition 1 Let party k make the proposal at $t = 1$, party h make the proposal at $t = 2$ and party j make the proposal at $t = 3$. We will prove only the cases in which no party has the majority in the house.

(a) Party M is the first to make a proposal:

Suppose that $w_M > w_R > w_L$.¹ From the Proposition 1, the optimum proposal at $t=3$ is $C_3 = \{L, M\}$, $y_3 = P_{ML}$, $g_L = G - (P_{ML} - P_M)^2$ and $g_M = (P_{ML} - P_M)^2$ for $G \geq 2(P_{ML} - P_M)^2$. Otherwise, no agreement is met. The opportunity cost for each party at $t = 2$ will be $v_L^2 = G - 2(P_{ML} - P_M)^2$, $v_M^2 = 0$ and $v_R^2 = -(P_{ML} - P_R)^2$ if $G \geq 2(P_{ML} - P_M)^2$ and $v_L^2 = v_M^2 = v_R^2 = 0$ if $G < 2(P_{ML} - P_M)^2$.

For each interval of G , party R faces a different maximization problem to choose the best proposal and coalition. It has two options: to make a coalition with party M

¹ Note that $w_M > w_L > w_R$ has a similar analysis.

(Prob.2RM) or with party L (Prob.2RL). If $G \geq 2(P_{ML} - P_M)^2 = (d_L)^2/2$, the best proposal for party R in (Prob.2RM) maximizes the following problem:

$$\begin{aligned}
& \max_{\mathbf{g}, y} \quad g_R - (y - P_R)^2 \\
& \text{s.t.} \quad g_R - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\
& \quad \quad g_M - (y - P_M)^2 = 0 \\
& \quad \quad g_R + g_M = G \\
& \quad \quad g_R, g_M \geq 0
\end{aligned}$$

Consider the simplified version:

$$\begin{aligned}
& \max_y \quad G - (y - P_M)^2 - (y - P_R)^2 \\
& \text{s.t.} \quad G - (y - P_M)^2 \geq 0
\end{aligned} \tag{6.2}$$

$$G - (y - P_M)^2 - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \tag{6.3}$$

where $g_R = G - (y - P_M)^2$ and $g_M = (y - P_M)^2$

The global optimum, where $y_2 = P_{RM}$, $g_{2R} = G - (P_{RM} - P_M)^2$ and $g_{2M} = (P_{RM} - P_M)^2$, is the solution for the following ideological distance configuration and its respective interval of G :

- i. $d_R \geq \sqrt{2}d_L$ and $G \geq (d_R)^2/4$;
- ii. $\sqrt{2}d_L > d_R$ and $G \geq (d_L)^2/2$.

Note that the constraint (6.2) is satisfied for $G \geq (d_R)^2/4$:

$$G - (P_{RM} - P_M)^2 = G - \left(\frac{d_R}{2}\right)^2 \geq 0$$

and the constraint 6.3 is always satisfied:

$$\begin{aligned}
G - 2(P_{RM} - P_M)^2 + (P_{ML} - P_R)^2 &= \\
G - 2\left(\frac{P_R - P_M}{2}\right)^2 + \left(P_R - \frac{P_M + P_L}{2}\right)^2 &= \\
G - \frac{2(d_R)^2}{4} + \frac{(2d_R + d_L)^2}{4} &= \\
G + \frac{2(d_R)^2 + 4d_R d_L + (d_L)^2}{4} &\geq 0
\end{aligned}$$

The ideological distance setting will define which interval of G is more restrictive. Thus, if $\sqrt{2}d_L \geq d_R$, we have $(d_L)^2/2 \geq (d_R)^2/4$ and so this proposal will only be possible for $G \geq (d_L)^2/2$. Otherwise, the relevant interval of G will be $G \geq (d_R)^2/4$.

If none of the previous conditions is valid (i.e. $(d_L)^2/2 \leq G < (d_R)^2/4$), the solution will be $y_2 = P_M + \sqrt{G}$ and $g_{2M} = G$. For this interval of G , constraint (6.2) would not be satisfied with $y = P_{RM}$. Since only $y \in [P_M - \sqrt{G}, P_M + \sqrt{G}]$ can satisfy this restriction and it is a quadratic function in y , the closest value of P_{RM} in this interval, $y = P_M + \sqrt{G}$, maximizes the objective function and also satisfies restriction (6.3):

$$\begin{aligned}
G - (P_M + \sqrt{G} - P_M)^2 - (P_M + \sqrt{G} - P_R)^2 + (P_{ML} - P_R)^2 &= \\
-(d_R - \sqrt{G})^2 + \left(\frac{2d_R + d_L}{2}\right)^2 &\geq 0,
\end{aligned}$$

by contradiction, we would have $(d_R - \sqrt{G})^2 > (2d_R + d_L)^2/4 \implies -\sqrt{G} > d_L/2$.

If $(d_L)^2/2 > G$, the maximization problem for party R in (Prob.2RM) will be:

$$\begin{aligned}
\max_{\mathbf{g}, y} \quad & g_R - (y - P_R)^2 \\
\text{s.t.} \quad & g_R - (y - P_R)^2 \geq 0 \\
& g_M - (y - P_M)^2 = 0 \\
& g_R + g_M = G \\
& g_R, g_M \geq 0
\end{aligned}$$

Note that this problem is similar to that faced at $t = 3$ and the solution is the same described in Proposition 1. If $G \geq (d_R)^2/2$, the optimum proposal will be $y_2 = P_{RM}$,

$g_{2M} = (P_{RM} - P_M)^2$ and $g_{2R} = G - (P_{RM} - P_M)^2$. Otherwise, no agreement will be met.

Therefore, with the ideological distance setting $d_R \geq d_L$, we have $(d_R)^2/2 \geq (d_L)^2/2 > G$ and then no agreement is met. Alternatively, with the setting $d_L > d_R$, we have $(d_L)^2/2 > (d_R)^2/2$. In this case, if $(d_L)^2/2 > G \geq (d_R)^2/2$, the solution will be $y_2 = P_{RM}$, $g_{2M} = (P_{RM} - P_M)^2$ and $g_{2R} = G - (P_{RM} - P_M)^2$, otherwise, no agreement will be met. The optimum settlement for (Prob.2RM) is summarized in Table 8.

Table 8 – Optimum settlement between party R and M at $t = 2$ (Prob.2RM)

$d_R > \sqrt{2}d_L$	$\sqrt{2}d_L \geq d_R \geq d_L$	$d_L > d_R$
$G \geq \frac{(d_R)^2}{4} :$ $y_2 = P_{RM}$ $g_{2R} = G - \frac{(d_R)^2}{4}$ $g_{2M} = \frac{(d_R)^2}{4}$	$G \geq \frac{(d_L)^2}{2} :$ $y_2 = P_{RM}$ $g_{2R} = G - \frac{(d_R)^2}{4}$ $g_{2M} = \frac{(d_R)^2}{4}$	$G \geq \frac{(d_R)^2}{2} :$ $y_2 = P_{RM}$ $g_{2R} = G - \frac{(d_R)^2}{4}$ $g_{2M} = \frac{(d_R)^2}{4}$
$\frac{(d_R)^2}{4} > G \geq \frac{(d_L)^2}{2} :$ $y_2 = P_M + \sqrt{G}$ $g_{2M} = G$	$\frac{(d_L)^2}{2} > G :$ No agreement is met.	$\frac{(d_R)^2}{2} > G :$ No agreement is met.
$\frac{(d_L)^2}{2} > G :$ No agreement is met.		

If party R decides to make a proposal to party L (Prob.2RL), when $G \geq (d_L)^2/2$ the maximization problem will be:

$$\begin{aligned}
 & \max_{\mathbf{g}, y} \quad g_R - (y - P_R)^2 \\
 & \text{s.t.} \quad g_R - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\
 & \quad \quad g_L - (y - P_L)^2 = G - 2(P_{ML} - P_M)^2 \\
 & \quad \quad g_R + g_L = G \\
 & \quad \quad g_R, g_L \geq 0
 \end{aligned}$$

We can simplify this problem to:

$$\begin{aligned} \max_y \quad & 2(P_{ML} - P_M)^2 - (y - P_L)^2 - (y - P_R)^2 \\ \text{s.t.} \quad & G - 2(P_{ML} - P_M)^2 + (y - P_L)^2 \geq 0 \end{aligned} \quad (6.4)$$

$$2(P_{ML} - P_M)^2 - (y - P_L)^2 \geq 0 \quad (6.5)$$

$$2(P_{ML} - P_M)^2 - (y - P_L)^2 - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \quad (6.6)$$

where $g_L = G - 2(P_{ML} - P_M)^2 + (y - P_L)^2$ and $g_R = 2(P_{ML} - P_M)^2 - (y - P_L)^2$.

The optimum proposal will also depend on the ideological distance setting and on the value of G . The solution for the unconstrained version of this problem is $y = P_{RL}$ and it also satisfies the constraints (6.4), (6.5) and (6.6) if we have $d_L(\sqrt{2} - 1) \geq d_R$ and $G \geq (d_L)^2/2$. For restriction (6.4), we have:

$$\begin{aligned} G - 2(P_{ML} - P_M)^2 + (P_{RL} - P_L)^2 = \\ G - \frac{(d_L)^2}{2} + \frac{(d_R + d_L)^2}{4} > G - \frac{(d_L)^2}{2} \geq 0 \end{aligned}$$

for restriction (6.5):

$$\begin{aligned} 2(P_{ML} - P_M)^2 - (P_{RL} - P_L)^2 = \\ \frac{(d_L)^2}{2} - \frac{(d_R + d_L)^2}{4} \geq 0, \end{aligned}$$

since $d_L(\sqrt{2} - 1) \geq d_R \implies \sqrt{2}d_L \geq d_R + d_L \implies \frac{(d_L)^2}{2} \geq \frac{(d_R + d_L)^2}{4}$, and for restriction (6.6):

$$\begin{aligned} 2(P_{ML} - P_M)^2 - 2(P_{RL} - P_L)^2 + (P_{ML} - P_R)^2 = \\ \frac{2(d_L)^2}{4} - \frac{2(d_R + d_L)^2}{4} + \frac{(2d_R + d_L)^2}{4} = \\ \frac{2(d_R)^2 + (d_L)^2}{4} > 0 \end{aligned}$$

The solution for this case will then be $y_2 = P_{RL}$, $g_{2L} = G - 2(P_{ML} - P_M)^2 + (P_{RL} - P_L)^2$ and $g_{2R} = 2(P_{ML} - P_M)^2 - (P_{RL} - P_L)^2$.

