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Essays on Endogenous Economic Growth and Inequality

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Essays on Endogenous Economic Growth and Inequality

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Resumo

Esta tese é composta por dois ensaios em crescimento econômico endógeno com acumulação de capital humano e agentes heterogêneos. Nos dois ensaios, estudamos a relação entre o crescimento econômico e a dinâmica da desigualdade. No primeiro ensaio, intitulado “Private versus public education in a two-stage human capital model”, estudamos os impactos de longo prazo de diferentes regimes educacionais sobre o crescimento e desigualdade, usando um modelo com acumulação de capital humano em dois estágios. A importância de considerar os estágios educacionais está em reconhecer a natureza hierárquica da educação e suas implicações dinâmicas no longo-prazo. Em cada estágio educacional, básico ou avançado, o financiamento da educação pode ser privado ou público. O segundo artigo, intitulado "The Lucas model under heterogeneous agents", estuda o padrão das dinâmicas distribucionais em um modelo de crescimento econômico com acumulação de capital humano. Nesse artigo, estudamos as propriedades da dinâmica da desigualdade implícitas em uma versão simplificada do modelo de Lucas (1988). Para isso, consideramos que os agentes diferem em suas dotações iniciais de capital humano.

Códigos JEL: I25, O15, O40

Palavras-chaves: Crescimento, Desigualdade, Educação, Capital Humano, Agentes heterogêneos

Abstract

This thesis is composed of two essays on endogenous economic growth with human capital accumulation and heterogeneous agents. In both essays, we study the relationship between economic growth and the dynamics of inequality. In the first paper, entitled “Private versus public education in a two-stage human capital model,” we study the long-term impacts of different educational regimes on growth and inequality using a two-stage human capital accumulation model. The importance of accounting for educational stages is to recognize the hierarchical nature of education and its dynamic implications in the long-run. At each educational stage, basic or advanced, funding may either be public or private. The second paper, entitled “The Lucas model under heterogeneous agents,” studies the pattern of distributional dynamics on an economic growth model with human capital accumulation. We explore the implicit properties of the dynamics of inequality in a simplified form of the Lucas (1988) model, in which agents differ in their initial endowments of human capital.

JEL Code: I25, O15, O40

Key-words: Growth, Inequality, Education, Human capital, Heterogeneous Agents

Sumário

Capítulo I. Private versus public education in a two-stage human capital model.....	9
1.1 Introduction.....	9
1.2 The Model Economy.....	12
1.3 The Balanced Growth Path.....	26
1.4 Conclusion.....	35
Appendix.....	37
References.....	42
 Capítulo II. The Lucas model under heterogeneous agentes.....	 43
2.1 Introduction.....	43
2.2 The Decentralized Problem.....	44
2.3 The Central Planner Problem.....	53
2.4 Uzawa's Human Capital Accumulation.....	57
2.5 Conclusion.....	58
Appendix.....	59
References.....	72

List of Figures

Figure 1: Economic growth and parameter values.....	33
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CAPÍTULO I

Private versus public education in a two-stage human capital model

Abstract

This paper studies the long-term impacts of different educational regimes on economic growth and income inequality using a two-stage human capital accumulation model with heterogeneous agents. At each stage, the basic (elementary and middle school) and the advanced one (high school, undergraduate and graduate degree), education funding may either be public or private. The importance of accounting for educational stages is to recognize the hierarchical nature of education, which has dynamic implications that must be addressed to analyze the impacts of different educational regimes. Our results show that comparing the four educational regimes, the one with the highest growth rate is always associated with private basic education. Moreover, for reasonable parameter values of the elasticities of human capital, there is a trade-off between economic growth and reduction of inequality. The completely public regime is the regime in which inequality vanishes faster, but it leads to the lowest economic growth. On the other hand, completely private educational regimes produces the highest economic growth rates, but the slowest reduction of inequality. For a low value of the elasticities of human capital, other patterns may be observed, depending on the interest rate and the amount of time that parents spend to raise their children. In such a case, the trade-off between economic growth and the reduction of inequality could no longer happen.

JEL Classification: O40, O15, I25

Key-words: Growth, Inequality, Education, Human capital

1.1 Introduction

This paper studies the long-term impacts of different educational regimes on the economic growth and income inequality using a two-stage human capital accumulation model with heterogeneous agents. The goal is to understand which form of educational funding results in the fastest reduction of inequality and which educational regime generates higher economic growth in the long run. The paper also analyzes the relationship between economic growth rate and the dynamics of inequality across these different educational regimes.

We considered that human capital accumulation includes two periods: basic and advanced education. In each stage, the average human capital of the economy produces an externality on the human capital accumulation process. This externality justifies public intervention on education. Therefore, we consider that, at each stage, education funding may either be public or private, without the possibility of two regimes coexisting at the same education stage.

These two forms of educational funding give rise to four different educational regimes: completely private, completely public and two mixed educational regimes (private basic and public advanced education or public basic and private advanced education regimes).

The importance of considering educational stages relies on the hierarchical nature of education that models with a single education phase are not able to capture. These models treat investment in schooling throughout the learning process as perfect substitutes. However, the empirical evidence suggests that skill formation is a complex process and that not only there are critical periods for learning specific pieces of knowledge, but also there is a dynamic complementarity between investments in education at different stages (see Heckman and Cunha (2007)). The hierarchy of education has implications that we must address to analyze the impacts of different educational regimes on economic growth and inequality. Therefore, our work generalizes the De la Croix and Doepke (2004) model by reckoning the hierarchical education process.

Our paper is part of the vast literature that studies the link between inequality and growth. Caselli and Ventura (2000) included different forms of heterogeneity in representative-agent growth models and show that a wide range of distributive dynamics can emerge from these models. De la Croix e Doepke (2003) allowed fertility and education decisions to be interdependent and found that higher inequality levels lower economic growth. De la Croix and Doepke (2004) compared the implication of public and private educational regimes for inequality and growth. Their results showed that private education leads to higher growth when inequality is low, however when inequality is high, public education generates higher growth.

Our paper is also closely related to the literature that accounts for the hierarchical nature of education to study the effects of private and public educational regimes on economic growth. However, this literature does not explore the dynamics of inequality under different educational regimes and the link between inequality and growth, which is some of our contributions. Arcalean and Schiopu (2010) examined the relationship between private and public spending across different educational stages. In contrast to our paper, these authors allow the coexistence of private and public spendings at the same stage. They suggest that to maximize growth, the share of public spending devoted to basic education should be high because it induces higher private education spending overall. Although our paper does not allow the coexistence of private and public spendings, our result is in line with their paper, in the sense that government spendings should be focus on the basic education stage. Abington and Blankenau (2013) study the effects of government expenditures across early and late childhood on the output level. Sarid (2017) shows that, if the goal is to induce growth, the share of public resources allocated to basic education declines as the economy grows and that, even after all individuals acquire higher education, growth-enhancing policies still subsidize higher education. Our result goes in the opposite direction and shows that, if governments can only funding one stage of the educational process, for higher economic growth, government spending should be concentrated in basic education.

Our results show that the speed of convergence to the long-term equilibrium is highly associated with elasticities of human capital. We compared educational regimes and found that the completely public regime is the regime in which inequality vanishes faster. The reason for this is that, in completely public regimes, governments invest equally in the education of all students in the same educational stage. The opposite happens when educational regimes are entirely private. In this case, the reduction of inequality is the slowest among the regimes because the investment in each educational stage depends on the human capital of parents. The greater the parental human wealth, the more parents want to invest in the education of their children in both educational periods.

When educational regimes are mixed, the speed of convergence depends on the basic and advanced education elasticities of human capital. The mixed educational regime with the fastest convergence is the one in which governments fund the educational stage with the highest elasticity. Heckman, Cunha and Schennach (2010) suggest that investment in early childhood education has the highest productivity for human capital formation. In this case, our results indicate that public funding of basic education is more efficient in reducing inequality than the public funding of advanced education. Again, the reason is that, when education is public, governments invest equally in each student of the same stage. When only one stage is public, to reduce inequality, it would be better for governments to fund the stage with the highest productivity, that is, the most important stage for human capital accumulation.

In addition to the study on the dynamics of inequality, this paper also analyzes long-term economic growth. For the society as a whole, regimes that take a little longer to overcome inequality but show a permanently higher economic growth rate could be better than regimes with a fast reduction of inequality but low economic growth. In such case, everyone would be equally poor. Our results show that, comparing the four educational regimes, the one with the highest growth rate is always associated with private basic education. Basic education sets the stage for productive advanced education. Therefore, in regard to economic growth, it is best that parents make private decisions about investing in basic schooling. The results also suggest that although a completely public regime converges faster to the equilibrium without inequality, it is never the education regime with the highest growth.

Other comparisons about economic growth rates under different education regimes depend on the parameter values, and a few patterns may be observed. These patterns are strongly related to the elasticities of human capital concerning basic and advanced education and the human capital produced during childhood. For values of these elasticities close to those found in the empirical literature, the regimes with higher growth rate have the advanced education privately funded. The completely private regimes present the highest economic growth and the completely public regimes present the lowest one. Therefore, there is a trade-off between economic growth and the reduction of inequality between regimes fully funded in one way. In mixed educational regimes, the trade-off disappears. Basic public and advanced private education regimes present faster reductions of inequality and higher economic growth rates

when compared to regimes in which basic education is private and advanced education is public.

For a low value of elasticities of human capital, other patterns may occur, depending on the interest rate and the amount of time parents spend with their children. As we shall see, when the elasticities of human capital concerning basic and advanced education and the human capital produced during childhood are low, the externality on the human capital production during youth is high. Besides, if the interest rate is low, the advanced public education could generate higher growth. This is because savings are used to finance private advanced education. When the interest rate is low, the remuneration of savings falls, and advanced education becomes more costly. In such a case, public advanced education, which is financed by a labor income tax, by internalizing the externality and does not depends on the interest rate, could generate higher growth.

If in addition to the interest rate and elasticities being low, parents dedicate much of their time to raise a child, the completely public regime can grow more than the completely private one. In such a case, the trade-off between economic growth and reduction of inequality no longer happens.

1.2 The Model Economy

Consider an economy populated by overlapping generations who live for four periods of fifteen years (s): childhood (c), youth (y), adulthood (a) and old age (o). Households are indexed by i and are distinguished only by their endowment of human capital, $h_t^{s,i}$. We assume that there are only two groups of people $i = A, B$: low-skilled, group A, and high-skilled, group B.

Regarding the life cycle, adults work, save and give birth to n_t^i children. In the next period, adults become elderly and children become young. The elderly spend their savings in consumption and education of the young. When the elderly die, their offspring become adults and the cycle restarts.

Let us suppose that at the beginning of time, $(P_0^{a,i})_{i=A,B}$ adults are born. In the next period, $(P_1^{a,i} = \sqrt{n_0^i} P_0^{a,i})_{i=A,B}$ adults are born. While these first adults exogenously arise, they give birth to their offspring. In these first two first periods, adults receive an initial endowment of human capital, $h_0^{a,i} = h_1^{a,i} > 0$. We assume that group A families are smarter than group B families, $h_0^{a,B} > h_0^{a,A}$.

From time one onwards, there are always children, young people, adults and elderly people co-existing. The law of motion for the adult population is:

$$P_{t+1}^{a,i} = n_{t-1}^i P_{t-1}^{a,i} \quad (1)$$

The relative size of the group-A adult population:

$$\zeta_t^a = \frac{P_t^{a,A}}{P_t^{a,A} + P_t^{a,B}} \quad (2)$$

Adults make all the decisions. They care about the number of children, n_t^i , family consumption during their adulthood, $c_t^{a,i}$, and old age stage, $c_{t+1}^{o,i}$, and the human capital that each child accumulates during the education phase $h_{t+2}^{a,i}$. The adults' utility function is given by:

$$u_t^{a,i}(c_t^{a,i}, c_{t+1}^{o,i}, n_t^i, h_{t+2}^{a,i}) = \ln c_t^{a,i} + \rho \ln c_{t+1}^{o,i} + \gamma \ln (n_t^i h_{t+2}^{a,i}), \quad (3)$$

in which the parameter $\rho > 0$ is the psychological discount factor and $\gamma > 0$ is the altruism factor.

The education phase occurs during the first two stages of life: childhood and youth. During this phase, individuals only study. Children attend basic schooling (elementary school and middle school) and young individuals attend advanced education (high school and undergraduate education). These two stages have distinct features. We assume that each step of the education phase has specific human-capital technologies.

The human capital production function during childhood, $h_{t+1}^{y,i}$, depends on parental human capital, $h_t^{a,i}$, investments in basic education (amount of time), $e_t^{c,i}$, and the quality of teachers, $\bar{h}_t$. To raise each child, adults spend a fixed fraction $\phi \in (0, 1)$ of their time. Therefore, we assume that parental human capital affects the human capital of children due to hereditary traits and to the quality of the time that children spend with parents at home.

Let η_1^c , η_2^c and $\eta_3^c \in (0, 1)$ denote the elasticity of the human capital of children regarding parental human capital, investments in basic education and the quality of teachers, respectively. Parameter $\mu^c > 0$ denotes a multiplicative constant in human capital technology during childhood.

$$h_{t+1}^{y,i} = f_t^c(h_t^{a,i}, e_t^{c,i}, \bar{h}_t) = \mu^c (h_t^{a,i})^{\eta_1^c} (e_t^{c,i})^{\eta_2^c} (\bar{h}_t)^{\eta_3^c} \quad (4)$$

This children's human capital production function is a particular case of the human capital formation technology used by Heckman and Cunha (2007). We utilize this particular case because it allows more comparability between this paper and other papers that address economic growth through human capital accumulation. Comparing our children's human capital production function with the technology of the human capital production function in De la Croix and Doepke (2004), we observed that the only difference between them is the non-inclusion of an additive constant to investment in education. De la Croix and Doepke (2004) argue that the presence of the additive constant guarantees that human capital remains positive even if parents do not invest in education. However, the adult's utility function ensures that corner solutions are not optimal; therefore, parents would not choose not to invest in childhood education.

The technology of human capital formation during youth, $h_{t+2}^{a,i}$, depends on the human capital produced during childhood, $h_{t+1}^{y,i}$, the investment in advanced education, $e_t^{y,i}$, and the qual-

ity of teachers, $\bar{h}_{t+1}$. The human capital acquired in childhood is required for acquiring new knowledge in advanced education. Moreover, childhood education sets the stage for productive education in youth. Let η_1^y, η_2^y and $\eta_3^y \in (0, 1)$ denote the human capital of children, investment in advanced education and quality of teachers elasticity of the human capital of the young, respectively. Let $\mu^y > 0$ denote a multiplicative constant in the human capital technology of the young.

$$h_{t+2}^{a,i} = f_{t+1}^y (h_{t+1}^{y,i}, e_{t+1}^{y,i}, \bar{h}_{t+1}) = \mu^y (h_{t+1}^{y,i})^{\eta_1^y} (e_{t+1}^{y,i})^{\eta_2^y} (\bar{h}_{t+1})^{\eta_3^y} \quad (5)$$

The human capital accumulated by an adult that was born in period t is given below. To simplify, let μ denote the aggregate multiplicative constant $\mu^y (\mu^c)^{\eta_1^y}$.

$$h_{t+2}^{a,i} = \mu (h_t^{a,i})^{\eta_1^c \eta_1^y} (e_t^{c,i})^{\eta_2^c \eta_1^y} (\bar{h}_t)^{\eta_3^c \eta_1^y} (e_{t+1}^{y,i})^{\eta_2^y} (\bar{h}_{t+1})^{\eta_3^y} \quad (6)$$

The combined parameter $\eta_2^c \eta_1^y < 1$ is the effective basic education elasticity of the human capital of adults. This human capital technology has three special features, as highlighted by Heckman and Cunha (2007): Self-productivity, dynamic complementarity and critical periods¹.

Let us make an assumption on the values of the elasticities of human capital accumulation. We assume that the human capital technology of children has constant returns to scale in parental human capital and teacher human capital. As mentioned by De la Croix and Doepke (2004), this commonly assumed in the literature, in which the spillover effect of average human capital is considered crucial for human capital convergence. As to the young people human capital production function, we assumed technology generates diminishing returns to the human capital produced during childhood but constant returns to scale in the human capital produced during childhood and the human capital of teachers.

Assumption 1: $\eta_1^c + \eta_3^c = 1$ and $\eta_1^y + \eta_3^y = 1$ (whence $\eta_1^c \eta_1^y + \eta_3^c \eta_1^y + \eta_3^y = 1$).

Educational regimes can be private or public. However, regimes cannot co-exist at the same educational stage. Consequently, there are four possible educational regimes: basic and advanced private education; basic and advanced public education; private basic and public advanced education; public basic and private advanced education. Educational regimes have the same quality, only distinguished by the source of the funding.

¹Self-productivity arises when human capital produced during childhood, $h_{t+1}^{y,i}$, increases the human capital produced during youth. Dynamic complementarity means that human capital accumulated during childhood, $h_{t+1}^{y,i}$, increases the productivity of investments in advanced education. This implies that investment in the schooling of children boosts investments in advanced education. Childhood (youth) is a critical period for investment in basic (advanced) schooling. Because basic (advanced) education is productive in childhood (youth) but is not in youth (childhood).

1.2.1 Consumption good and the education markets

There are two markets in our economy: the education market and the market of consumption good. There is free mobility of labor between them, so adults can work in both markets. A single firm produces consumption good that uses physical capital and labor as inputs. Let K_t^d be the demand for physical capital and L_t^d the efficient demanded labor. The production technology has constant returns to scale.

$$Y_t = F(K_t^d, L_t^d) = L_t^d f(\kappa_t^d) \quad (7)$$

Wherein $f(\cdot) = F(\cdot, 1)$. Let us define the physical capital - labor ratio as $\kappa_t^d := \frac{K_t^d}{L_t^d}$. The firm chooses the quantity of physical capital and efficient labor to maximize its profit, π_t . Let r_t be the rental rate of physical capital and w_t the wage per unit of human capital.

$$\pi_t = L_t^d f(\kappa_t^d) - r_t K_t^d - w_t L_t^d$$

We assumed that this economy is small and open, one in which physical capital has perfect mobility. The firm can borrow unlimited physical capital from abroad. The aggregate physical capital supply is composed of the total domestic savings and the external savings, S_t^x . We assumed that the physical capital depreciates completely every period.

$$K_{t+1} = P_t^{a,A} S_t^A + P_t^{a,B} S_t^B - S_t^x \quad (8)$$

We assumed that the world interest rate is constant over time, $r > 0$. The free mobility of physical capital guarantees that the domestic interest rate equals the world interest rate.

$$r_t = r \quad (9)$$

The first-order condition of the firm's problem implies that the physical capital-labor ratio is constant over time κ_t^d . It also means that the wage per human capital unit, w , is constant over time.

The efficient labor supply available to the representative firm is reduced by the amount of time devoted by adults to work in the education market. Since the total education time is fixed, it does not matter who provides education in equilibrium. Without loss of generality, we assume that adults are picked randomly in each group, so the average human capital of teachers equals the average human capital of adults, $\bar{h}_t^a$.

$$\bar{h}_t = \bar{h}_t^a = \frac{P_t^{a,A} h_t^{a,A} + P_t^{a,B} h_t^{a,B}}{P_t^{a,A} + P_t^{a,B}} \quad (10)$$

Besides that, adults dedicate a fraction of their time, ϕ , to raise each child. Therefore, the time spent with children, ϕn_t^i , reduces the time available to work:

$$\begin{aligned}
L_t = & P_t^{a,A} \left(h_t^{a,A} (1 - \phi n_t^A) - e_t^{c,A} n_t^A \bar{h}_t^a \right) - P_{t-1}^{a,A} \left(e_t^{y,A} n_{t-1}^A \bar{h}_t^a \right) + \\
& + P_t^{a,B} \left(h_t^{a,B} (1 - \phi n_t^B) - e_t^{c,B} n_t^B \bar{h}_t^a \right) - P_{t-1}^{a,B} \left(e_t^{y,B} n_{t-1}^B \bar{h}_t^a \right)
\end{aligned} \tag{11}$$

Since there is free mobility of labor, the wage per unit of human capital, w , is the same across markets.

1.2.2 Basic and advanced private education

Adults work and earn the constant wage per unit of human capital, w . To face the costs in their elderly years, adults need to save. The savings of adults are used as physical capital by the firm, which pays a rental rate, r . When the educational regime is completely private, parents pay for the education of each child in both periods. The total cost to educate the offspring in each stage of the education phase is $wn_t^i e_t^{s,i} \bar{h}_t^a$. The budget constraint faced by adults of the group i is:

$$c_t^{a,i} + s_t^i = wh_t^{a,i} (1 - \phi n_t^i) - wn_t^i e_t^{c,i} \bar{h}_t^a \tag{12}$$

The budget constraint for the elderly is:

$$c_{t+1}^{o,i} = (1 + r)s_t^i - wn_t^i e_{t+1}^{y,i} \bar{h}_{t+1}^a \tag{13}$$

Equilibrium under the basic and advanced private education regime

Definition 1: Given initial endowments of human capital $\{h_0^{a,i}, h_1^{a,i}\}_{i=A,B}$, initial group size $\{P_0^{a,i}, \sqrt{n_0^i} P_0^{a,i}\}_{i=A,B}$, an initial stock of physical capital (K_0), an equilibrium under basic and advanced private educational regime consists of a sequence of prices $\{r_t, w_t\}$, aggregate quantities $\{L_t, K_{t+1}, \bar{h}_t^a\}$, group sizes $\{P_{t+1}^{a,i}\}_{i=A,B}$, and decision rules $\{c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i, e_t^{c,i}, e_{t+1}^{y,i}, h_{t+2}^{a,i}\}_{i=A,B}$ such that:

- 1 - the households decision rules $c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i, e_t^{c,i}, e_{t+1}^{y,i}$ and $h_{t+2}^{a,i}$ maximize utility function (3) subject to the constraints (12), (13), (6);
- 2 - the firm's choice of L_t^d and K_t^d maximize profits;
- 3 - prices w_t, r_t are such that markets clear, i.e., (8), (9) and (11) hold;
- 4 - group populations evolve according to (1);
- 5 - the aggregate variables $\zeta_t^a, \bar{h}_t^a$ are given by (2) and (10).

Let $x_t^{a,i}$ denote the relative adult human capital of group i and g_t the average human capital of adults:

$$x_t^{a,i} = \frac{h_t^{a,i}}{\bar{h}_t^a} \quad (14)$$

$$g_t = \frac{\bar{h}_{t+1}^a}{\bar{h}_t^a} \quad (15)$$

The solution to the household decision problem is interior, and the first-order conditions imply:

$$n = \frac{\gamma (1 - \eta_2^y - \eta_2^c \eta_1^y)}{\phi(1 + \rho + \gamma)} \quad (16)$$

$$e_t^{c,i} = \frac{\eta_2^c \eta_1^y \phi x_t^{a,i}}{(1 - \eta_2^y - \eta_2^c \eta_1^y)} \quad (17)$$

$$e_{t+1}^{y,i} = \frac{\eta_2^y (1 + r) \phi x_t^{a,i}}{g_t (1 - \eta_2^y - \eta_2^c \eta_1^y)} \quad (18)$$

$$s_t^i = \frac{(\rho + \gamma \eta_2^y) w h_t^{a,i}}{(1 + \rho + \gamma)} \quad (19)$$

To prevent negative decisions on fertility in a completely private education sytem, we need to make one more assumption. The sum of the effective basic education and the advanced education elasticities of the human capital of adults is less than one.