If $d_R > d_L(\sqrt{2} - 1)$, constraint (6.5) will not be satisfied, since $y = P_{RL}$ will be

outside the interval of y that can satisfy this restriction:

$$\begin{aligned} 2(P_{ML} - P_M)^2 &\geq (y - P_L)^2 \\ \frac{\sqrt{2}}{2}d_L &\geq |y - P_L| \\ P_L - \frac{\sqrt{2}}{2}d_L &\leq y \leq P_L + \frac{\sqrt{2}}{2}d_L \end{aligned}$$

The solution in this case is $y_2 = P_L + \sqrt{2}d_L/2$, that is the closest value of this interval to the solution from the unconstrained version of this problem, $y = P_{RL}$, and it also satisfies restriction (6.4):

$$\begin{aligned} G - 2(P_{ML} - P_M)^2 + \left(P_L + \frac{\sqrt{2}d_L}{2} - P_L\right)^2 &= \\ G - \frac{2(d_L)^2}{4} + \frac{2(d_L)^2}{4} &= \\ G &\geq 0 \end{aligned}$$

and restriction (6.6):

$$\begin{aligned} 2(P_{ML} - P_M)^2 - \left(P_L + \frac{\sqrt{2}}{2}d_L - P_L\right)^2 - \left(P_L + \frac{\sqrt{2}}{2}d_L - P_R\right)^2 + (P_{ML} - P_R)^2 &= \\ \frac{2(d_L)^2}{4} - \frac{2(d_L)^2}{4} - \left(d_R + \left(1 - \frac{\sqrt{2}}{2}\right)d_L\right)^2 + \frac{(2d_R + d_L)^2}{4} &= \\ -(d_R)^2 - 2\left(1 - \frac{\sqrt{2}}{2}\right)d_Rd_L - \left(1 - \frac{\sqrt{2}}{2}\right)^2(d_L)^2 + (d_R)^2 + d_Rd_L + \frac{(d_L)^2}{4} &= \\ (\sqrt{2} - 1)d_Rd_L + \left(\sqrt{2} - \frac{5}{4}\right)(d_L)^2 &\geq 0 \end{aligned}$$

When $G < (d_L)^2/2$, note that the problem will be similar to that described in Proposition 1. Since $(d_R + d_L)^2/2 > (d_L)^2/2 > G$, the magnitude of G cannot support an agreement between parties R and L , and the optimum will thus be a no deal situation.

The optimum settlement agreement for (Prob.2RL) is summarized in Table 9.

Table 9 – Optimum settlement between party R and L at $t = 2$ (Prob.2RL)

$d_R > d_L(\sqrt{2} - 1)$	$d_L(\sqrt{2} - 1) \geq d_R$
$\mathbf{G} \geq \frac{(\mathbf{d}_L)^2}{2} :$ $y_2 = P_L + \frac{\sqrt{2}d_L}{2}$ $g_{2L} = G$	$\mathbf{G} \geq \frac{(\mathbf{d}_L)^2}{2} :$ $y_2 = P_{RL}$ $g_{2R} = \frac{(d_L)^2}{2} - \frac{(d_R+d_L)^2}{4}$ $g_{2L} = G - \frac{(d_L)^2}{2} + \frac{(d_R+d_L)^2}{4}$
$\frac{(\mathbf{d}_L)^2}{2} > \mathbf{G} :$ No agreement is met.	$\frac{(\mathbf{d}_L)^2}{2} > \mathbf{G} :$ No agreement is met.

Table 10 – Utility of party R in both options of coalition for each region at $t = 2$ ($w_M > w_R > w_L$).

$\mathbf{d}_R > \sqrt{2}\mathbf{d}_L$	$\sqrt{2}\mathbf{d}_L \geq \mathbf{d}_R \geq \mathbf{d}_L$
$\mathbf{G} \geq \frac{(\mathbf{d}_R)^2}{4} :$ Prob.2RM: $u_R = G - \frac{(d_R)^2}{2}$ Prob.2RL: $u_R = -\left(d_R + \left(\frac{2-\sqrt{2}}{2}\right)d_L\right)^2$	$\mathbf{G} \geq \frac{(\mathbf{d}_L)^2}{2} :$ Prob.2RM: $u_R = G - \frac{(d_R)^2}{2}$ Prob.2RL: $u_R = -\left(d_R + \left(\frac{2-\sqrt{2}}{2}\right)d_L\right)^2$
$\frac{(\mathbf{d}_R)^2}{4} > \mathbf{G} \geq \frac{(\mathbf{d}_L)^2}{2} :$ Prob.2RM: $u_R = -(d_R - \sqrt{G})^2$ Prob.2RL: $u_R = -\left(d_R + \left(\frac{2-\sqrt{2}}{2}\right)d_L\right)^2$	$\frac{(\mathbf{d}_L)^2}{2} > \mathbf{G} :$ No agreement is met.
$\frac{(\mathbf{d}_L)^2}{2} > \mathbf{G} :$ No agreement is met.	
$\mathbf{d}_L > \mathbf{d}_R > \mathbf{d}_L(\sqrt{2} - 1)$	$\mathbf{d}_L(\sqrt{2} - 1) \geq \mathbf{d}_R$
$\mathbf{G} \geq \frac{(\mathbf{d}_L)^2}{2} :$ Prob.2RM: $u_R = G - \frac{(d_R)^2}{2}$ Prob.2RL: $u_R = -\left(d_R + \left(\frac{2-\sqrt{2}}{2}\right)d_L\right)^2$	$\mathbf{G} \geq \frac{(\mathbf{d}_L)^2}{2} :$ Prob.2RM: $u_R = G - \frac{(d_R)^2}{2}$ Prob.2RL: $u_R = \frac{(d_L)^2}{2} - \frac{(d_R+d_L)^2}{2}$
$\frac{(\mathbf{d}_L)^2}{2} > \mathbf{G} \geq \frac{(\mathbf{d}_R)^2}{2} :$ Prob.2RM: $u_R = G - \frac{(d_R)^2}{2}$ Prob.2RL: no deal ($u_R = 0$)	$\frac{(\mathbf{d}_L)^2}{2} > \mathbf{G} \geq \frac{(\mathbf{d}_R)^2}{2} :$ Prob.2RM: $u_R = G - \frac{(d_R)^2}{2}$ Prob.2RL: no deal ($u_R = 0$)
$\frac{(\mathbf{d}_R)^2}{2} > \mathbf{G} :$ No agreement is met.	$\frac{(\mathbf{d}_R)^2}{2} > \mathbf{G} :$ No agreement is met.

Comparing the utility obtained by party R in each option (see Table 10), we see that it will have a greater utility if the coalition is between party R and M . Remember that when the party is indifferent between forming a coalition or staying in a situation

with no agreement, it will choose form a coalition. Hence, Table 8 also summarizes the optimum proposal of $t = 2$.

At last, party M makes a proposal at $t = 1$, anticipating all optimum proposals for the other parties. If it is one of the following cases:

- i. $d_R > \sqrt{2}d_L$ and $G \geq (d_R)^2/4$;
- ii. $\sqrt{2}d_L \geq d_R \geq d_L$ and $G \geq (d_L)^2/2$;
- iii. $d_L > d_R$ and $G \geq (d_R)^2/2$.

the optimum proposal for a coalition between party M and L (Prob.1ML) maximizes the problem:

$$\begin{aligned}
 & \max_{\mathbf{g}, y} \quad g_M - (y - P_M)^2 \\
 & \text{s.t.} \quad g_M - (y - P_M)^2 \geq 0 \\
 & \quad \quad g_L - (y - P_L)^2 \geq -(P_{RM} - P_L)^2 \\
 & \quad \quad g_L + g_M = G \\
 & \quad \quad g_L, g_M \geq 0
 \end{aligned}$$

However, if it is the case where $d_R > \sqrt{2}d_L$ and $(d_R)^2/4 > G \geq (d_L)^2/2$, the maximization problem will be:

$$\begin{aligned}
 & \max_{\mathbf{g}, y} \quad g_M - (y - P_M)^2 \\
 & \text{s.t.} \quad g_M - (y - P_M)^2 \geq 0 \\
 & \quad \quad g_L - (y - P_L)^2 \geq -(P_M - P_L + \sqrt{G})^2 \\
 & \quad \quad g_L + g_M = G \\
 & \quad \quad g_L, g_M \geq 0
 \end{aligned}$$

Note that the solution for both problems above will be $y_1 = P_M$ and $g_{1M} = G$, since this is the best agreement for party M considering an unconstrained version of this problem and it also gives a greater utility for party L .

However, if it is one of the cases where no agreement is met at $t = 2$:

- i. $d_R \geq d_L$ and $(d_L)^2/2 > G$;
- ii. $d_L > d_R$ and $(d_R)^2/2 > G$

the maximization problem will be similar to that described in Proposition 1:

$$\begin{aligned}
 & \max_{\mathbf{g}, y} \quad g_M - (y - P_M)^2 \\
 & \text{s.t.} \quad g_M - (y - P_M)^2 \geq 0 \\
 & \quad \quad g_L - (y - P_L)^2 = 0 \\
 & \quad \quad g_L + g_M = G \\
 & \quad \quad g_L, g_M \geq 0
 \end{aligned}$$

Since $G < (d_L)^2/2$ for those cases, no agreement will be met.

The optimum settlement agreement for (Prob.1ML) at $t = 1$ is summarized in the following table.