Assumption 2: $\eta_2^c \eta_1^y + \eta_2^y < 1$.

The fertility decision is constant over time and does not change between population groups. The number of children is negatively related to the effective basic education elasticity of the human capital of children, $\eta_2^c \eta_1^y$, and the advanced education elasticity of the human capital of young, η_2^y .

The investments in the education of children and young people increases with parental relative human capital. This means that richer (or more skilled) parents will invest more in the education of their offspring. Moreover, note that, in equilibrium, both basic and advanced education increases with basic and advanced education elasticities, $\eta_2^c \eta_1^y$ and η_2^y , respectively $\left(\frac{\partial e_t^{c,i}}{\partial (\eta_2^c \eta_1^y)} > 0, \frac{\partial e_t^{c,i}}{\partial \eta_2^y} > 0, \frac{\partial e_{t+1}^{y,i}}{\partial (\eta_2^c \eta_1^y)} > 0, \frac{\partial e_{t+1}^{y,i}}{\partial \eta_2^y} > 0 \right)$. Since parents want to invest more in basic and advanced schooling when these elasticities are high, they will have fewer children so they are able to afford higher education for them.

1.2.3 Basic and advanced public education

The average human capital of the economy causes external effects on the production of human capital. This dynamic externality justifies government interventions to internalize this

spillover effect. The government chooses how much to invest at each education phase knowing that this education policy affects the average human capital, thereby internalizing the externality.

In an exclusively public educational regime, schooling is provided only publicly, which is the only role for the government. Governments, therefore, levy a proportional income tax on adults, ν_t , and use it to finance education in both education periods. Although parents do not directly choose how the investment in the education of their offspring is made, adults influence the choice of educational policy through voting. We assume there is an electoral process in which adults choose the optimal tax rate, which means that adults decide how much of their income they are willing to invest in the education of the society as a whole. The budget constraint of an adult parent in a exclusive public model is:

$$c_t^{a,i} + s_t^i = (1 - \nu_t) w h_t^{a,i} (1 - \phi n_t^i) \quad (20)$$

The elderly are not taxed and do not pay for the education of young people. The savings of the elderly are used in the consumption of the family in this stage.

$$c_{t+1}^{o,i} = (1 + r) s_t^i \quad (21)$$

Governments do not distinguish investments in education between groups; investments in education are equal for all students of the same stage. However, educational expenditures can vary between stages. The human capital production function in an exclusive public regime is:

$$h_{t+2}^{a,i} = \mu (h_t^{a,i})^{\eta_1^c \eta_1^y} (\bar{e}_t^c)^{\eta_2^c \eta_1^y} (\bar{h}_t^a)^{\eta_3^c \eta_1^y} (\bar{e}_{t+1}^y)^{\eta_2^y} (\bar{h}_{t+1}^a)^{\eta_3^y} \quad (22)$$

Only adults participate in the electoral process. They vote for the size of government by choosing the labor income tax, ν_t^i , which maximizes their indirect utility function. Governments must keep a balanced budget in every period, which means the total expenditure on education has to equal total tax receipts at time t. Adults know that and consider it when they choose the income tax.

$$\begin{aligned} w \bar{e}_t^c \bar{h}_t^a (P_t^{a,A} n_t^A + P_t^{a,B} n_t^B) + w \bar{e}_t^y \bar{h}_t^a (P_{t-1}^{a,A} n_{t-1}^A + P_{t-1}^{a,B} n_{t-1}^B) = \\ = w \nu_t (P_t^{a,A} h_t^{a,A} (1 - \phi n_t^A) + P_t^{a,B} h_t^{a,B} (1 - \phi n_t^B)) \end{aligned} \quad (23)$$

Government terms last only one period. During its term, the public authority is concerned with the well-being of all living individuals in society; therefore, it chooses the level of education at each stage to maximize the welfare of adults and seniors. In doing so, governments rely on parents to establish how much a child is worth, that is, the parent's altruistic factor. In its maximization problem, governments must respect their budget constraints and the adults' choice of income tax.

$$\max_{\bar{e}_t^c, \bar{e}_t^y} W_t = P_t^{a,A} u_t^A + P_t^{a,B} u_t^B + P_{t-1}^{a,A} u_{t-1}^A + P_{t-1}^{a,B} u_{t-1}^B \quad (24)$$

Equilibrium under the basic and advanced public education regime

Definition 2: Given initial endowments of human capital $(h_0^{a,i}, h_1^{a,i})_{i=A,B}$, initial group size $(P_0^{a,i}, \sqrt{n_0^i} P_0^{a,i})_{i=A,B}$, an initial stock of physical capital (K_0) , an equilibrium under basic and advanced public educational regime consists of a sequence of prices $\{r_t, w_t\}$, aggregate quantities $\{L_t, K_{t+1}, \bar{h}_t^a\}$, group sizes $\{P_{t+1}^{a,i}\}_{i=A,B}$, decision rules $\{c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i\}_{i=A,B}$, and policy variables $\{\bar{e}_t^c, \bar{e}_{t+1}^y, \nu_t\}$ such that:

- 1 - households decision rules $c_t^{a,i}$, $c_{t+1}^{o,i}$, s_t^i and n_t^i maximize utility function (3) subject to the constraints (20), (20) and (22);
- 2 - governments' budget constraint (23) is satisfied;
- 3 - given the decision rules, the labor income tax ν_t^i is chosen by adults to maximize their indirect utility;
- 4 - given the decision rules and the result of the electoral process ν_t , governments decide $\bar{e}_t^c$, $\bar{e}_{t+1}^y$ to maximize population welfare (24);
- 5 - the firm chooses L_t^d and K_t^d to maximize profits;
- 6 - prices w_t, r_t are such that markets clear, i.e., (8), (9) and (11) hold;
- 7 - group populations evolve according to (1);
- 8 - the aggregate variables $\zeta_t^a, \bar{h}_t^a$ are given by (2) and (10).

Let us denote population growth rate as:

$$N_t^a = \frac{P_{t+1}^{a,A} + P_{t+1}^{a,B}}{P_t^{a,A} + P_t^{a,B}} \quad (25)$$

The solution of the household decision problem is interior, and the first-order conditions imply:

$$n = \frac{\gamma}{\phi(1 + \rho + \gamma)} \quad (26)$$

$$s_t^i = \left(\frac{\rho}{1 + \rho + \gamma} \right) (1 - \nu_t) w h_t^{a,i} \quad (27)$$

Again, the fertility decision is constant over time and does not change between family groups. Contrary to the completely private educational regime, in the public regime, fertility does not depend on the elasticities of human capital production function. When governments fund schooling, parents choose to have more children than in a completely private regime. This happens because parents do not pay for education directly. Given the choice of number of children, the government budget constraint is simplified to:

$$\bar{e}_t^y = \frac{N_{t-1}^a (1 + \rho) \phi}{\gamma} \nu_t - N_{t-1}^a \bar{e}_t^c \quad (28)$$

Adults vote for the labor income tax to maximize its indirect utility function, considering their equilibrium choices and government budget constraint:

$$\begin{aligned} \max_{\nu_t^i} u_t^{a,i} = & \ln \left((1 - \nu_t^i) w h_t^{a,i} \left(\frac{1}{1 + \rho + \gamma} \right) \right) + \rho \ln \left((1 + r) \rho (1 - \nu_t^i) \left(\frac{w h_t^{a,i}}{1 + \rho + \gamma} \right) \right) + \gamma \ln \left(\frac{\gamma}{(1 + \rho + \gamma) \phi} \right) + \\ & + \gamma \ln \left(\mu (h_t^{a,i})^{\eta_1^c \eta_1^y} \left(\frac{N_{t-1}^a (1 + \rho) \phi \nu_t^i - \gamma \bar{e}_t^y}{\gamma N_{t-1}^a} \right)^{\eta_2^c \eta_1^y} (\bar{h}_t)^{\eta_3^c \eta_1^y} \left(\frac{N_t^a ((1 + \rho) \phi \nu_{t+1} - \gamma \bar{e}_{t+1}^c)}{\gamma} \right)^{\eta_2^y} (\bar{h}_{t+1})^{\eta_3^y} \right) \end{aligned} \quad (29)$$

As the fertility choice is constant over time and groups, the population growth path is:

$$N_t^a = \frac{P_{t+1}^{a,A} + P_{t+1}^{a,B}}{P_t^{a,A} + P_t^{a,B}} = \sqrt{n} = \left(\frac{\gamma}{(1 + \rho + \gamma) \phi} \right)^{\frac{1}{2}} \quad (30)$$

The first-order condition of the election process implies:

$$\nu_t = \frac{\left(\frac{\gamma}{(1 + \rho + \gamma) \phi} \right)^{\frac{1}{2}} \gamma \eta_2^c \eta_1^y + \bar{e}_t^y \gamma}{1 + \rho + \gamma \eta_2^c \eta_1^y} \quad (31)$$

Since we assumed logarithmic utility, it turns out that all adults prefer the same tax rate. Note that the labor income tax of the next period is not a choice variable for current adults, who will be elderly in the next period and will not vote. After the electoral process, government budget constraint can be simplified to:

$$\bar{e}_t^c = \frac{\eta_2^c \eta_1^y (1 + \rho) \left(\frac{\gamma}{(1 + \rho + \gamma) \phi} \right)^{\frac{1}{2}} \phi - \gamma \eta_2^c \eta_1^y \bar{e}_t^y}{\left(\frac{\gamma}{(1 + \rho + \gamma) \phi} \right)^{\frac{1}{2}} (1 + \rho + \gamma \eta_2^c \eta_1^y)} \quad (32)$$

Once governments know their revenue, they maximize the welfare of the living population in their terms by choosing the level of basic and advanced education provided:

$$\max_{\bar{e}_t^y, \bar{e}_t^c} W_t = P_t^{a,A} u_t^{a,A} + P_t^{a,B} u_t^{a,B} + P_{t-1}^{a,A} u_{t-1}^{a,A} + P_{t-1}^{a,B} u_{t-1}^{a,B},$$

subject to (31) and (32).

Governments are aware that their choices can affect future variables and, therefore, the intertemporal utility of current living adults. When governments enhance the ongoing investment in the education of young people, $\bar{e}_t^y$, they boost the average human capital of living adults in the following period, $\bar{h}_{t+1}^a$. Consequently, in the future, teachers will be more qualified, which increases the human capital accumulated by the offspring of the current living adults, $h_{t+2}^{a,i}$. An

increase of current investment in the education of young will affect the utility of current adults positively. Therefore, the public authority internalizes the dynamic externality. The first-order condition of the welfare maximization problem is:

The first-order condition of the welfare maximization problem is:

$$\bar{e}_t^c = \frac{\eta_2^c \eta_1^y (1 + \rho) \phi \left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}}}{(1 + \rho + \gamma (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y)) \left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}} + \gamma \eta_2^y} \quad (33)$$

$$\bar{e}_t^y = \frac{\left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}} \phi (1 + \rho) \left(\eta_3^y \eta_2^y \left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}} + \eta_2^y \right)}{(1 + \rho + \gamma (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y)) \left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}} + \gamma \eta_2^y} \quad (34)$$

Therefore, the labor income tax is:

$$\nu_t = \frac{\gamma \left(\left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}} (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y) + \eta_2^y \right)}{(1 + \rho + \gamma (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y)) \left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}} + \gamma \eta_2^y} \quad (35)$$

Our results show an adverse effect of the elasticity of human capital associated with one stage of education on the investments in the education of the other stage. An increase in the elasticity of human capital concerning basic (advanced) education decreases the investment in advanced (basic) education $\left(\frac{\partial \bar{e}_t^y}{\partial (\eta_2^c \eta_1^y)} < 0; \frac{\partial \bar{e}_t^c}{\partial \eta_2^y} < 0 \right)$. As expected, the elasticity associated with one stage of education has positive effects on the investment in education on the same stage $\left(\frac{\partial \bar{e}_t^c}{\partial (\eta_2^c \eta_1^y)} > 0, \frac{\partial \bar{e}_t^y}{\partial \eta_2^y} > 0 \right)$.

1.2.4 Private basic and public advanced education

In this mixed educational regime, parents directly pay for the education of their children and, during youth, individuals attend public schools or universities. To fund advanced public education, governments levy a proportional labor income tax on adults. The budget constraint of adults and older adults are given below.

$$c_t^{a,i} + s_t^i = (1 - \nu_t) w h_t^{a,i} (1 - \phi n_t^i) - w n_t^i e_t^{c,i} \bar{h}_t^a \quad (36)$$

$$c_{t+1}^{o,i} = (1 + r) s_t^i \quad (37)$$

$$c_t^{a,i} \geq 0, c_{t+1}^{o,i} \geq 0$$

Again, adults make all the decisions. In order to maximize the utility function, adults choose the level of family consumption in both periods, as well as the number of children they will have and the investments they will make in childhood education. The human capital production function in this model considers that basic education varies between children and advanced education is funded by governments and equal among young people.

$$h_{t+2}^{a,i} = \mu \left(h_t^{a,i} \right)^{\eta_1^c \eta_1^y} \left(e_t^{c,i} \right)^{\eta_2^c \eta_1^y} \left(\bar{h}_t \right)^{\eta_3^c \eta_1^y} \left(\bar{e}_{t+1}^y \right)^{\eta_2^y} \left(\bar{h}_{t+1} \right)^{\eta_3^y} \quad (38)$$

In this educational regime, an electoral process in which only adults vote is not possible. This failure occurs because agents cannot internalize the externality of average human capital on the human capital production function. Adults are unable to decide the optimal labor income tax, or, in other words, the maximization of adults' indirect utility regarding income tax rate has no optimal solution. In such case, only the government can internalize this externality and choose the optimal investment in education.

To understand this failure, note that the tax revenue from adult income taxation is used to finance the studies of young people, the offspring of seniors. But the education of the young people are not directly included in the utility function of adults. In this case, adults would like to choose a corner solution $\nu^i = 0$. By choosing $\nu^i = 0$, governments would not fund advanced education and the human capital of new adults, which would become new teachers, would be null, $\bar{h}_{t+1} = 0$. In the next period, the offspring of adults will be the new young people and still attend school, which implies that their human capital will be null too and that the utility of adults would converge to minus infinity. Therefore, the government must intercede in this education market and maximize the welfare of the population alive by choosing the optimal investment in the education of each young.

$$\max_{\bar{e}_t^y} W_t = P_t^{a,A} u_t^A + P_t^{a,B} u_t^B + P_{t-1}^{a,A} u_{t-1}^A + P_{t-1}^{a,B} u_{t-1}^B \quad (39)$$

Adults will pay the income tax necessary to keep the government budget constraint satisfied:

$$w \bar{e}_t^y \bar{h}_t \left(P_{t-1}^{a,A} n_{t-1}^A + P_{t-1}^{a,B} n_{t-1}^B \right) = w \nu_t \left(P_t^{a,A} h_t^{a,A} (1 - \phi n_t^A) + P_t^{a,B} h_t^{a,B} (1 - \phi n_t^B) \right) \quad (40)$$

Equilibrium under the private basic and public advanced educational regime

Definition 3: Given initial endowments of human capital $(h_0^{a,i}, h_1^{a,i})_{i=A,B}$, initial group size $(P_0^{a,A}, \sqrt{n_0^i} P_0^{a,i})_{i=A,B}$, an initial stock of physical capital (K_0) , an equilibrium with public basic and advanced education consists of a sequence of prices $\{r_t, w_t\}$, aggregate quantities $\{L_t, K_{t+1}, \bar{h}_t^a\}$, group sizes $\{P_{t+1}^{a,i}\}_{i=A,B}$, decision rules $\{c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i, e_t^{c,i}\}_{i=A,B}$, and policy variables $\{\bar{e}_{t+1}^y, \nu_t\}$ such that:

1 - adults decision rules $c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i$ and $e_t^{c,i}$ maximize utility function (3) subject to the

constraints (36), (37), (38);

2 - given the decision rules, governments' choice $\bar{e}_{t+1}^y$ maximize the welfare of population (39);

3 - governments' budget constraint (40) is satisfied;

4 - the firm's choices L_t^d and K_t^d maximize profits;

5 - prices w_t, r_t are such that markets clear, i.e., (8), (9) and (11) hold;

6 - group populations evolve according to (1);

7 - the aggregate variables $\zeta_t^a, \bar{h}_t^a$ are given by (2) and (10).

The solution of household decision problem is interior and the first-order conditions imply:

$$n = \frac{\gamma (1 - \eta_2^c \eta_1^y)}{\phi (1 + \rho + \gamma)} \quad (41)$$

$$s_t^i = \frac{\rho (1 - \nu_t) w h_t^{a,i}}{1 + \rho + \gamma} \quad (42)$$

$$e_t^{c,i} = \frac{\eta_2^c \eta_1^y (1 - \nu_t) \phi x_t^{a,i}}{1 - \eta_2^c \eta_1^y} \quad (43)$$

The fertility choice is, once again, constant over time and between groups. However, in this private-public regime, the effective basic education elasticity of human capital, $\eta_2^c \eta_1^y$, has an adverse effect in fertility choice $\left(\frac{\partial n}{\partial (\eta_2^c \eta_1^y)} < 0 \right)$. This elasticity has also a positive impact on the private choice of investments in basic education $\left(\frac{\partial e_t^{c,i}}{\partial (\eta_2^c \eta_1^y)} > 0 \right)$. The more productive the basic education, the more parents invest in the education of their offspring in this stage. The advanced education elasticity of human capital, η_2^y , has no effect on decision choices in this mixed model. Given equation (41), government budget is constraint simplified to:

$$\bar{e}_t^y = \frac{\phi (1 + \rho + \gamma \eta_2^c \eta_1^y) \left(\frac{\gamma (1 - \eta_2^c \eta_1^y)}{\phi (1 + \rho + \gamma)} \right)^{\frac{1}{2}}}{\gamma (1 - \eta_2^c \eta_1^y)} \nu_t \quad (44)$$

Given the optimal choices for the consumer problem, the public authority maximizes the welfare of living people by choosing the investments in the education of young individuals. The first-order condition of the government problem is:

$$\bar{e}_t^y = \frac{\left(\left(\frac{\gamma (1 - \eta_2^c \eta_1^y)}{\phi (1 + \rho + \gamma)} \right)^{\frac{1}{2}} \eta_3^y \eta_2^y + \eta_2^y \right) \phi (1 + \rho + \gamma \eta_2^c \eta_1^y) \left(\frac{\gamma (1 - \eta_2^c \eta_1^y)}{\phi (1 + \rho + \gamma)} \right)^{\frac{1}{2}}}{(1 - \eta_2^c \eta_1^y) \left((1 + \rho + \gamma (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y)) \left(\frac{\gamma (1 - \eta_2^c \eta_1^y)}{\phi (1 + \rho + \gamma)} \right)^{\frac{1}{2}} + \gamma \eta_2^y \right)} \quad (45)$$

Given this level of education, the labor income tax is set to satisfy the government budget constraint:

$$\nu_t = \frac{\gamma \left(\left(\frac{\gamma(1-\eta_2^c \eta_1^y)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} \eta_3^y \eta_2^y + \eta_2^y \right)}{(1 + \rho + \gamma (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y)) \left(\frac{\gamma(1-\eta_2^c \eta_1^y)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} + \gamma \eta_2^y} \quad (46)$$

Basic education investments may be finally found by replacing the optimal income tax rate (46) in (43):

$$e_t^{c,i} = \frac{(1 + \rho + \gamma \eta_2^c \eta_1^y) \left(\frac{\gamma(1-\eta_2^c \eta_1^y)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} \eta_2^c \eta_1^y \phi x_t^{a,i}}{(1 - \eta_2^c \eta_1^y) \left((1 + \rho + \gamma (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y)) \left(\frac{\gamma(1-\eta_2^c \eta_1^y)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} + \gamma \eta_2^y \right)} \quad (47)$$

1.2.5 Public basic and private advanced education

When basic education is public, parents do not choose the level of investment in the education of each child. Instead, the government will choose the same amount of education for all children in the economy. During their offspring's youth, parents can choose how much to invest in advanced education. To fund basic public education, governments levy a proportional labor income tax on adults. The budget constraint of adults and the elderly people are given below.

$$c_t^{a,i} + s_t^i = (1 - \nu_t) w h_t^{a,i} (1 - \phi n_t^i) \quad (48)$$

$$c_{t+1}^{o,i} = (1 + r) s_t^i - w n_t^i e_{t+1}^{y,i} \bar{h}_{t+1}^a \quad (49)$$

The human capital production function considers not only that all children will be served the same amount of education during the period of basic public education, but also that the choice of investing in advanced education is private and, therefore, varies according to the decisions of adults.

$$h_{t+2}^{a,i} = \mu (h_t^{a,i})^{\eta_1^c \eta_1^y} (\bar{e}_t^c)^{\eta_2^c \eta_1^y} (\bar{h}_t)^{\eta_3^c \eta_1^y} (e_{t+1}^{y,i})^{\eta_2^y} (\bar{h}_{t+1})^{\eta_3^y} \quad (50)$$

As in the completely public regime, only adults participate in the electoral process. They vote for the size of government by choosing the income tax, ν_t^i , that maximizes their indirect utility function. On the election, adults consider that the government must respect its budget constraint.

$$w\bar{e}_t^c\bar{h}_t \left(P_t^{a,A}n_t^A + P_t^{a,B}n_t^B \right) = w\nu_t \left(P_t^{a,A}h_t^{a,A} (1 - \phi n_t^A) + P_t^{a,B}h_t^{a,B} (1 - \phi n_t^B) \right) \quad (51)$$

Equilibrium under the public basic and private advanced educational regime

Definition 4: Given initial endowments of human capital $(h_0^{a,i}, h_1^{a,i})_{i=A,B}$, initial group size $(P_0^{a,i}, \sqrt{n_0^i}P_0^{a,i})_{i=A,B}$, an initial stock of physical capital (K_0) , an equilibrium under public basic and private advanced education consists of a sequence of prices $\{r_t, w_t\}$, aggregate quantities $\{L_t, K_{t+1}, \bar{h}_t^a\}$, group sizes $\{P_{t+1}^{a,i}\}_{i=A,B}$, decision rules $\{c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i, e_{t+1}^{y,i}\}_{i=A,B}$, and policy variables $\{\bar{e}_t^c, \nu_t\}$ such that:

- 1 - the adults' decision rules $c_t^{a,i}, c_{t+1}^{o,i}, s_t^i, n_t^i$ and $e_{t+1}^{y,i}$ maximize utility function (3) subject to constraints (48), (49), (50);
- 2 - given the decision rules, the labor income tax ν_t^i is chosen by adults to maximize indirect utility;
- 3 - the government's budget constraint (51) is satisfied;
- 4 - the firm's choice L_t^d and K_t^d maximize profits;
- 5 - the prices w_t, r_t are such that markets clear, i.e., (8), (9) and (11) hold;
- 6 - group populations evolve according to (1);
- 7 - the aggregate variables $\zeta_t^a, \bar{h}_t^a$ are given by (2) and (10).

The first-order condition of adult problem decision is:

$$n = \frac{\gamma(1 - \eta_2^y)}{\phi(1 + \rho + \gamma)} \quad (52)$$

$$s_t^i = \frac{(\rho + \gamma\eta_2^y)(1 - \nu_t)wh_t^{a,i}}{1 + \rho + \gamma} \quad (53)$$

$$e_{t+1}^{y,i} = \frac{\phi\eta_2^y(1 + r)(1 - \nu_t)x_t^{a,i}}{g_t(1 - \eta_2^y)} \quad (54)$$

Note that, we checked a posteriori that, in all educational regimes, the time spent by parents to raise their children is not binding, i.e., ϕn is on the unit interval $(0, 1)$ in all educational regimes.