Table 11 – Optimum settlement between party M and L at $t = 1$ (Prob.1ML)

$\mathbf{d_R} \geq \mathbf{d_L}$	$\mathbf{d_L} > \mathbf{d_R}$
$\mathbf{G} \geq \frac{(\mathbf{d_L})^2}{2} :$ $y_1 = P_M$ $g_{1M} = G$	$\mathbf{G} \geq \frac{(\mathbf{d_R})^2}{2} :$ $y_1 = P_M$ $g_{1M} = G$
$\frac{(\mathbf{d_L})^2}{2} > \mathbf{G} :$ No agreement is met.	$\frac{(\mathbf{d_R})^2}{2} > \mathbf{G} :$ No agreement is met.

Note that the agreement with party L gives the highest possible utility, $u_M(G, P_M) = G$. Any other proposal made to party R gives a lower utility, since this is the optimum for an unrestricted maximization of utility.

Furthermore, party R would never accept a coalition that supports the government policy as P_M and it would then gives all the benefits G for party M , since this proposal would provide a lower utility for party R than it would have at the following period, $t = 2$.

Depending on ideological distance setting and the magnitude of G , party R 's utility would be $u_R = G - (d_R)^2/2$ or $u_R = -(d_R - \sqrt{G})^2$ at $t=2$. Both possibilities are greater than the agreement with party M at $t = 1$:

$$\begin{aligned} G - \frac{(d_R)^2}{2} &> -(d_R)^2 = u_R(0, P_M) \\ G + \frac{(d_R)^2}{2} &> 0 \end{aligned}$$

and

$$\begin{aligned} -(d_R - \sqrt{G})^2 &> -(d_R)^2 = u_R(0, P_M) \\ (d_R - \sqrt{G})^2 - (d_R)^2 &< 0 \end{aligned}$$

Therefore, the equilibrium for this bargaining game is also resumed at Table 11.

(b) Party M is the second to make a proposal:

Suppose that $w_R > w_M > w_L$.² From Proposition 1, the optimum proposal at $t = 3$ is $C_3 = \{L, M\}$, $y_3 = P_{ML}$, $g_L = G - (P_{ML} - P_M)^2$ and $g_M = (P_{ML} - P_M)^2$ for $G \geq 2(P_{ML} - P_M)^2$, otherwise, no agreement is met. Therefore, the opportunity cost at $t = 2$ is $v_L^2 = G - 2(P_{ML} - P_M)^2$, $v_M^2 = 0$ and $v_R^2 = -(P_{ML} - P_R)^2$ for $G \geq 2(P_{ML} - P_M)^2$, otherwise $v_L^2 = v_L^2 = v_L^2 = 0$.

For each interval of G , party M faces a different maximization problem. First, we will consider a coalition between party M and R (Prob.2MR). If $G \geq 2(P_{ML} - P_M)^2$, the optimum proposal maximizes the following problem:

$$\begin{aligned} \max_{\mathbf{g}, y} \quad & g_M - (y - P_M)^2 \\ \text{s.t.} \quad & g_M - (y - P_M)^2 \geq 0 \\ & g_R - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\ & g_M + g_R = G \\ & g_M, g_R \geq 0 \end{aligned}$$

² Note that $w_L > w_M > w_R$ has a similar analysis.

Note that the solution for the unconstrained version of this problem also satisfies all the restrictions described above. The optimum solution is thus $y_2 = P_M$ and $g_{2M} = G$.

If $G < 2(P_{ML} - P_M)$, the maximization problem faced by party M is similar to that described in Proposition 1, but the strategy depends on the ideological distance setting. When $d_L \leq d_R$, G will not be sufficiently large to support an agreement between party M and R . However, when $d_L > d_R$, the optimum proposal is as stated at the Proposition 1: if $2(P_{ML} - P_M)^2 > G \geq 2(P_{RM} - P_M)^2$, the optimum proposal will be $y_2 = P_{RM}$, $g_{2R} = (P_{RM} - P_M)^2$ and $g_{2M} = G - (P_{RM} - P_M)^2$; and if $G < 2(P_{RM} - P_M)^2$, no agreement will be met.

If party M chooses party L to make the coalition (Prob.2ML), the maximization problem will be:

$$\begin{aligned}
& \max_{g_M, y} \quad g_M - (y - P_M)^2 \\
& \text{s.t.} \quad g_M - (y - P_M)^2 \geq 0 \\
& \quad \quad G - g_M - (y - P_L)^2 = G - 2(P_{ML} - P_M)^2 \\
& \quad \quad 0 \leq g_M \leq G
\end{aligned} \tag{6.7}$$

note that from restriction (6.7) we have $g_M = 2(P_{ML} - P_M)^2 - (y - P_L)^2$, thus we can simplify this problem to:

$$\begin{aligned}
& \max_y \quad 2(P_{ML} - P_M)^2 - (y - P_L)^2 - (y - P_M)^2 \\
& \text{s.t.} \quad (y - P_L)^2 + (y - P_M)^2 \leq 2(P_{ML} - P_M)^2 \\
& \quad \quad 0 \leq 2(P_{ML} - P_M)^2 - (y - P_L)^2 \leq G
\end{aligned}$$

The solution for the unrestricted version of this problem, $y_2 = P_{ML}$, also satisfies the restrictions above. This is the same proposal made at $t = 3$ and the utility obtained by party M will be the same obtained at $t = 3$. Comparing the utilities obtained in both options (see Table 12), we conclude that it is better to make a coalition with party R .

Table 12 – Utility of party M in both options of coalition for each region at $t = 2$
 $(w_R > w_M > w_L)$.

$\mathbf{d_L} > \mathbf{d_R}$	$d_L \leq d_R$
$\mathbf{G} \geq \frac{(\mathbf{d_L})^2}{2}$: Prob.2MR: $u_M = G$ Prob.2ML: $u_M = 0$	$\mathbf{G} \geq \frac{(\mathbf{d_L})^2}{2}$: Prob.2MR: $u_M = G$ Prob.2ML: $u_M = 0$
$\frac{(\mathbf{d_L})^2}{2} > \mathbf{G} \geq \frac{(\mathbf{d_R})^2}{2}$: Prob.2MR: $u_M = G - \frac{(d_R)^2}{2}$ Prob.2ML: no deal ($u_M = 0$)	$\frac{(\mathbf{d_L})^2}{2} > \mathbf{G}$: No agreement is met.
$\frac{(\mathbf{d_R})^2}{2} > \mathbf{G}$: No agreement is met.	

At $t = 1$, party R makes a proposal. Considering that it chooses a coalition with party L (Prob.1RL), if $G \geq 2(P_{ML} - P_M)^2$, the maximization problem will be the following, regardless the ideological distance setting.

$$\begin{aligned} \max_{g_R, y} \quad & g_R - (y - P_R)^2 \\ \text{s.t.} \quad & g_R - (y - P_R)^2 \geq -(P_M - P_R)^2 \end{aligned} \tag{6.8}$$

$$G - g_R - (y - P_L)^2 \geq -(P_M - P_L)^2 \tag{6.9}$$

$$0 \leq g_R \leq G$$

Let's suppose first that the solution for this problem is $g_{1R} < G$. If (6.9) is an active constraint, we can express g_R in terms of y , $g_R = G + (P_M - P_L) - (y - P_L)^2$, and simplify the previous optimization. However, the optimum government problem will be $y_1 = P_{RL}$, that gives $g_R = G - (P_{RL} - P_L)^2 + (P_M - P_L)^2 > G$.

Analyzing the Kuhn–Tucker condition of this problem, $\lambda_2 = \lambda_1 + 1$, where λ_1 and λ_2 are the multipliers of the constraints (6.8) and (6.9), respectively, if constraint (6.9) is not active, i.e. $\lambda_2 = 0$, we will have $\lambda_1 = -1$, that does not satisfy a necessary condition of maximization.

Thereby, since there is no other feasible proposal, the optimum is $g_R = G$, where the closest policy y to party R ideal that satisfies (6.9) is $y_1 = P_M$.

If $d_L > d_R$ and $(d_L)^2/2 > G \geq (d_R)^2/2$, the maximization problem will be:

$$\begin{aligned} \max_{g_R, y} \quad & g_R - (y - P_R)^2 \\ \text{s.t.} \quad & g_R - (y - P_R)^2 \geq G - 2(P_{RM} - P_R)^2 \end{aligned} \quad (6.10)$$

$$G - g_R - (y - P_L)^2 \geq -(P_{RM} - P_L)^2 \quad (6.11)$$

$$0 \leq g_R \leq G$$

Let's also suppose first that $g_R < G$. Again, one of the Kuhn–Tucker conditions is $\lambda_2 = \lambda_1 + 1$, where λ_1 and λ_2 are the multipliers of constraints (6.10) and (6.11), respectively. Thus, only the second restriction is active or all of them are binding.

The other Kuhn–Tucker condition is $(y - P_R)(\lambda_1 + 1) = (y - P_L)(-\lambda_2)$. Therefore, if only restriction (6.11) is active, i.e. $\lambda_1 = 0$ and $\lambda_2 = 1$, we will have $y_1 = P_{RL}$. Since we have $g_R = G + (P_{RM} - P_L)^2 - (y - P_L)^2$ by (6.11), the proposal would be $g_{1R} = G + (P_{RM} - P_L)^2 - (P_{RL} - P_L)^2$, but this is greater than G .

Alternatively, if all of the restrictions are active, i.e. $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, we will have $g_{1R} = G - 2(P_{RM} - P_R)^2 + (y_1 - P_R)^2$ by (6.10), where y_1 also satisfies (6.11) with equality, i.e. $(y_1 - P_R)^2 + (y_1 - P_L)^2 = 2(P_{RM} - P_R)^2 + (P_{RM} - P_L)^2$. Note that this proposal gives the utility $u_R = G - 2(P_{RM} - P_R)^2$ to party R .

Now, with $g_{1R} = G$, we must have $y \in [P_L, P_{RM}]$ to satisfy (6.11) and $y \in [P_R - d_R/\sqrt{2}, P_R]$ to satisfy (6.10). Since $P_{MR} = P_R - d_R/2 > P_R - d_R/\sqrt{2}$, the intersection of those intervals is not empty and the closest value of y to party R 's ideal that belongs to both intervals is the optimum government policy. That agreement would give the utility $u_R = G - (P_{RM} - P_R)^2$ for party R .