The advanced education elasticity of human capital plays an important role in the choice of fertility and investments in advanced education. The increase in elasticity makes parents invest more in advanced education because schooling has become more productive $\left(\frac{\partial e_{t+1}^{y,i}}{\partial \eta_2^y} > 0\right)$. On the other hand, as the elasticity concerning advanced education rises, the number of children desired by adults decreases $\left(\frac{\partial n}{\partial \eta_2^y} < 0\right)$. It happens because adults must satisfy the budget constraint. In this education regime, effective basic education does not affect the decisions of

adults. Given the choice of quantity of children, the government budget constraint is simplified to:

$$\nu_t = \frac{\bar{e}_t^c \gamma (1 - \eta_2^y)}{\phi (1 + \rho + \gamma \eta_2^y)} \quad (55)$$

Adults vote for government size given the equilibrium choices of their decision problem and considering the government budget constraint. The labor income tax is:

$$\nu_t = \frac{\gamma \eta_2^c \eta_1^y}{1 + \rho + \gamma (\eta_2^y + \eta_2^c \eta_1^y)} \quad (56)$$

Note that labor income tax increases with the elasticity concerning effective basic education ($\frac{\partial \nu_t}{\partial \eta_2^c \eta_1^y} > 0$), while the opposite is true regarding the advanced education elasticity of human capital ($\frac{\partial \nu_t}{\partial \eta_2^y} < 0$). Only basic education is funded by labor income tax, so as basic education becomes more productive, more adults are willing to pay taxes.

Since government revenue is already known, the level of investment in basic education is set to keep the government constraint satisfied. The same result was found in the case the government maximizes the welfare of the population alive given its budget constraint.

$$\bar{e}_t^c = \frac{\eta_2^c \eta_1^y \phi (1 + \rho + \gamma \eta_2^y)}{(1 - \eta_2^y) (1 + \rho + \gamma (\eta_2^y + \eta_2^c \eta_1^y))} \quad (57)$$

1.3 The Balanced Growth Path

We are interested in studying a particular solution of this model in which the growth rate of the economy is constant in the equilibrium. Since physical capital completely depreciates at each period, the economic growth is equal to the human capital growth g_t .

To investigate long-term dynamics of the economy under each educational regime, it would be best to rewrite the equilibrium decisions as functions of constant variables on the balanced growth path. We used the following auxiliary variables to describes long- term dynamics: the relative size of the population of group A, ζ_t^a , the relative human capital, $x_t^{a,i}$, and the average human capital growth rate, g_t .

In all educational regimes, the fertility choice is constant across time and group of families. Therefore, the motion law for the relative size of group A, ζ_t^a , is:

$$\zeta_{t+1}^a = \frac{P_{t+1}^{a,A}}{P_{t+1}^{a,A} + P_{t+1}^{a,B}} = \frac{P_{t-1}^{a,A} n}{P_{t-1}^{a,A} n + P_{t-1}^{a,B} n} = \frac{P_{t-1}^{a,A}}{P_{t-1}^{a,A} + P_{t-1}^{a,B}} = \zeta_{t-1}^a \quad (58)$$

Using the motion law of the relative size of group A and the assumption about the initial size of the population, we find that:

$$\zeta_3^a = \frac{P_3^{a,A}}{P_3^{a,A} + P_3^{a,B}} = \frac{P_1^{a,A}}{P_1^{a,A} + P_1^{a,B}} = \frac{\sqrt{n} P_0^{a,A}}{\sqrt{n} P_0^{a,A} + \sqrt{n} P_0^{a,B}} = \zeta_0^a$$

By induction, the relative size of group A, ζ_t^a , is constant over time and equal to the initial exogenous relative size, ζ_0^a .

1.3.1 Basic and advanced private educational regime:

We replace the optimal education choices (17) and (18) in the production function of human capital of adults (6):

$$x_{t+1}^{a,i} = \mu (x_{t-1}^a)^{\eta_1^y(\eta_1^c + \eta_2^c) + \eta_2^y} \left(\frac{1}{g_{t-1}} \right)^{1 - \eta_3^y + \eta_2^y} \left(\frac{1}{g_t} \right) \left(\frac{\phi}{1 - \eta_2^c \eta_1^y - \eta_2^y} \right)^{\eta_2^c \eta_1^y + \eta_2^y} (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} (\eta_2^y (1+r))^{\eta_2^y} \quad (59)$$

Given the definition of the relative size of population of group A, (2), and the relative human capital, (14), we know that:

$$1 = \zeta_0^a x_{t+1}^{a,A} + (1 - \zeta_0^a) x_{t+1}^{a,B} \quad (60)$$

We found the average human capital growth rate using the law motion of relative human capital of each group (60):

$$g_t = \mu \left(\frac{\eta_2^c \eta_1^y \phi}{1 - \eta_2^c \eta_1^y - \eta_2^y} \right)^{\eta_2^c \eta_1^y} \left(\frac{1}{g_{t-1}} \right)^{1 - \eta_3^y + \eta_2^y} \left(\frac{\eta_2^y \phi (1+r)}{1 - \eta_2^c \eta_1^y - \eta_2^y} \right)^{\eta_2^y} \left\{ \zeta_0^a (x_{t-1}^{a,A})^{\eta_1^y(\eta_1^c + \eta_2^c) + \eta_2^y} + (1 - \zeta_0^a) (x_{t-1}^{a,B})^{\eta_1^y(\eta_1^c + \eta_2^c) + \eta_2^y} \right\} \quad (61)$$

Using (59), (60) and (61), we found the motion law of the relative human capital of group A:

$$x_{t+1}^{a,A} = \frac{1}{\zeta_0^a + (1 - \zeta_0^a)^{1 - \eta_1^y(\eta_1^c + \eta_2^c) - \eta_2^y} \left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^{\eta_1^y(\eta_1^c + \eta_2^c) + \eta_2^y}} \quad (62)$$

The average human capital growth rate under a completely private educational regime, therefore, is:

$$g_t = \mu \left(\frac{\eta_2^c \eta_1^y \phi}{1 - \eta_2^c \eta_1^y - \eta_2^y} \right)^{\eta_2^c \eta_1^y} \left(\frac{1}{g_{t-1}} \right)^{1 - \eta_3^y + \eta_2^y} \left(\frac{\eta_2^y \phi (1+r)}{1 - \eta_2^c \eta_1^y - \eta_2^y} \right)^{\eta_2^y} \times \left\{ \zeta_0^a (x_{t-1}^{a,A})^{\eta_1^y(\eta_1^c + \eta_2^c) + \eta_2^y} + (1 - \zeta_0^a)^{1 - \eta_1^y(\eta_1^c + \eta_2^c) - \eta_2^y} \left(1 - \zeta_0^a x_{t-1}^{a,A} \right)^{\eta_1^y(\eta_1^c + \eta_2^c) + \eta_2^y} \right\} \quad (63)$$

1.3.2 Basic and advanced public educational regime:

We did the same we had done in the previous subsection and found the following dynamic system for an economy under a completely public educational regime:

$$\zeta_{t-1}^a = \zeta_{t+1}^a = \zeta_0^a \quad (64)$$

$$x_{t+1}^{a,A} = \frac{1}{\zeta_0^a + (1 - \zeta_0^a)^{1-\eta_1^c \eta_1^y} \left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^{\eta_1^c \eta_1^y}} \quad (65)$$

$$g_t = \mu \left(\frac{(1 + \rho) \phi \left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}}}{(1 + \rho + \gamma (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y)) \left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}} + \gamma \eta_2^y} \right)^{\eta_2^c \eta_1^y + \eta_2^y} (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} \left(\frac{1}{g_{t-1}} \right)^{1-\eta_3^y} \times \quad (66)$$

$$\times \left(\left(\frac{\gamma}{(1 + \rho + \gamma)\phi} \right)^{\frac{1}{2}} \eta_3^y \eta_2^y + \eta_2^y \right)^{\eta_2^y} \left(\zeta_0^a (x_{t-1}^{a,A})^{\eta_1^c \eta_1^y} + (1 - \zeta_0^a)^{1-\eta_1^c \eta_1^y} (1 - \zeta_0^a x_{t+1}^{a,A})^{\eta_1^c \eta_1^y} \right)$$

On a completely public regime, only the parental human capital elasticity, η_1^c and childhood human capital elasticity, η_1^y matters for the dynamic of the relative human capital.

1.3.3 Private basic and public advanced educational regime:

The following dynamic system describes the economy under a private basic and public advanced educational regime:

$$\zeta_{t+1}^a = \zeta_{t-1}^a = \zeta_0^a \quad (67)$$

$$x_{t+1}^{a,A} = \frac{1}{\zeta_0^a + (1 - \zeta_0^a)^{1-\eta_1^y (\eta_1^c + \eta_2^c)} \left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^{\eta_1^y (\eta_1^c + \eta_2^c)}} \quad (68)$$

$$g_t = \mu \left(\frac{\phi (1 + \rho + \gamma (\eta_2^c \eta_1^y)) \left(\frac{\gamma (1 - \eta_2^c \eta_1^y)}{\phi (1 + \rho + \gamma)} \right)^{\frac{1}{2}}}{(1 - \eta_2^c \eta_1^y) \left((1 + \rho + \gamma (\eta_2^c \eta_1^y + \eta_3^y \eta_2^y)) \left(\frac{\gamma (1 - \eta_2^c \eta_1^y)}{\phi (1 + \rho + \gamma)} \right)^{\frac{1}{2}} + \gamma \eta_2^y \right)} \right)^{\eta_2^c \eta_1^y + \eta_2^y} (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} \left(\frac{1}{g_{t-1}} \right)^{1-\eta_3^y} \times \quad (69)$$

$$\times \left(\left(\frac{\gamma (1 - \eta_2^c \eta_1^y)}{\phi (1 + \rho + \gamma)} \right)^{\frac{1}{2}} \eta_3^y \eta_2^y + \eta_2^y \right)^{\eta_2^y} \left(\zeta_0^a (x_{t-1}^{a,A})^{\eta_1^y (\eta_1^c + \eta_2^c)} + (1 - \zeta_0^a)^{1-\eta_1^y (\eta_1^c + \eta_2^c)} \left(\frac{1 - \zeta_0^a x_{t-1}^{a,A}}{1 - \zeta_0^a} \right)^{\eta_1^y (\eta_1^c + \eta_2^c)} \right)$$

1.3.4 Public basic and private advanced educational regime:

The following dynamic system describes the economy under a mixed educational regime in which basic schooling is public and advanced education is private:

$$\zeta_{t+1}^a = \zeta_{t-1}^a = \zeta_0^a \quad (70)$$

$$x_{t+1}^{a,A} = \frac{1}{\zeta_0^a + (1 - \zeta_0^a)^{1-\eta_1^c\eta_1^y-\eta_2^y} \left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^{\eta_1^c\eta_1^y+\eta_2^y}} \quad (71)$$

$$g_t = \mu \left(\frac{\phi(1 + \rho + \gamma\eta_2^y)}{(1 - \eta_2^y)(1 + \rho + \gamma(\eta_2^y + \eta_2^c\eta_1^y))} \right)^{\eta_2^c\eta_1^y+\eta_2^y} (\eta_2^c\eta_1^y)^{\eta_2^c\eta_1^y} \left(\frac{1}{g_{t-1}} \right)^{1-\eta_3^y+\eta_2^y} (\eta_2^y(1+r))^{\eta_2^y} \times \quad (72)$$

$$\times \left(\zeta_0^a \left(x_{t-1}^{a,A} \right)^{\eta_1^c\eta_1^y+\eta_2^y} + (1 - \zeta_0^a)^{1-(\eta_1^c\eta_1^y+\eta_2^y)} \left(1 - \zeta_0^a x_{t+1}^{a,A} \right)^{\eta_1^c\eta_1^y+\eta_2^y} \right)$$

The dynamic systems under different educational regimes are quite similar. They differ only on the powers in the denominator of $x_{t+1}^{a,A}$. Therefore, we can analyze the stability of a representative system below and extend the analysis to all models:

$$\zeta_{t+1}^a = \zeta_{t-1}^a = \zeta_0^a$$

$$x_{t+1}^{a,A} = \frac{1}{\zeta_0^a + (1 - \zeta_0^a)^{1-z} \left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^z} \quad (73)$$

This system has two fixed points: $(\zeta_0^a, 1)$ and $(\zeta^a, 0)$. The first one is the equilibrium in which inequality vanishes, which means that all the population has an equal amount of human capital. The second one is the equilibrium in which inequality is as severe as possible.

The behavior of the dynamic equation (73) crucially depends on the value of z , that is, the combination of a few elasticities of the human capital accumulation function. As already discussed, this combination of elasticities changes in different educational regimes. The value of z determines if the economy will converge to a balanced growth path in which inequality vanishes or to a balanced growth path with the highest inequality. If $z < 1$, then the fixed-point $(\zeta_0^a, 1)$ is globally stable. If $z > 1$, then the fixed-point $(\zeta^a, 0)$ is globally stable. If $z = 1$, the economy stays at its initial point and, therefore, there is an infinite number of fixed-points in the range $[0, 1]$, depending on the initial relative size of the population and the initial human capital of each group. Let us make one more assumption to ensure that the economy in all educational regimes has $z < 1$:

Assumption 3: $\eta_1^c\eta_1^y + \eta_2^c\eta_1^y + \eta_2^y < 1$.

Proposition 1. *Under assumption 3, the economy will converge to a globally stable balanced growth path with no inequality in all educational regimes.*

Proof. See Appendix □

As stated in Proposition 1, under Assumption 3, all educational regimes converge to an equilibrium in which inequality vanishes. However, in the absence of Assumption 3, the economy under some educational regimes could converge to an equilibrium in which the inequality is the most severe, while under other educational regimes, the economy could converge to equilibrium with no inequality. It is worth to note that in a completely public regime, the global stability to the balanced growth path with no inequality does not depend on Assumption 3, and always converges to equilibrium with no inequality, because of $\eta_1^c \eta_1^y < 1$.

This result is closely related the one found in De la Croix and Doepke (2004). They found that both under private and public education, there exists a balanced growth path in which all inequality has vanished. However, on a public education regime, the equilibrium is globally stable, while in the private regime the stability is only local. In such a case, the economy does not converge to a balanced growth path, and the income difference between the two groups can grow without bounds. In our model, in the absence of Assumption 3, the completely private regime could converge to the balanced growth path if the highest inequality, but, as in De la Croix and Doepke (2004), the completely public regime always converge to the equilibrium with no inequality.

1.3.5 The implications for income inequality:

We are interested in comparing the four educational regimes regarding the speed of convergence for its balanced growth path in which inequality vanishes. For this, we used the concept of half-life, that is, the time it takes for the economy to move halfway to long-term equilibrium.

Once more, the analysis relies on the value of the power z in each educational regime. The lower the amount of z , the faster the convergence to the equilibrium without inequality and the lower the half-life.

Proposition 2. *Under assumption 3:*

- (i) *The completely public educational regime is the one with the fastest convergence to the balanced growth path with no inequality;*
- (ii) *The completely private educational regime is the one with the slowest convergence to the balanced growth path with no inequality;*
- (iii) *For mixed education regimes: (a) if $\eta_1^y \eta_2^c > \eta_2^y$, then the public basic and private advanced education regime converge faster than the private basic and public educational regime to the balanced growth path with no inequality; (b) if $\eta_1^y \eta_2^c < \eta_2^y$, then the private basic and*

public educational regime converge faster than the public basic and private advanced education regime to the balanced growth path with no inequality.

Proof. See Appendix □

The completely private educational regime has the highest value of $z = \eta_1^y (\eta_1^c + \eta_2^c) + \eta_2^y$, therefore the convergence to the balanced growth path is the slowest among educational regimes. On the other hand, the completely public regime has the lowest value of $z = \eta_1^c \eta_1^y$, which means that the economy under this education regime converges to the equilibrium faster than any other educational regime.

For mixed educational regimes, the convergence speed depends on who funds the education stage with the highest elasticity. As the empirical evidence suggests, basic education is the schooling stage with the highest productivity, i.e, the effective basic education elasticity is higher than the elasticity of advanced education, $\eta_2^c \eta_1^y > \eta_2^y$. In this case, the power of the denominator of the relative human capital dynamics is lower when basic education is public and the advanced stage is private, $\eta_2^c \eta_1^y + \eta_2^y$, when compared to private basic education and public advanced education, $\eta_2^c \eta_1^y + \eta_2^c \eta_1^y$. Therefore, inequality vanishes faster when basic education is public.

1.3.6 Growth on Balanced Growth Path:

Besides studying the speed of convergence for the equilibrium without inequality, it is crucial to examine the long-term economic growth rate. For the society as a whole, a regime that takes a little longer to overcome inequality but shows permanently higher levels of economic growth could do better than a regime with a fast reduction of inequality but low economic growth. In such a case, everyone would be equally poor. In this section, we studied the average human capital growth on the Balanced Growth Path in each educational regime and tried to establish the relationship between economic growth and the dynamics of inequality. Once the balanced growth is reached, the human capital growth is:

Basic and advanced private educational regime:

$$g^{priv-priv} = \left(\mu \left(\frac{\phi}{1 - \eta_2^c \eta_1^y - \eta_2^y} \right)^{\eta_2^c \eta_1^y + \eta_2^y} (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} (\eta_2^y (1 + r))^{\eta_2^y} \right)^{\frac{1}{1 + \eta_1^y + \eta_2^y}} \quad (74)$$

Basic and advanced public educational regime:

$$g^{pub-pub} = \left(\mu \left(\frac{(1 + \rho) \phi \left(\frac{\gamma}{(1 + \rho + \gamma) \phi} \right)^{\frac{1}{2}}}{(1 + \rho + \gamma (\eta_1^y (\eta_2^c - \eta_2^y) + \eta_2^y)) \left(\frac{\gamma}{(1 + \rho + \gamma) \phi} \right)^{\frac{1}{2}} + \gamma \eta_2^y} \right)^{\eta_2^c \eta_1^y + \eta_2^y} (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} \left(\left(\frac{\gamma}{(1 + \rho + \gamma) \phi} \right)^{\frac{1}{2}} (1 - \eta_1^y) \eta_2^y + \eta_2^y \right)^{\eta_2^y} \right)^{\frac{1}{1 + \eta_1^y}} \quad (75)$$

Basic private education and advanced public educational regime:

$$g^{priv-pub} = \left(\mu \left(\frac{(1 + \rho + \gamma \eta_2^c \eta_1^y) \phi \left(\frac{\gamma(1 - \eta_2^c \eta_1^y)}{\phi(1 + \rho + \gamma)} \right)^{\frac{1}{2}}}{(1 - \eta_2^c \eta_1^y) \left((1 + \rho + \gamma (\eta_1^y (\eta_2^c - \eta_2^y) + \eta_2^y)) \left(\frac{\gamma(1 - \eta_2^c \eta_1^y)}{\phi(1 + \rho + \gamma)} \right)^{\frac{1}{2}} + \gamma \eta_2^y \right)} \right) \right)^{\eta_2^c \eta_1^y + \eta_2^y} (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} \right)^{\frac{1}{1 + \eta_1^y}} \times$$

$$\times \left(\left(\frac{\gamma(1 - \eta_2^c \eta_1^y)}{\phi(1 + \rho + \gamma)} \right)^{\frac{1}{2}} (1 - \eta_1^y) \eta_2^y + \eta_2^y \right)^{\frac{\eta_2^y}{1 + \eta_1^y}} \quad (76)$$

Basic public education and advanced private educational regime:

$$g^{pub-priv} = \left(\mu \left(\frac{\phi(1 + \rho + \gamma \eta_2^y)}{(1 - \eta_2^y)(1 + \rho + \gamma(\eta_2^y + \eta_2^c \eta_1^y))} \right)^{\eta_2^c \eta_1^y + \eta_2^y} (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} (\eta_2^y (1 + r))^{\eta_2^y} \right)^{\frac{1}{1 + \eta_1^y + \eta_2^y}} \quad (77)$$

Proposition 3. *Under Assumption 3, given any type of advanced education funding, the economy grows more when basic education is private:*

- (i) $g^{priv-priv} > g^{pub-priv}$;
- (ii) $g^{priv-pub} > g^{pub-pub}$.

Proof. See Appendix □

Proposition 3 states that, given the type of advanced education, in the long-term, under private basic education regimes, the economy will grow permanently more. Therefore, for a higher economic growth, basic education must be private. Moreover, according to Proposition 3, the completely public regime and the mixed regime, in which basic schooling is public and advanced education is private, are not the educational regimes that drive the economy to higher growth level. Although the completely public regime is the one in which the inequality vanishes the fastest, regarding economic growth, there will always be at least one better educational regime to adopt.

All other comparisons between economic growth rates under different educational regimes depend on parameter values. However, some regularities may be established. Let us argue, based on Proposition 3, that there are only six possibilities of ordering educational regimes by resulting economic growth rate, which are:

- (i) $g^{priv-priv} > g^{pub-priv} \geq g^{priv-pub} > g^{pub-pub}$;
- (ii) $g^{priv-pub} \geq g^{priv-priv} > g^{pub-priv} \geq g^{pub-pub}$;

- (iii) $g^{priv-pub} > g^{pub-pub} \geq g^{priv-priv} > g^{pub-priv}$;
- (iv) $g^{priv-priv} \geq g^{priv-pub} > g^{pub-priv} \geq g^{pub-pub}$;
- (v) $g^{priv-priv} \geq g^{priv-pub} > g^{pub-pub} \geq g^{pub-priv}$;
- (vi) $g^{priv-pub} \geq g^{priv-priv} > g^{pub-pub} \geq g^{pub-priv}$.

To compare growth rates, we first set the values of some parameters. The aggregate multiplicative constant μ does not affect decisions and growth rate differentials, so it's set to $\mu = 1$. Although discount factor ρ affects choices, it does not qualitatively influence growth rate differentials; and, as De la Croix and Doepke (2003), we chose a standard value in real-business-cycle literature, $\rho = 0.55$ (i.e., 0.99 per quarter). The interest rate, r , the elasticities of human capital, η_s , the time allocated by parents to raise a child, ϕ , and the altruism factor, γ , all influence growth rate differentials qualitatively, so we do not set a value for them.

Figure 1: Economic growth and parameter values

Figure 1 shows the ordering of the growth rates for different parameter values. Note that only the elasticities of human capital concerning basic education, η_2^c , advanced education, η_2^y ,

and human capital produced during childhood, η_1^y , matter to economic growth rates on the Balanced Growth Path. The first regularity we observed is that, for reasonable values of elasticities of human capital, $\eta_s = (\eta_2^c, \eta_1^y, \eta_2^y)$, the rank (i) is the only one that occurs, regardless of the values of other parameters. After a comparison between panel (a) and other panels in Figure 1, we concluded that there is a threshold for elasticities of human capital beyond which the order of growth rates (i) is unique. This threshold changes very little with the altruism factor, γ . For instance, if parents consider their children as important as themselves, $\gamma = 1$, the threshold of elasticities would be 0.09.²

According to a sequence (i), the economy under a completely private regimes, the economy shows the highest growth rates when compared to all other educational regimes. Moreover, sequence (i) suggests that private advanced education is always better for growth. The completely public regime is responsible for the lowest economic growth in equilibrium. Therefore, there is a trade-off between economic growth and the reduction of inequality. Although completely private systems take the longest to approach the equilibrium without inequality, it is the education system that drives the economy to the highest growth. The opposite happens when education systems are public. Reduction of inequality is fast, but growth rates are permanently lower when compared to other educational regimes. For mixed educational regimes, this trade-off does not happen. As already discussed, the empirical literature suggests that basic education is the stage with the highest productivity for investments in education. In such case, inequality is reduced faster, and the economy grows more under a mixed model in which primary education is public.