Note that if $g_{1R} = G$, party R will obtain a higher utility. Therefore, the equilibrium for these values of G and this ideological distance setting is $g_{1R} = G$ and $y_1 = P_{RM}$.

Finally, if we consider one of the following cases, where there is no agreement at $t = 2$:

- i. $d_L > d_R$ and $G < (d_R)^2/2$;

ii. $d_L \leq d_R$ and $G < (d_L)^2/2$,

the maximization problem will be similar to that described at Proposition 1. Thus, party R will prefer a no deal situation, since the value of G is not large enough to support a coalition with party L . The optimum proposals depending on the values of G and the ideological distances setting with a coalition between party R and L (Prob.1RL) are described at the following table.

Table 13 – Optimum settlement between party R and L at $t = 1$ (Prob.1RL)

$d_L > d_R$	$d_L \leq d_R$
$G \geq \frac{(d_L)^2}{2} :$ $y_1 = P_M$ $g_{1R} = G$	$G \geq \frac{(d_L)^2}{2} :$ $y_1 = P_M$ $g_{1R} = G$
$\frac{(d_L)^2}{2} > G \geq \frac{(d_R)^2}{2} :$ $y_1 = P_{RM}$ $g_{1R} = G$	$\frac{(d_L)^2}{2} > G :$ No agreement is met.
$\frac{(d_R)^2}{2} > G :$ No agreement is met.	

In the case of a coalition between party R and M (Prob.1RM), the maximization problem will be:

$$\begin{aligned}
& \max_{g_R, y} \quad g_R - (y - P_R)^2 \\
& \text{s.t.} \quad g_R - (y - P_R)^2 \geq -(P_M - P_R)^2 \\
& \quad \quad G - g_R - (y - P_M)^2 = G \\
& \quad \quad 0 \leq g_R \leq G
\end{aligned} \tag{6.12}$$

and, from restriction (6.12), we have $g_R = -(y - P_M)^2$, but, since $g_R \geq 0$, $g_R = 0$ is the only possible value that satisfies this restriction. Thus the optimum proposal will be $g_{1R} = 0$ and $y_1 = P_M$ – the same proposal made at $t = 2$.

Comparing the utility of party R in each coalition, we can easily verify that all the optimum proposal made in (Prob.1RL) gives a greater utility. Therefore, the equilibrium

of this case is also resumed at Table 13.

(c) Party M is the last party to make a proposal:

Suppose that $w_R > w_L > w_M$.³ At $t = 3$, party M faces a problem similar to that described in Proposition 1. The optimum proposal for this period will depend on the ideological distance setting. If party L is the closest to party M , $d_L < d_R$, the solution will be $C_3 = \{M, L\}$, $y_3 = P_{ML}$, $g_{3M} = G - (P_{ML} - P_M)^2$ and $g_{3L} = (P_{ML} - P_M)^2$ for $G \geq (P_{ML} - P_M)^2$, otherwise, party M will prefer a no deal situation. However, if party R is the closest, $d_R \leq d_L$, the solution will be $C_3 = \{R, M\}$, $y_3 = P_{RM}$, $g_{3M} = G - (P_{MR} - P_M)^2$ and $g_{3R} = (P_{MR} - P_M)^2$ for $G \geq (P_{RM} - P_M)^2$, otherwise no agreement is met.

At $t = 2$, if party L decides to make a proposal for party R (Prob.2LR), the maximization problem for $G \geq 2(P_{ML} - P_L)^2$ and $d_L < d_R$ is:

$$\begin{aligned} \max_{g_L, y} \quad & g_L - (y - P_L)^2 \\ \text{s.t.} \quad & g_L - (y - P_L)^2 \geq 0 \\ & G - g_L - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\ & 0 \leq g_L \leq G \end{aligned}$$

Considering that $g_L < G$, since the Kuhn–Tucker condition at which one arrives when differentiation is done with respect to g_L is $\lambda_2 = \lambda_1 + 1$, there are only two possibilities: only the second restriction holds with equality or both restrictions hold with equality.

If $\lambda_1 = 0$ and $\lambda_2 = 1$ (the first case), the optimum government policy will be $y_2 = P_{RL}$ by the other Kuhn–Tucker condition: $(y - P_L)(\lambda_1 + 1) = (y - P_R)(-\lambda_2)$. However, note that with the second constraint active, we will have $g_{2L} = G + (P_{ML} - P_R)^2 - (P_{RL} - P_R)^2 > G$, which is not a possible equilibrium. Regarding the second case, if both constraints are active, the utility of party L will be $u_L = g_{2L} + (y_2 - P_L)^2 = 0$.

Considering now that $g_{2L} = G$, the interval of government policy y that satisfies the second condition is $y \in [P_{ML}, P_R]$. Since $y_2 = P_{ML}$ is the closest value of the ideal

³ Note that $w_L > w_R > w_M$ has a similar analysis.

policy of party L that belongs to this interval and it also satisfies the first condition, the utility obtained with this proposal will be greater: $u_L = G - (P_{ML} - P_L)^2 > 0$.

If $G \geq 2(P_{RM} - P_M)^2$ and $d_R \leq d_L$, the maximization problem will be:

$$\begin{aligned} \max_{g_L, y} \quad & g_L - (y - P_L)^2 \\ \text{s.t.} \quad & g_L - (y - P_L)^2 \geq -(P_{RM} - P_L)^2 \\ & G - g_L - (y - P_R)^2 = 0 \\ & 0 \leq g_L \leq G \end{aligned}$$

By the second constraint, we have $g_L = G - (y - P_R)^2$ and we can simplify this problem to:

$$\begin{aligned} \max_y \quad & G - (y - P_R)^2 - (y - P_L)^2 \\ \text{s.t.} \quad & G - (y - P_R)^2 - (y - P_L)^2 \geq -(P_{RM} - P_L)^2 \\ & 0 \leq G - (y - P_R)^2 \end{aligned}$$

Considering the Kuhn–Tucker condition $(y - P_R)(-1 - \lambda_1 - \lambda_2) = (y - P_L)(1 + \lambda_1)$, the analysis of all possible equilibria are:

- If $\lambda_1 = 0$ and $\lambda_2 = 0$, we will have $y_2 = P_{RL}$ by the condition described above and $g_L = G - (P_{RL} - P_R)^2$. Note that it satisfies the first constraint:

$$\begin{aligned} G - (P_{RL} - P_R)^2 - (P_{RL} - P_L)^2 + (P_{RM} - P_L)^2 &= \\ G - 2(P_{RL} - P_R)^2 + (P_{RM} - P_L)^2 &= \\ G - \frac{2(d_R + d_L)^2}{4} + \frac{(d_R + 2d_L)^2}{4} &= \\ G - \frac{2(d_R)^2 + 4d_Rd_L + 2(d_L)^2}{4} + \frac{(d_R)^2 + 4d_Rd_L + 4(d_L)^2}{4} &= \\ G + \frac{2(d_L)^2 - (d_R)^2}{4} &> 0 \end{aligned}$$

and only satisfies the second restriction if $G \geq (d_R + d_L)^2/4$

$$G - (P_{RL} - P_R)^2 = G - \frac{(d_R + d_L)^2}{4} \geq 0$$

The utility obtained by party L with this proposal will be $u_L = G - 2(P_{RL} - P_L)^2$.

- If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, the optimum government policy would be $y_2 = P_{RL}$, but, with the first restriction active, we would have $G = 2(P_{RL} - P_L)^2 - (P_{RM} - P_L)^2 < 0$, that is not possible.
- If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, by the second restriction, the optimum government policy would be $y_2 = P_R - \sqrt{G}$ and it also satisfies the first restriction if $G \leq (4d_L + 3d_R)^2/4$:

$$\begin{aligned}
& G - (P_R - \sqrt{G} - P_R)^2 - (P_R - \sqrt{G} - P_L)^2 + (P_{RM} - P_L)^2 = \\
& \quad - ((d_R + d_L) - \sqrt{G})^2 + \frac{(d_R + 2d_L)^2}{4} = \\
& \quad - ((d_R + d_L)^2 - 2(d_R + d_L)\sqrt{G} + G) + \frac{(d_R)^2 + 4d_Rd_L + 4(d_L)^2}{4} = \\
& \quad - (d_R)^2 - 2d_Rd_L - (d_L)^2 + 2(d_R + d_L)\sqrt{G} - G + \frac{(d_R)^2}{4} + d_Rd_L + (d_L)^2 = \\
& \quad \text{(for } G \leq (4d_L + 3d_R)^2/4) \quad - G + 2(d_R + d_L)\sqrt{G} - \frac{3(d_R)^2}{4} - d_Rd_L \geq 0
\end{aligned}$$

However, the utility obtained by party L with this proposal, $u_L = -(P_R - \sqrt{G} - P_L)^2$ is lower than the proposal suggested in the first item:

$$\begin{aligned}
& (G - 2(P_{RL} - P_L)^2) - \left(- (P_R - \sqrt{G} - P_L)^2 \right) = \\
& \quad \left(G - \frac{(d_R + d_L)^2}{2} \right) - \left(- (d_R + d_L - \sqrt{G})^2 \right) = \\
& \quad \left(G - \frac{(d_R + d_L)^2}{2} \right) - \left(- (d_R + d_L)^2 + 2\sqrt{G}(d_R + d_L) - G \right) = \\
& \quad 2G - 2\sqrt{G}(d_R + d_L) + \frac{(d_R + d_L)^2}{2} = \\
& \quad \frac{((d_R + d_L) - 2\sqrt{G})^2}{2} \geq 0
\end{aligned}$$

The optimum proposal at (Prob.2LR) is:

- If $G \geq (d_L)^2/2$ and $d_L < d_R$, $y_2 = P_{ML}$ and $g_{2L} = G$;
- If $G \geq (d_R + d_L)^2/4$ and $d_R \leq d_L$, $y_2 = P_{RL}$, $g_{2L} = G - (P_{RL} - P_R)^2$ and $g_{2R} = (P_{RL} - P_R)^2$;
- If $(d_R + d_L)^2/4 > G \geq (d_R)^2/2$ and $d_R \leq d_L$, $y_2 = P_R - \sqrt{G}$ and $g_{2R} = G$.

However, if G and the ideological distance are such that there was no agreement at $t = 3$, the maximization problem would be similar to that described in Proposition 1, and the value of G will not support an agreement between party L and R .

Considering now a coalition between party L and M (Prob.2LM), if $G \geq 2(P_{ML} - P_L)^2$ and $d_L < d_R$, the maximization problem will be:

$$\begin{aligned} \max_{g_L, y} \quad & g_L - (y - P_L)^2 \\ \text{s.t.} \quad & g_L - (y - P_L)^2 \geq 0 \\ & G - g_L - (y - P_M)^2 = G - 2(P_{ML} - P_M)^2 \\ & 0 \leq g_L \leq G \end{aligned}$$

By the second restriction, $g_L = 2(P_{ML} - P_M)^2 - (y - P_M)^2$ and this problem can be simplified to:

$$\begin{aligned} \max_y \quad & 2(P_{ML} - P_M)^2 - (y - P_M)^2(y - P_L)^2 \\ \text{s.t.} \quad & 2(P_{ML} - P_M)^2 - (y - P_M)^2 - (y - P_L)^2 \geq 0 \\ & (y - P_M)^2 \leq 2(P_{ML} - P_M)^2 \end{aligned}$$

The Kuhn–Tucker condition is $(y - P_M)(-1 - \lambda_1 - \lambda_2) = (y - P_L)(1 + \lambda_1)$, which analysis of cases are:

- If $\lambda_1 = 0$ and $\lambda_2 = 0$, the optimum policy would be $y_2 = P_{ML}$ and $g_{2L} = (P_{ML} - P_M)^2$, that is, the same agreement proposed at $t = 3$. Thus, the utility obtained by party L is also $u_L = 0$.
- If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, the government policy would also be $y_2 = P_{ML}$ and the analysis is similar to the previous item.

- For $\lambda_1 = 0$ and $\lambda_2 \neq 0$, we have by the second restriction:

$$\begin{aligned} |y - P_M| &= \sqrt{2} |P_{ML} - P_M| \\ y &= P_M + \frac{\sqrt{2}}{2} d_L, \text{ if } y > P_M \\ y &= P_M - \frac{\sqrt{2}}{2} d_L, \text{ if } y < P_M \end{aligned}$$

The government policy will then be $y_2 = P_M - \frac{\sqrt{2}}{2} d_L$, since this is the closest value to P_L . However, it does not satisfy the first restriction:

$$\begin{aligned} 2(P_{ML} - P_M)^2 - (P_M - \frac{\sqrt{2}}{2} d_L - P_M)^2 - (P_M - \frac{\sqrt{2}}{2} d_L - P_L)^2 = \\ \frac{(d_L)^2}{2} - \frac{(d_L)^2}{2} - (d_L)^2 \left(\frac{(2 - \sqrt{2})}{2} \right)^2 < 0 \end{aligned}$$

Thus, this is not a possible equilibrium.

- For $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, there will be no possible equilibrium, since there is no y that satisfies both restrictions with equality.

Therefore, the only possible strategy is hold the same agreement at $t = 3$.

If $G \geq 2(P_{RM} - P_M)^2$ and $d_L \geq d_R$, the maximization problem will be:

$$\begin{aligned} \max_{g_L, y} \quad & g_L - (y - P_L)^2 \\ \text{s.t.} \quad & g_L - (y - P_L)^2 \geq -(P_{RM} - P_L)^2 \\ & G - g_L - (y - P_M)^2 = G - 2(P_{RM} - P_M)^2 \\ & 0 \leq g_L \leq G \end{aligned}$$

Since $g_L = 2(P_{RM} - P_M)^2 - (y - P_M)^2$ from the second constraint, we can simplify this problem to:

$$\begin{aligned} \max_{g_L, y} \quad & 2(P_{RM} - P_M)^2 - (y - P_M)^2 - (y - P_L)^2 \\ \text{s.t.} \quad & 2(P_{RM} - P_M)^2 - (y - P_M)^2 - (y - P_L)^2 \geq -(P_{RM} - P_L)^2 \\ & (y - P_M)^2 \leq 2(P_{RM} - P_M)^2 \end{aligned}$$

The Kuhn–Tucker condition is $(y - P_M)(-1 - \lambda_1 - \lambda_2) = (y - P_L)(1 + \lambda_1)$, which analysis

of cases are:

- If $\lambda_1 = 0$ and $\lambda_2 = 0$, the government policy will be $y_2 = P_{ML}$. Note that this policy will always satisfy the first constraint:

$$\begin{aligned} 2(P_{RM} - P_M)^2 - 2(P_{ML} - P_L)^2 + (P_{RM} - P_L)^2 &= \\ \frac{2(d_R)^2}{4} - \frac{2(d_L)^2}{4} + \frac{(d_R + 2d_L)^2}{4} &= \\ \frac{2(d_R)^2 - 2(d_L)^2 + (d_R)^2 + 4d_Rd_L + 4(d_L)^2}{4} &= \\ \frac{3(d_R)^2 + 2(d_L)^2 + 4d_Rd_L}{4} &> 0 \end{aligned}$$

and will only satisfy the second constraint if $\sqrt{2}d_R \geq d_L$:

$$\begin{aligned} (P_{ML} - P_M)^2 - 2(P_{RM} - P_M)^2 &= \\ \frac{(d_L)^2}{4} - \frac{2(d_R)^2}{4} &\leq 0, \text{ for } \sqrt{2}d_R \geq d_L \end{aligned}$$

The utility obtained by party L with this agreement would be $u_L = 2(P_{RM} - P_M)^2 - 2(P_{ML} - P_M)^2$.

- If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, the optimum government policy will also be $y_2 = P_{ML}$, thus the analysis is similar to the previous item.
- For $\lambda_1 = 0$ and $\lambda_2 \neq 0$, the government policies that satisfy the second constraint with equality are:

$$\begin{aligned} (y - P_M)^2 &= 2(P_{RM} - P_M)^2 \\ |y - P_M| &= \frac{\sqrt{2}}{2}d_R \\ y &= P_M + \frac{\sqrt{2}}{2}d_R, \text{ if } y > P_M \\ y &= P_M - \frac{\sqrt{2}}{2}d_R, \text{ if } y < P_M \end{aligned}$$

Since $y = P_M - \frac{\sqrt{2}}{2}d_R$ is closer to the ideal policy of party L , this will be the best

option of proposal. Note that this policy also satisfies the other constraint:

$$\begin{aligned}
2(P_{RM} - P_M)^2 - \left(P_M - \frac{\sqrt{2}}{2}d_R - P_M\right)^2 - \left(P_M - \frac{\sqrt{2}}{2}d_R - P_L\right)^2 + (P_{RM} - P_L)^2 = \\
\frac{(d_R)^2}{2} - \frac{(d_R)^2}{2} - \left(d_L - \frac{\sqrt{2}}{2}d_R\right)^2 + \frac{(2d_L + d_R)^2}{4} = \\
-\frac{(2d_L - \sqrt{2}d_R)^2}{4} + \frac{(2d_L + d_R)^2}{4} > 0
\end{aligned}$$

The utility obtained with this proposal is $u_L = -(P_M - \frac{\sqrt{2}}{2}d_R - P_L)^2$ and it is less than the utility obtained with the previous proposal:

$$\begin{aligned}
\left(-\left(P_M - \frac{\sqrt{2}d_R}{2} - P_L\right)^2\right) - \left(2(P_{RM} - P_M)^2 - 2(P_{ML} - P_M)^2\right) = \\
-\left(d_L - \frac{\sqrt{2}d_R}{2}\right)^2 - \left(\frac{(d_R)^2}{2} - \frac{(d_L)^2}{2}\right) = \\
\text{(for } \sqrt{2}d_R > d_L\text{)} \quad \sqrt{2}d_R d_L - (d_R)^2 - \frac{(d_L)^2}{2} < 0
\end{aligned}$$

However, it was only possible if $\sqrt{2}d_R > d_L$.

- For $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, there will be no possible equilibrium, since there is no y that satisfies both restrictions with equality.

Again, note that if G and the ideological distance are such that there was no agreement at $t = 3$, party L will prefer a no deal situation, since the condition does not support any agreement between party L and M (see Proposition 1).

To know which will be the optimum coalition for party L , we have to compare the utility obtained in each options of coalition (see Table 14).

Table 14 – Utility of party L in both options of coalition for each region at $t = 2$
 $(w_R > w_L > w_M)$.

$d_L < d_R$	$d_R \leq d_L \leq \sqrt{2}d_R$	$\sqrt{2}d_R < d_L$
$G \geq \frac{(d_L)^2}{2}$: Prob.2LR: $u_L = G - \frac{(d_L)^2}{4}$ Prob.2LM: no deal ($u_L = 0$)	$G \geq \frac{(d_R+d_L)^2}{4}$: Prob.2LR: $u_L = G - \frac{(d_L+d_R)^2}{2}$ Prob.2LM: $u_L = \frac{(d_R)^2}{2} - \frac{(d_L)^2}{2}$	$G \geq \frac{(d_R+d_L)^2}{4}$: Prob.2LR: $u_L = G - \frac{(d_L+d_R)^2}{2}$ Prob.2LM: $u_L = -(d_L - \frac{\sqrt{2}}{2}d_R)^2$
$\frac{(d_L)^2}{2} > G$: No agreement is met.	$\frac{(d_R+d_L)^2}{4} > G \geq \frac{(d_R)^2}{2}$: Prob.2LR: $u_L = -((d_L + d_R) - \sqrt{G})^2$ Prob.2LM: $u_L = \frac{(d_R)^2}{2} - \frac{(d_L)^2}{2}$	$\frac{(d_R+d_L)^2}{4} > G \geq \frac{(d_R)^2}{2}$: Prob.2LR: $u_L = -((d_L + d_R) - \sqrt{G})^2$ Prob.2LM: $u_L = -(d_L - \frac{\sqrt{2}}{2}d_R)^2$
	$\frac{(d_R)^2}{2} > G$: No agreement is met.	$\frac{(d_R)^2}{2} > G$: No agreement is met.