When these elasticities are low, growth rates may be ordered differently. The low elasticities mean that the productivity of investments in education on both stages is very low. Moreover, given Assumption 1, when these elasticities are low, the human capital of teachers, and thus the externality of average human capital, is the most important input to human capital production during youth. In such case, since governments internalize this externality during youth, advanced public education could generate higher growth.

Even so, other patterns only occur if interest rates are low. The interest rate in Figure 1 is the interest rate accumulated in fifteen years. For instance, $r = 0.56$ is equivalent to an interest rate of 3% a.a.. At low interest rates, the remuneration of savings is low, so income drops in old age. The combination of low interest rates and low elasticity of human capital causes parents to reduce investments in the advanced education of their offspring when advanced education is private. In such case, the mixed education system in which advanced education is public, $g^{priv-pub}$, presents the highest economic growth. The reason for that is that government internalize this externality on the second stage of education and, as adults pay income tax, the low interest rates do not affect the level of public investment in advanced education.

²Heckman, Cunha, and Schennach (2010) estimate these elasticities for different stages of childhood. Their econometric results shows that the elasticity of human capital concerning the investments in early childhood education is 0,231, the elasticity of human capital of adults concerning to the stock of human capital accumulated during childhood stage is 0,902, and the elasticity of human capital of children concerning parental human capital is 0,05.

The black area is the combination of parameters in which sequence (iii) happens. On sequence (iii), it is always better if governments finance advanced education. In such case, completely public education systems grow more than the completely private education regimes. Sequence (iii) happens when parents dedicate much of their time to raise their children, interest rates and elasticities of human capital are low. For countries wherein parents culturally spent more time with their offspring, completely public education systems could be better for growth than completely private education systems. In such case, the trade-off between growth and reduction of inequality disappears. Note that, when the educational regimes are completely private, the investment in advanced education increases with ϕ at a constant rate on the BGP $\left(\frac{\partial e_{t+1}^y}{\partial \phi} > 0; \frac{\partial^2 e_{t+1}^y}{\partial \phi^2} = 0\right)$. However, in public educational regimes, the investment in advanced education increases with ϕ at a decreasing rate $\left(\frac{\partial \bar{e}_{t+1}^y}{\partial \phi} > 0; \frac{\partial^2 \bar{e}_{t+1}^y}{\partial \phi^2} < 0\right)$. If the elasticities of human capital are low and ϕ is high, it could be better to invest less in advanced education. Note that, as the altruism factor increases the black area in Figure 1 increases.³

1.4 Conclusion

The aim of this paper was to explore the impacts of different educational regimes on economic growth and the dynamic of inequality in a framework that accounts for the hierarchical nature of education. The analysis leads to two main conclusions. First, considering the reduction of inequality, we found that the completely public regime converge faster than all others to the equilibrium with no inequality. The opposite happens when educational regimes are entirely private. In this case, if the economy converges to the equilibrium with no inequality, the reduction of inequality is the slowest among the regimes. When educational regimes are mixed, our results suggest that for a faster reduction of inequality, governments should fund the educational stage with the highest elasticity.

Second, comparing the educational regimes regarding the economic growth rate in the equilibrium with no inequality, our results show that the regime with the highest growth rate is always associated with private basic education. Thus, if the goal of society is to maximize economic growth, then our results suggest that it should be privately funded. However, if the basic education is the most productive one, as suggested in the literature, then it should be funded publicly to a faster reduction of inequality. Therefore, there is a trade-off between economic growth and the reduction of inequality. The choice of the regime would rely on the social welfare function adopted. Sen (1973) and Foster and Sen (1997) suggested a social welfare function that considers the inequality of society. Both of these social welfare function consider the average income and an inequality index. The first one uses the Gini Index, while the second uses the Atkinson's Indexes. If the social welfare function considers inequality, it could be better to choose an education regime in which the convergence to the equilibrium with no inequality is

³For United State, De la Croix and Doepke (2004) argue that the time-cost parameter $\phi = 0.075$.

fast.

An interesting extension would be allowing young people to work and use their labor income to finance advanced education, since the interest rate, which is determined exogenously, qualitatively influences growth rate differentials. In the same direction, another extension would be let parents choose how much of time they want to spend raising a child.

Appendix

A: Global Stability

Proof of Proposition 1:

First we need to prove that if $x_t^{a,A} < 1$, $x_{t+\tau}^{a,A} > 1 \forall \tau > 0$ do not happen. Suppose that $x_{t-1}^{a,A} < 1$ and to the contrary that $x_{t+1}^{a,A} > 1$. Replacing $x_{t+1}^{a,A}$ for its definition (74):

$$\frac{1}{\zeta_0^a + (1 - \zeta_0^a)^{1-z} \left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^z} > 1$$

$$(1 - \zeta_0^a)^{1-z} \left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^z < 1 - \zeta_0^a$$

$$\left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^z < (1 - \zeta_0^a)^z$$

$$\left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right) < (1 - \zeta_0^a)$$

This implies that the inverse of $x_{t-1}^{a,A}$ is smaller than 1:

$$\frac{1}{x_{t-1}^{a,A}} < 1$$

So, $x_{t-1}^{a,A} > 1$, which is a contradiction.

Therefore, we can study the dynamic of relative human capital of the low-skilled people $x_{t+1}^{a,A}$ in the range $[0, 1]$. Let $z < 1$, suppose that $x_{t+1}^{a,A} < x_{t-1}^{a,A}$. Then,

$$x_{t+1}^{a,A} = \frac{1}{\zeta_0^a + (1 - \zeta_0^a)^{1-z} \left(\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a \right)^z} < x_{t-1}^{a,A} \Rightarrow$$

$$\frac{1}{\zeta_0^a + (1 - \zeta_0^a)^{1-z} \left(\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} \right)^z} < x_{t-1}^{a,A} \Rightarrow \zeta_0^a + (1 - \zeta_0^a)^{1-z} \left(\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} \right)^z > \frac{1}{x_{t-1}^{a,A}} \Rightarrow$$

$$\left(\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} \right)^z > \frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a}$$

This implies that:

$$\left(\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} \right)^{1-z} < 1,$$

which is a contradiction.

If $z < 1$, then $x_{t+1}^{a,A} \geq x_{t-1}^{a,A}$, $\forall x_{t-1}^{a,A} > 0$. This means that the dynamic system converges to the fixed-point $(\zeta_0^a, 1)$, regardless the initial endowments in the economy. Therefore, the fixed-point $(\zeta_0^a, 1)$ is globally stable.

Now, let $z > 1$, suppose that $x_{t-1}^{a,A} < x_{t+1}^{a,A}$, then

$$x_{t-1}^{a,A} < \frac{1}{\zeta_0^a + (1 - \zeta_0^a) \left(\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} \right)^z} \Rightarrow$$

$$\zeta_0^a + (1 - \zeta_0^a) \left(\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} \right)^z < \frac{1}{x_{t-1}^{a,A}} \Rightarrow \left(\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} \right)^z < \frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a}$$

This implies that

$$\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} < 1$$

Which is a contradiction.

Therefore, $x_{t+1}^{a,A} \leq x_{t-1}^{a,A}$, $\forall x_{t-1}^{a,A} \in [0, 1]$, the economy converges to a balanced growth path with the highest inequality.

If $z = 1$, then $x_{t+1}^{a,A} = x_{t-1}^{a,A}$, $\forall x_{t-1}^{a,A} \in [0, 1]$. This means that, in equilibrium, the economy is at its starting point.

B: Speed of Convergence

Proof of Proposition 2:

Let the dynamic equation of the relative human capital of an adult of group A be $\Theta^j(x_{t-1}^{a,A}, \zeta_0^a, z) = \frac{1}{\left\{ \zeta_0^a + (1 - \zeta_0^a)^{1-z} \left(\frac{\frac{1}{x_{t-1}^{a,A}} - \zeta_0^a}{1 - \zeta_0^a} \right)^z \right\}}$, where j indicates the education regime. First, we want to prove that the curves Θ^j of the distinct educational regimes do not cross each other. The only difference among the education regimes is the value of z . Suppose that $x_{t-1}^{a,A} \in (0, 1)$ and $\zeta_0^a \in (0, 1)$. The first-order derivative with relation to z is negative.

$$\frac{\partial \Theta^j}{\partial z} = \left(\ln \left(\frac{1 - \zeta_0^a x_{t-1}^{a,A}}{x_{t-1}^{a,A} - \zeta_0^a x_{t-1}^{a,A}} \right) \right) (\zeta_0^a - 1) \frac{\left(\frac{1 - \zeta_0^a x_{t-1}^{a,A}}{x_{t-1}^{a,A} - \zeta_0^a x_{t-1}^{a,A}} \right)^z}{\left(\zeta_0^a + \left(\frac{\zeta_0^a x_{t-1}^{a,A} - 1}{x_{t-1}^{a,A} (\zeta_0^a - 1)} \right)^z - \zeta_0^a \left(\frac{\zeta_0^a x_{t-1}^{a,A} - 1}{x_{t-1}^{a,A} (\zeta_0^a - 1)} \right)^z \right)^2} < 0$$

The lower z , the larger Θ^j in every point of the considered range. Therefore, the dynamic of the different educational regime does not cross each other.

Now, we can prove that the lower u , the faster the economy converges to a balanced growth path with no inequality. We define half-life to the balanced growth path in which inequality is vanished as

$$x_{t+half-life(x_t^{a,A})}^{a,A} = \frac{x_t^{a,A} + 1}{2}$$

As the dynamic of the relative human capital is described by an second-order difference equation, if we calculate the composition of Θ^j and Θ^j by $\frac{half-life(x_t^{a,A})}{2}$ times, we will find a $x^{a,A}$ very close to $x_{t+half-life(x_t^{a,A})}^{a,A}$, such that:

$$(\Theta^j)^{\frac{half-life(x_t^{a,A})}{2}} \geq \frac{x_t^{a,A} + 1}{2} \text{ and } (\Theta^j)^{\frac{half-life(x_t^{a,A})}{2} - 1} < \frac{x_t^{a,A} + 1}{2}$$

In particular, if the economy is close enough to equilibrium, we can trace a linear function that passes through the point $x^{a,A} = 1$ and has the same slope of Θ^j in this point:

$$f(x) = 1 + a(x - 1), \text{ where } a = \frac{\partial \Theta^j}{\partial x_{t-1}^{a,A}}(1, \zeta_0^a, z).$$

The composition of $f(\cdot)$ in $f(\cdot)$ are:

$$f^2(x) = 1 + a^2(x - 1)$$

$$f^3(x) = 1 + a^3(x - 1)$$

By induction:

$$f^n(x) = 1 + a^n(x - 1)$$

Therefore, the composition of $f(\cdot)$ in $f(\cdot)$ by $\frac{half-life(x_t^a)}{2}$ times is:

$$f^{\frac{half-life(x)}{2}}(x) = 1 + a^{\frac{half-life(x)}{2}}(x - 1) = \frac{x + 1}{2}$$

This means that:

$$a^{\frac{half-life(x)}{2}} = \frac{1}{2} \Rightarrow \frac{half-life(x)}{2} = \log_a \frac{1}{2} \Rightarrow half-life(x) = \log_a \frac{1}{4} = \frac{\ln \frac{1}{4}}{\ln a}$$

But $a = \frac{\partial \Theta^j}{\partial x_{t-1}^{a,A}} (1, \zeta_0^a, z) = z$. Therefore, z and half-life vary in the same direction. We can conclude that the lower z , the lower is the half-life and the faster is the convergence to the long-term equilibrium.