When $d_L < d_R$ and $G = (d_L)^2/2$, party L will be indifferent between make a coalition with party R or M . Thus, by an ad hoc hypothesis, party L will choose an agreement with the closest party in terms of ideology distance, party M .

When $G \geq (d_R + d_L)^2/4$ and $d_R \leq d_L \leq \sqrt{2}d_R$, the optimum coalition will depend on the value of G again:

$$\begin{aligned}
 & \left(G - \frac{(d_L + d_R)^2}{2} \right) - \left(\frac{(d_R)^2}{2} - \frac{(d_L)^2}{2} \right) = \\
 & G - \frac{(d_L)^2}{2} - d_R d_L - \frac{(d_R)^2}{2} - \frac{(d_R)^2}{2} + \frac{(d_L)^2}{2} = \\
 & G - d_R d_L - (d_R)^2 \begin{cases} > 0, & \text{if } G > (d_R)^2 + d_R d_L \\ \leq 0, & \text{if } (d_R)^2 + d_R d_L \geq G \geq \frac{(d_R+d_L)^2}{4} \end{cases}
 \end{aligned}$$

Therefore, party L will prefer (Prob.2LR) if $G > (d_R)^2 + d_R d_L$ and (Prob.2LM) if $(d_R)^2 + d_R d_L \geq G \geq (d_R + d_L)^2/4$.

When $G \geq (d_R + d_L)^2/4$ and $\sqrt{2}d_R < d_L$, the optimum coalition will also depend on

the region of G and ideological distance setting:

$$\begin{aligned} & \left(G - \frac{(d_L + d_R)^2}{2} \right) - \left(- \left(d_L - \frac{\sqrt{2}}{2} d_R \right)^2 \right) = \\ & G - \frac{(d_L)^2}{2} - d_R d_L - \frac{(d_R)^2}{2} + (d_L)^2 - \sqrt{2} d_R d_L + \frac{(d_R)^2}{2} = \\ & G - \left(d_R d_L (1 + \sqrt{2}) - \frac{(d_L)^2}{2} \right) \end{aligned}$$

This difference between utilities will be strictly positive, i.e. (Prob.2LR) will be the optimum, if $G > \left(d_R d_L (1 + \sqrt{2}) - (d_L)^2/2 \right)$ and it will be non positive, i.e. (Prob.2LM) will be the optimum, if $\left(d_R d_L (1 + \sqrt{2}) - (d_L)^2/2 \right) \geq G \geq (d_R + d_L)^2/4$.

When $d_R \leq d_L \leq \sqrt{2} d_R$ and $(d_R + d_L)^2/4 > G \geq (d_R)^2/2$, the difference between utilities will be:

$$\begin{aligned} & \left(- \left((d_R + d_L) - \sqrt{G} \right)^2 \right) - \left(\frac{(d_R)^2 - (d_L)^2}{2} \right) = \\ & - \left((d_R + d_L)^2 - 2(d_R + d_L)\sqrt{G} + G \right) - \left(\frac{(d_R)^2 - (d_L)^2}{2} \right) = \\ & -G + 2(d_R + d_L)\sqrt{G} - (d_R)^2 - 2d_R d_L - (d_L)^2 - \frac{(d_R)^2}{2} + \frac{(d_L)^2}{2} = \\ & -G + 2(d_R + d_L)\sqrt{G} - 2d_R d_L - \frac{3(d_R)^2}{2} - \frac{(d_L)^2}{2} \end{aligned}$$

Using computational techniques, we can verify that this difference will be never positive at the relevant intervals of G and ideological distance setting. The optimum proposal will be (Prob.2LM). The same is true when $\sqrt{2} d_R < d_L$ and $(d_R + d_L)^2/4 > G \geq (d_R)^2/2$.

Table 15 – Optimum settlements at $t = 2$ ($w_R > w_L > w_M$)

$d_L < d_R$	$d_R \leq d_L \leq \sqrt{2}d_R$	$\sqrt{2}d_R < d_L$
$G > \frac{(d_L)^2}{2} :$ $C_2 = \{R, L\}$ $y_2 = P_{ML}$ $g_{2L} = G$	$G > (d_R)^2 + d_R d_L :$ $C_2 = \{R, L\}$ $y_2 = P_{RL}$ $g_{2L} = G - \frac{(d_R + d_L)^2}{4}$ $g_{2R} = \frac{(d_R + d_L)^2}{4}$	$G > d_R d_L (1 + \sqrt{2}) - \frac{(d_L)^2}{2} :$ $C_2 = \{R, L\}$ $y_2 = P_{RL}$ $g_{2L} = G - \frac{(d_R + d_L)^2}{4}$ $g_{2R} = \frac{(d_R + d_L)^2}{4}$
$\frac{(d_L)^2}{2} = G :$ $C_2 = \{M, L\}$ $y_2 = P_{ML}$ $g_{2L} = \frac{(d_L)^2}{4}$ $g_{2M} = \frac{(d_L)^2}{4}$	$(d_R)^2 + d_R d_L \geq G \geq \frac{(d_R)^2}{2} :$ $C_2 = \{M, L\}$ $y_2 = P_{ML}$ $g_{2L} = \frac{(d_R)^2}{2} - \frac{(d_L)^2}{4}$ $g_{2M} = G - g_{2L}$	$d_R d_L (1 + \sqrt{2}) - \frac{(d_L)^2}{2} \geq G \geq \frac{(d_R)^2}{2} :$ $C_2 = \{M, L\}$ $y_2 = P_M - \frac{\sqrt{2}d_R}{2}$ $g_{2M} = G$
$\frac{(d_L)^2}{2} > G :$ No agreement.	$\frac{(d_R)^2}{2} > G :$ No agreement.	$\frac{(d_R)^2}{2} > G :$ No agreement.

At $t = 1$, party R should decide its proposal. Since party M has always the lowest opportunity cost and it is also the closest party to R in terms of ideological distance, there is no trade off between an agreement with party L and M . Therefore, we will focus only on the proposals made for party M that surely will give a greater utility to party R .

The optimum proposal is analyzed by region:

i. If $d_L < d_R$ and $G > (d_L)^2/2$, the maximization problem will be:

$$\begin{aligned}
& \max_{g_R, y} \quad g_R - (y - P_R)^2 \\
& \text{s.t.} \quad g_R - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\
& \quad \quad G - g_R - (y - P_M)^2 \geq -(P_{ML} - P_M)^2 \\
& \quad \quad 0 \leq g_R \leq G
\end{aligned}$$

Considering first the point in which the Kuhn–Tucker conditions are not qualified,

$y_1 = P_M$, we have from the first restriction:

$$\begin{aligned} g_R &\geq (P_M - P_R)^2 - (P_{ML} - P_R)^2 \\ g_R &\geq (d_R)^2 - \frac{(2d_R + d_L)^2}{4} \\ g_R &\geq -d_L d_R - \frac{(d_L)^2}{4} \end{aligned}$$

and from the second restriction: $g_R \leq G + (P_{ML} - P_M)^2$, but, since $0 \leq g_R \leq G$, note that $g_{1R} = G$ will satisfy all constraints. The utility obtained by party R will thus be $u_R = G - (P_R - P_M)^2$.

Considering the points where the Kuhn-Tucker conditions are qualified and that $g_{1R} = G$, the interval of y that satisfies the second constraint is $2P_M - P_{ML} \leq y \leq P_{ML}$. Since $y = P_{ML}$ is the closest value to the ideal policy of R , this would be the optimum proposal and the utility obtained by party R would be $u_R = G - (P_{ML} - P_R)^2$, also satisfying the first restriction. However, note that this utility is lower than the previous proposal.

For $g_R < G$, we have to analyze the Kuhn-Tucker conditions: $\lambda_2 = 1 + \lambda_1$ and $(y - P_R)(\lambda_1) = (y - P_M)(-\lambda_2)$. If $\lambda_1 = 0$, we will have $\lambda_2 = 1$. Thus, $y_1 = P_{RM}$ from the second condition and $g_{1R} = G - (P_{RM} - P_M)^2 + (P_{ML} - P_M)^2$ from the second constraint. Note that this also satisfies the first constraint:

$$\begin{aligned} G - (P_{RM} - P_M)^2 + (P_{ML} - P_M)^2 - (P_{RM} - P_R)^2 + (P_{ML} - P_R)^2 &= \\ G - \frac{(d_R)^2}{2} + \frac{(d_L)^2}{4} + \frac{(2d_R + d_L)^2}{4} &= \\ G - \frac{(d_R)^2}{2} + \frac{(d_L)^2}{4} + (d_R)^2 + d_R d_L + \frac{(d_L)^2}{4} &= \\ G + \frac{(d_R)^2}{2} + \frac{(d_L)^2}{2} + d_R d_L &> 0 \end{aligned}$$

and the utility obtained by party R will be $u_R = G - 2(P_{RM} - P_M)^2 + (P_{ML} - P_M)^2$,

that is higher than the first proposal:

$$\begin{aligned} \left(G - 2(P_{RM} - P_M)^2 + (P_{ML} - P_M)^2 \right) - \left(G - (P_M - P_R)^2 \right) = \\ -\frac{(d_R)^2}{2} + \frac{(d_L)^2}{4} + (d_R)^2 = \\ \frac{(d_R)^2}{2} + \frac{(d_L)^2}{4} > 0 \end{aligned}$$

If $\lambda_2 = 0$, we have $\lambda_1 = -1$, but that does not satisfy a necessary condition of maximization, and if $\lambda_2 \neq 0$ and $\lambda_1 \neq 0$, both constraints will be active and the utility obtained by party R will be $u_R = -(P_{ML} - P_R)^2$, but this is lower than with the last proposal.

Therefore, the optimum proposal in this region is $y^* = P_{RM}$, $g_R^* = G - (P_{RM} - P_M)^2 + (P_{ML} - P_M)^2$ and $g_M^* = (P_{RM} - P_M)^2 - (P_{ML} - P_M)^2$.

ii. When $d_L < d_R$ and $G = (d_L)^2/2$, the maximization problem would be:

$$\begin{aligned} \max_{g_R, y} \quad & g_R - (y - P_R)^2 \\ \text{s.t.} \quad & g_R - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\ & G - g_R - (y - P_M)^2 = 0 \\ & 0 \leq g_R \leq G \end{aligned}$$

From the second restriction, we have $g_R = G - (y - P_M)^2$ and the problem can be simplified as:

$$\begin{aligned} \max_y \quad & G - (y - P_M)^2 - (y - P_R)^2 \\ \text{s.t.} \quad & G - (y - P_M)^2 - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\ & (y - P_M)^2 \leq G \end{aligned}$$

The Kuhn–Tucker conditions of this problem is: $(y - P_M)(-1 - \lambda_1 - \lambda_2) = (y - P_R)(1 + \lambda_1)$. If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, we will have $g_R = 0$ and $G = (y - P_M)^2$ from the second restriction. The values of y that satisfies this restriction are $y = P_M + \sqrt{G}$ and $y = P_M - \sqrt{G}$. Since $y = P_M + \sqrt{G}$ is the closest value of P_R and it also satisfies

the first restriction: $(P_M + \sqrt{G} - P_R)^2 \leq (P_{ML} - P_R)^2$, it is a candidate for optimum proposal. The utility obtained by party R would be $u_R = -(d_R + \sqrt{G})^2$.

If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, from the equality of the first restriction, we know that the utility obtained by party R would be $u_R = -(P_{ML} - P_R)^2$, that is lower than the utility obtained with the previous proposal.

If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, both restrictions should be satisfied with equality. However, the values of y , $P_M + \sqrt{G}$ and $P_M - \sqrt{G}$ that satisfies the second restriction would not satisfy the first one with equality.

Finally, if $\lambda_1 = 0$ and $\lambda_2 = 0$, the interval of y that satisfies the second restriction would be $P_M - \sqrt{G} \leq y \leq P_M + \sqrt{G}$ and the interval that satisfies the first restriction would be $(y - P_M)^2 + (y - P_R)^2 \leq G + (P_{ML} - P_R)^2$. Replacing $G = (d_L)^2/2$ at these intervals, we can show that the solution for the unrestricted version of this problem, $y = P_{RM}$, also satisfies both restrictions. From the first restriction, we have:

$$\begin{aligned} (P_{RM} - P_M)^2 + (P_{RM} - P_R)^2 &\leq \frac{(d_L)^2}{2} + (P_{ML} - P_R)^2 \\ \frac{(d_R)^2}{2} &\leq \frac{(d_L)^2}{2} + \frac{(d_L + 2d_R)^2}{4} \\ 0 &\leq \frac{-(d_R)^2 + (d_L)^2}{2} + \frac{(d_L)^2 + 4d_Ld_R + 4(d_R)^2}{4} \end{aligned}$$

and from the second restriction, we have:

$$\begin{aligned} P_M - \frac{d_L}{\sqrt{2}} &\leq P_{RM} \leq P_M + \frac{d_L}{\sqrt{2}} \\ -\frac{d_L}{\sqrt{2}} &\leq P_{RM} - P_M \leq \frac{d_L}{\sqrt{2}} \\ -\frac{d_L}{\sqrt{2}} &\leq \frac{d_R}{2} \leq \frac{d_L}{\sqrt{2}} \\ \frac{(d_R)^2}{4} &\leq \frac{(d_L)^2}{2} \end{aligned}$$

Note that it will be true only for $d_L < d_R \leq \sqrt{2}d_L$. Otherwise, the optimum policy that satisfies this interval would also be $y = P_M + \sqrt{G}$, the same proposal as before. However, the utility obtained with the proposal in which the government policy is

$y = P_{RM}$ would be higher:

$$\left(\frac{(d_L)^2}{2} - \frac{(d_R)^2}{2} \right) - \left(\frac{(d_L)^2}{2} - \left(P_M + \frac{d_R}{\sqrt{2}} - P_M \right)^2 - \left(P_M + \frac{d_R}{\sqrt{2}} - P_R \right)^2 \right) = - \left(P_M + \frac{d_R}{\sqrt{2}} - P_R \right)^2 < 0$$

Therefore, the optimum solution will be:

- $y^* = P_{RM}$ and $g_R^* = 2(d_L)^2 - (d_R)^2/4$ if $d_L < d_R \leq \sqrt{2}d_L$;
- $y^* = P_M + d_L/\sqrt{2}$ and $g_R^* = 0$ if $\sqrt{2}d_L < d_R$.

iii. If there is a region where the agreement is $y_2 = P_{RL}$ and $g_L = G - (P_{RL} - P_R)^2$ ($d_R \leq d_L$), the maximization problem will be:

$$\begin{aligned} \max_{g_R, y} \quad & g_R - (y - P_R)^2 \\ \text{s.t.} \quad & g_R - (y - P_R)^2 \geq 0 \\ & G - g_R - (y - P_M)^2 \geq -(P_{RL} - P_M)^2 \\ & 0 \leq g_R \leq G \end{aligned}$$

The Kuhn–Tucker conditions of this problem are: $\lambda_2 = 1 + \lambda_1$ and $(y - P_R)(\lambda_1 + 1) = (y - P_M)(-\lambda_2)$. When $\lambda_2 = 0$, we have $\lambda_1 = -1$, that does not satisfy the necessary condition of maximization.

When both restrictions are active, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, the optimum proposal is such that the utility obtained is $u_R = 0$.

When $\lambda_1 = 0$, we have $\lambda_2 = 1$. Thus, the government policy will be $y_1 = P_{RM}$ by the second condition and $g_{1R} = G - (P_{RM} - P_M)^2 + (P_{RL} - P_M)^2$. However, $g_{1R} \leq G$ only if $d_L \leq 2d_R$:

$$G - \left(G - (P_{RM} - P_M)^2 + (P_{RL} - P_M)^2 \right) = \frac{(d_R)^2}{4} - \frac{(d_R - d_L)^2}{4} \leq 0$$

and g_{1R} will be always greater than zero:

$$G - (P_{RM} - P_M)^2 + (P_{RL} - P_M)^2 =$$

$$G - \frac{(d_R)^2}{4} + \frac{(d_R - d_L)^2}{4} > 0,$$

since $G \geq (d_R)^2/2$.

Regarding the first restriction, we have:

$$G - (P_{RM} - P_M)^2 + (P_{RL} - P_M)^2 - (P_{RM} - P_R)^2 =$$

$$G - \frac{2(d_R)^2}{4} + \frac{(d_R - d_L)^2}{4} =$$

$$G - \frac{2(d_R)^2}{4} + \frac{(d_R)^2 - 2d_R d_L + (d_L)^2}{4} =$$

$$G - \frac{(d_R)^2 + 2d_R d_L}{4} + \frac{(d_L)^2}{4}$$

Note that this will be greater than zero, i.e. first restriction will be satisfied, if:

- $d_R \leq d_L \leq \sqrt{2}d_R$ and $G > (d_R)^2 + d_R d_L$; or
- $\sqrt{2}d_R < d_L < 2d_R$ and $G > (1 + \sqrt{2})d_R d_L - (d_L)^2/2$.

The utility obtained by party R with this proposal will be $u_R = G - (d_R)^2/2 + (d_R - d_L)^2/4 > 0$.

Now, considering that $g_{1R} = G$, the interval of y that satisfies the second restriction is:

$$(y - P_M)^2 \leq (P_M - P_{RL})^2$$

$$|y - P_M| \leq |P_M - P_{RL}|$$

$$P_{RL} \leq y \leq 2P_M - P_{RL}$$

If $3d_R \leq d_L$, the ideal policy of party R will belong to the interval:

$$\begin{aligned} 2P_M - P_{RL} - P_R &= \\ \frac{4P_M - P_R - P_L - 2P_R}{2} &= \\ \frac{(P_M - P_L) - 3(P_R - P_M)}{4} &= \\ \frac{d_L - 3d_R}{4} &\geq 0, \text{ if } 3d_R \leq d_L. \end{aligned}$$

In this case, the government policy will be $y_1 = P_R$. Note that it also satisfies the first condition, $G \geq (y_1 - P_R)^2 = 0$.

However, if $3d_R > d_L$, the ideal policy P_R will be outside this interval and the closest value, $y_1 = 2P_M - P_{RL}$, will be the government policy. Note that it also satisfies the first restriction for this ideological distance setting.

$$\begin{aligned} G - (2P_M - P_{RL} - P_R)^2 &= \\ G - \frac{(4P_M - P_R - P_L - 2P_R)^2}{4} &= \\ (\text{since } d_L < 3d_R) \quad G - \frac{(d_L - 3d_R)^2}{4} &> 0 \end{aligned}$$

However, note that this utility is lower than that obtained with the proposal in which $y_1 = P_{RM}$ and $g_{1R} = G - (d_R)^2/4 + (d_R - d_L)^2/4$:

$$\begin{aligned} \left(G - \frac{(d_R)^2}{4} + \frac{(d_R - d_L)^2}{4} \right) - \left(G - \frac{(d_L - 3d_R)^2}{4} \right) &= \\ -\frac{(d_R)^2}{4} + \frac{(d_R)^2 - 2d_R d_L + (d_L)^2}{4} + \frac{(d_L)^2 - 6d_L d_R + 9(d_R)^2}{4} &= \\ \frac{9(d_R)^2 - 8d_R d_L + 2(d_L)^2}{4} &> 0 \quad (\text{for } d_L \leq 2d_R) \end{aligned}$$

The equilibrium for these regions are summarized below:

- If $d_R \leq d_L \leq \sqrt{2}d_R$ and $G > (d_R)^2 + d_R d_L$, the optimum proposal is $y^* = P_{RM}$ and $g_R^* = G - (P_{RM} - P_M)^2 + (P_{ML} - P_M)^2$;
- If $\sqrt{2}d_R < d_L \leq 2d_R$ and $G > (1 + \sqrt{2})d_R d_L - (d_L)^2/2$, the optimum proposal is also $y^* = P_{RM}$ and $g_R^* = G - (P_{RM} - P_M)^2 + (P_{ML} - P_M)^2$;

- If $2d_R < d_L < 3d_R$ and $G > (1 + \sqrt{2})d_R d_L - (d_L)^2/2$, the optimum proposal will be $y^* = 2P_M - P_{RL}$ and $g_R^* = G$;
 - If $3d_R \leq d_L$ and $G > (1 + \sqrt{2})d_R d_L - (d_L)^2/2$, the optimum proposal will be $y^* = P_R$ and $g_R^* = G$.
- iv. If $d_R \leq d_L \leq \sqrt{2}d_R$ and $(d_R)^2 + d_R d_L \geq G \geq (d_R)^2/2$, the maximization problem will be:

$$\begin{aligned}
& \max_{g_R, y} \quad g_R - (y - P_R)^2 \\
& \text{s.t.} \quad g_R - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\
& \quad \quad G - g_R - (y - P_M)^2 = G - 2(P_{RM} - P_M)^2 \\
& \quad \quad 0 \leq g_R \leq G
\end{aligned}$$

By the second restriction, we have $g_R = 2(P_{RM} - P_M)^2 - (y - P_M)^2$, then we can simplify this problem to:

$$\begin{aligned}
& \max_y \quad 2(P_{RM} - P_M)^2 - (y - P_M)^2 - (y - P_R)^2 \\
& \text{s.t.} \quad 2(P_{RM} - P_M)^2 - (y - P_M)^2 - (y - P_R)^2 \geq -(P_{ML} - P_R)^2 \\
& \quad \quad (y - P_M)^2 \leq 2(P_{RM} - P_M)^2 \\
& \quad \quad 2(P_{RM} - P_M)^2 - (y - P_M)^2 \leq G
\end{aligned}$$

Analyzing the problem above, we observe three different scenarios: (1) the first restriction is active; (2) the second restriction is binding; or (3) none of the restrictions are satisfied with equality.

For the first case, where $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = 0$, the government policy is such that $u_R = -(P_{ML} - P_R)^2$. For the second one, where $\lambda_2 \neq 0$ and $\lambda_1 = \lambda_3 = 0$, we have:

$$\begin{aligned}
& |y - P_M| = \frac{\sqrt{2}}{2}d_R \\
& y = P_M + \frac{\sqrt{2}}{2}d_R, \text{ for } y > P_M \\
& y = P_M - \frac{\sqrt{2}}{2}d_R, \text{ for } y < P_M
\end{aligned}$$

Since $y = P_M + \frac{\sqrt{2}}{2}d_R$ is the closest value of the ideal policy of party R , this will be the optimum government policy. Note that it also satisfies the other restrictions:

$$\begin{aligned} 2(P_{RM} - P_M)^2 - \left(\frac{\sqrt{2}}{2}d_R\right)^2 - \left(P_R - P_M - \frac{\sqrt{2}}{2}d_R\right)^2 + (P_{ML} - P_R)^2 = \\ \frac{(d_R)^2}{2} - \frac{(d_R)^2}{2} - \frac{(d_R)^2}{4}(2 - \sqrt{2})^2 + \frac{(d_L + 2d_R)^2}{4} = \\ \frac{(d_L)^2 + 4d_Rd_L + 4(d_R)^2 - (2 - \sqrt{2})^2(d_R)^2}{4} = \\ \frac{(d_L)^2 + 4d_Rd_L + (4\sqrt{2} - 2)(d_R)^2}{4} > 0 \end{aligned}$$

and

$$\begin{aligned} 2(P_{RM} - P_M)^2 - \frac{2(d_R)^2}{4} = \\ \frac{2(d_R)^2}{4} - \frac{2(d_R)^2}{4} = 0 \leq G \end{aligned}$$

The utility obtained by party R with this proposal is $u_R = -(d_R)^2(2 - \sqrt{2})^2/4$. Note that this utility is greater than with the previous agreement:

$$\begin{aligned} -\frac{(d_R)^2(2 - \sqrt{2})^2}{4} + \frac{(d_L + 2d_R)^2}{4} = \\ \frac{(d_L)^2 + 4d_Rd_L + 4(d_R)^2 - (d_R)^2(2 - \sqrt{2})^2}{4} = \\ \frac{(d_L)^2 + 4d_Rd_L + (4\sqrt{2} - 2)(d_R)^2}{4} > 0 \end{aligned}$$

For the last scenario, $\lambda_1 = \lambda_2 = \lambda_3 = 0$. From the Kuhn-Tucker condition, $(y - P_M)(-1 - \lambda_1 - \lambda_2 + \lambda_3) = (y - P_R)(1 + \lambda_1)$, we have the government policy $y_1 = P_{RM}$. Since this policy satisfies all of restrictions:

$$\begin{aligned} 2(P_{RM} - P_M)^2 - 2(P_{RM} - P_M)^2 + (P_{ML} - P_R)^2 = \\ \frac{(d_L + 2d_R)^2}{4} > 0 \end{aligned}$$

and $2(P_{RM} - P_M)^2 - (P_{RM} - P_M)^2 = \frac{(d_R)^2}{4} \in [0, G]$, this will be the optimum proposal and the utility will be $u_R = 0$. Since this is greater than the utility from the previous

agreement, it will be the optimum for this region.

- v. If $\sqrt{2}d_R < d_L$ and $(1 + \sqrt{2})d_R d_L - (d_L)^2/2 \geq G \geq (d_R)^2/2$, the maximization problem will be:

$$\begin{aligned} \max_{g_R, y} \quad & g_R - (y - P_R)^2 \\ \text{s.t.} \quad & g_R - (y - P_R)^2 \geq - \left(d_R - \frac{\sqrt{2}d_R}{2} \right)^2 \\ & G - g_R - (y - P_M)^2 \geq G - \frac{(d_R)^2}{2} \\ & 0 \leq g_R \leq G \end{aligned}$$

Considering first the point where the Kuhn-Tucker conditions are not qualified, $y = P_{RM}$, the interval of g_R that satisfies the first restriction is:

$$\begin{aligned} g_R &\geq (P_{RM} - P_R)^2 - \frac{(d_R)^2}{4} (2 - \sqrt{2}) \\ g_R &\geq \frac{(d_R)^2}{4} (\sqrt{2} - 1) > 0 \end{aligned}$$

and the interval of g_R that satisfies the second restriction is $g_R \leq -(d_R)^2/4 + (d_R)^2/2 = (d_R)^2/4$. The value that satisfies both interval and also gives the highest utility for party R is $g_R = (d_R)^2/4$ – note that the utility obtained would be $u_R = (d_R)^2/4 - (d_R)^2/4 = 0$.

Considering now the points where the Kuhn-Tucker conditions are qualified, we only have the following possible cases: (1) second restriction is active and $0 < g_R < G$; (2) or second restriction is binding and $g_R = 0$.

The Kuhn-Tucker conditions of this problem are: $(y - P_R)(1 + \lambda_1) = (y - P_M)(-\lambda_2)$ and $1 + \lambda_1 = \lambda_2$. For the first case, we have $y = P_{RM}$ from the first Kuhn-Tucker condition, since $\lambda_2 \neq 0$, and $g_R = (d_R)^2/4$ from the second restriction. Note that this is the same proposal described previously.

For the second case, we also have $y = P_{RM}$ from the first Kuhn-Tucker condition, but $g_R = 0$. However, this proposal would give a lower utility, $u_R = -(P_{RM} - P_R)^2 < 0$.

The equilibrium for this region would be $y^* = P_{RM}$ and $g_R^* = (d_R)^2/4$.

Finally, if there was no agreement at $t = 2$, party R will also prefer a no deal situation, since the value of G will not support a coalition between party R and M . The equilibrium for this legislature configuration is summarized in the following table.

Table 16 – Optimum settlements at $t = 1$ ($w_R > w_L > w_M$)

$d_L < d_R \leq \sqrt{2}d_L$	$\sqrt{2}d_L < d_R$	$d_R \leq d_L \leq \sqrt{2}d_R$
$G > \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = G - g_M^*$ $g_M^* = \frac{(d_R)^2 - (d_L)^2}{4}$	$G > \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = G - g_M^*$ $g_M^* = \frac{(d_R)^2 - (d_L)^2}{4}$	$G > (d_R)^2 + d_R d_L :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = G - g_M^*$ $g_M^* = \frac{(d_R)^2 - (d_L)^2}{4}$
$G = \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = \frac{2(d_L)^2 - (d_R)^2}{4}$ $g_M^* = G - g_R^*$	$G = \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_M + \frac{d_L}{\sqrt{2}}$ $g_M^* = G$	$(d_R)^2 + d_R d_L \geq G \geq \frac{(d_R)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = \frac{(d_R)^2}{4}$ $g_M^* = G - g_R^*$
$\frac{(d_L)^2}{2} > G :$ No agreement.	$\frac{(d_L)^2}{2} > G :$ No agreement.	$\frac{(d_R)^2}{2} > G :$ No agreement.
$\sqrt{2}d_R < d_L \leq 2d_R$	$2d_R < d_L < 3d_R$	$3d_R \leq d_L$
$G > (1 + \sqrt{2})d_L d_R - \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = G - g_M^*$ $g_M^* = \frac{(d_R)^2 - (d_L)^2}{4}$	$G > (1 + \sqrt{2})d_L d_R - \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = 2P_M - P_{RL}$ $g_R^* = G$	$G > (1 + \sqrt{2})d_L d_R - \frac{(d_L)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_R$ $g_R^* = G$
$(1 + \sqrt{2})d_L d_R - \frac{(d_L)^2}{2} \geq G \geq \frac{(d_R)^2}{2} :$ $C^* = \{R, M\}$ $y^* = P_{RM}$ $g_R^* = \frac{(d_R)^2}{4}$ $g_M^* = G - g_R^*$		
$\frac{(d_R)^2}{2} > G :$ No agreement.		

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