Comparing the educational regimes, the completely public regime has the lowest $z = \eta_1^c \eta_1^y$, so it has the fastest convergence to a balanced growth path without inequality. In contrast, the completely private regime has the highest $z = \eta_1^y (\eta_1^c + \eta_2^c) + \eta_2^y$, so it has the slowest convergence to the long-term equilibrium. With regards to the two mixed regimes, we cannot conclude which one has the faster convergence. In the private basic and public advanced educational regime, $z = \eta_1^y (\eta_1^c + \eta_2^c)$. In the public basic and private advanced educational regime, $z = \eta_1^c \eta_1^y + \eta_2^y$.

C: Growth

Proof of Proposition 3:

Let us first prove part (i) of Proposition 3. Suppose that, in equilibrium, the growth rate under a completely private educational regime is less than or equal to the growth rate under public basic and private advanced education regime, $g^{priv-priv} \leq g^{pub-priv}$. Therefore,

$$\mu (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} (\eta_2^y (1+r))^{\eta_2^y} \left(\frac{\phi}{1 - \eta_2^c \eta_1^y - \eta_2^y} \right)^{\eta_2^c \eta_1^y + \eta_2^y} \leq \mu (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} (\eta_2^y (1+r))^{\eta_2^y} \left(\frac{\phi (1 + \rho + \gamma \eta_2^y)}{(1 + \rho + \gamma (\eta_2^y + \eta_2^c \eta_1^y)) (1 - \eta_2^y)} \right)^{\eta_2^c \eta_1^y + \eta_2^y}$$

$$\left(\frac{\phi}{1 - \eta_2^c \eta_1^y - \eta_2^y} \right)^{\eta_2^c \eta_1^y + \eta_2^y} \leq \left(\frac{\phi (1 + \rho + \gamma \eta_2^y)}{(1 + \rho + \gamma (\eta_2^y + \eta_2^c \eta_1^y)) (1 - \eta_2^y)} \right)^{\eta_2^c \eta_1^y + \eta_2^y}$$

By the definition of the parameters, we know that $\eta_2^c \eta_1^y + \eta_2^y > 0$. Moreover, under Assumption 2, $1 - \eta_2^c \eta_1^y - \eta_2^y > 0$, thus:

$$\frac{1}{1 - \eta_2^c \eta_1^y - \eta_2^y} \leq \frac{1 + \rho + \gamma \eta_2^y}{(1 + \rho + \gamma (\eta_2^y + \eta_2^c \eta_1^y)) (1 - \eta_2^y)}$$

$$(1 + \rho + \gamma (\eta_2^y + \eta_2^c \eta_1^y)) (1 - \eta_2^y) \leq (1 - \eta_2^c \eta_1^y - \eta_2^y) (1 + \rho + \gamma \eta_2^y)$$

$$\eta_2^c \eta_1^y (1 + \rho + \gamma) \leq 0$$

Which is a contradiction. Therefore, in equilibrium, the growth rate under a completely private educational regime is higher than under public basic and private advanced education regime, $g^{priv-priv} > g^{pub-priv}$.

Now let us prove part (ii) of Proposition 3, we define $g(q)$ as

$$g^j(q) = \left(\mu (\eta_2^c \eta_1^y)^{\eta_2^c \eta_1^y} \left(\left(\frac{\gamma(1-q)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} (1 - \eta_1^y) \eta_2^y + \eta_2^y \right)^{\eta_2^y} \right)^{\frac{1}{1+\eta_1^y}} \times$$

$$\times \left(\frac{(1+\rho+\gamma q) \phi \left(\frac{\gamma(1-q)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}}}{(1-q) \left((1+\rho+\gamma(\eta_2^c \eta_1^y + (1-\eta_1^y) \eta_2^y)) \left(\frac{\gamma(1-q)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} + \gamma \eta_2^y \right)} \right)^{\frac{\eta_2^c \eta_1^y + \eta_2^y}{1+\eta_1^y}}$$

Note that $g^j(\eta_2^c \eta_1^y) = g^{priv-pub} \mathbf{e} g^j(0) = g^{pub-pub}$.

We want to prove that $g^j(q)$ is increasing in q . In order to do that, we can apply logarithm in $g^j(q)$ and calculate the first-order derivative with relation to q .

$$\frac{\partial \ln(g^j(q))}{\partial q} = \frac{\gamma(\eta_2^c \eta_1^y + \eta_2^y)}{(1+\rho+\gamma q)} + \frac{1}{2} \frac{(\eta_2^c \eta_1^y + \eta_2^y)}{(1-q)} + \frac{1}{2} \left(\frac{\gamma}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} \times$$

$$\times * \left(\frac{\left(\frac{\gamma(1-q)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} (1+\rho+\gamma(\eta_2^c \eta_1^y + (1-\eta_1^y) \eta_2^y)) (1-\eta_1^y) (\eta_2^y \eta_2^c \eta_1^y) + \eta_2^y ((\eta_2^c \eta_1^y + \eta_2^y) (1+\rho+\gamma \eta_2^c \eta_1^y) + \gamma (\eta_2^y \eta_2^c \eta_1^y (1-\eta_1^y)))}{(1-\eta_1^y) \eta_2^y (1+\rho+\gamma(\eta_2^c \eta_1^y + (1-\eta_1^y) \eta_2^y)) \left(\frac{\gamma(1-q)}{\phi(1+\rho+\gamma)} \right) + \left(\frac{\gamma(1-q)}{\phi(1+\rho+\gamma)} \right)^{\frac{1}{2}} (2\gamma (\eta_2^y)^2 (1-\eta_1^y) + \eta_2^y (1+\rho+\gamma(\eta_2^c \eta_1^y))) + \gamma (\eta_2^y)^2} \right)$$

Note that all parameters are positive $\phi, \rho, \gamma > 0, q \in (0, 1) \Rightarrow 1 - q > 0$ and $\eta_1 \in (0, 1) \Rightarrow 1 - \eta_1 > 0$. As $\frac{\partial \ln(g^j(q))}{\partial q}$ is the sum of positive factors, then $\frac{\partial \ln(g^j(q))}{\partial q} > 0$. Therefore, $g^j(q)$ is increasing in q . We can conclude that $g^j(\eta_2^c \eta_1^y) = g^{priv-pub} > g^j(0) = g^{pub-pub}$.

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CAPÍTULO II

The Lucas model under heterogeneous agents

Abstract

This paper studies the pattern of distributional dynamics on an endogenous economic growth model with human capital accumulation. We explored the properties of the dynamics of inequality in a simplified form of the Lucas model (1988), in which agents differ in their initial endowments of human capital. Our results show that, along dynamic paths that converge to a Balanced Growth Path, despite the heterogeneity of initial human capital, agents initially choose to study the same amount of time and only one pattern of inequality dynamics emerge. The inequality is constant along dynamic paths that converge to a Balanced Growth Path.

JEL Classification: O40, O15, I25

Key-words: Growth, Heterogeneous Agents, Human Capital, Inequality

2.1 Introduction

This paper studies the pattern of distributional dynamics on a economic growth model with human capital accumulation. Human capital distribution was characterized along dynamic paths that converge to an equilibrium. Our goal was to understand the distributive dynamics that emerge from endogenous economic growth models with a representative consumer in which human capital is the engine of growth.

Economic growth models generally focus on the dynamic behavior of aggregate quantities, but only a few address long-term distributive properties and implications for inequality. However, the empirical literature highlighted the adverse effect of inequality on economic growth (Alesina and Rodrick (1994); Persson and Tabellini (1994)). Moreover, Benabou (1996) concluded that the link between income distribution and growth emerges precisely through investments in human capital. This is particularly pertinent in light of observed long-run changes in wealth and income inequality (Piketty and Saez (2003)). Therefore, understanding the mechanisms behind growth models and their implicit effects on inequality patterns is important, especially with human capital accumulation.

Our paper is part of the literature that studies implicit implications of economic growth models for wealth and income distribution. Stiglitz (1969) characterized the dynamics of inequality in the Solow model and showed that if the balanced growth path is stable, wealth and income are asymptotically evenly distributed. Caselli and Ventura (2000) studied the distributive dynamics that emerge from the Ramsey-Cass-Koopmans model and the Arrow-Romer model and

showed that a wide range of distributive dynamics could arise as the equilibrium outcome. They showed that Representative Consumer per se places very few restrictions on the distribution observed. Garcia-Penãlosa and Turnovsky (2007) studied an AK growth model with elastic labor supply, in which agents differ in their initial endowments of physical capital. They showed that wealth distribution remains fixed at its exogenously given initial level, and income distribution, although endogenously determined, remains constant over time. Our contribution to this literature is to study the dynamics of inequality in an endogenous economic growth model that allows human capital accumulation.

We explored the properties of the dynamics of inequality in a simplified form of the Lucas model (1988), in which agents differ in their initial endowments of human capital. On the one hand, this paper generalizes the Lucas model (1988) by allowing heterogeneity, but, on the other hand, to make this extension treatable, it was necessary to simplify it by not considering physical capital. In the absence of physical capital, it was possible to characterize the solution dynamics from the decentralized and centralized problem.

We showed that the Lucas model with heterogeneity allows a Representative Consumer. For aggregate quantities, the results of the Lucas model (1988) in the absence of physical capital remains. Comparing decentralized and centralized equilibriums, we found that individuals sub-accumulate human capital in the decentralized problem because of the externality of human capital in the production function.

As in the original Lucas model (1988), agents chose their optimal allocation of time between work and study. The only way to accumulate human capital is dedicating time to study. Therefore, by allowing heterogeneity, individuals could accumulate human capital at different growth rates, and the differential of human capital could change over time. However, our results show that, along dynamic paths that converge to a Balanced Growth Path, despite the heterogeneity of initial human capital, agents initially choose to study the same amount of time. Therefore, in contrast to Caselli and Ventura (2000), our results show that, in the decentralized problem, only one pattern of inequality dynamics can emerge. Inequality is constant along dynamic paths that converge to a Balanced Growth Path. In the centralized problem, we could not recover the individual dynamics of the allocation of time to work. Thus, we are not able to characterize the dynamics of inequality.

2.2 The Decentralized Problem

We modified the Lucas Model (1988) to allow heterogeneity between individuals. The source of heterogeneity is the initial endowment of human capital. For simplification, let us assume the existence of two groups of individuals, groups 1 and 2. Within groups, individuals are equal, which means that they received the same endowment of intelligence or skills. How-

ever, between groups, agents are different. We assumed that individuals in group 2 are initially smarter than those in group 1, $h_2(0) > h_1(0)$.

We also assumed that individuals have an infinite horizon or the ability to live forever. The population was defined on the continuum $(0, 1)$. We denoted ϕ as the proportion of individuals in group 1 and $(1 - \phi)$ in group 2. Population size does not change over time; therefore, the share of individuals in each group is constant.

Each agent produces an intermediate good, $z_j(t)$, which is used by firms to produce their final goods. To produce the intermediate commodity, individuals dedicate $u_j(t)$ hours to work and their skill level, $h_j(t)$. The production function of the intermediate good is:

$$z_j(t) = u_j(t) h_j(t) \quad (1)$$

This technology implies that the productivity of working time is increasing on the human capital of individuals. The higher the human capital of individuals, the fewer agents will have to work to produce a certain quantity of intermediate good.

In this model, there are no leisure choices, so the time that is not allocated to work is dedicated to studies. Study time is the only way to accumulate human capital. The human capital accumulation technology depends on effort devoted to education, $(1 - u_j)$, and the level of human capital, h_j . As Lucas (1988), we considered that human capital accumulation is linear on time dedicated to studying. Let δ be a positive constant parameter on the human capital accumulation technology.

$$\dot{h}_j = \delta (1 - u_j) h_j \quad (2)$$

The average human capital is $h = \int_0^1 h_j dj$. Since agents are equal within the group, the average human capital is calculated as below:

$$h = \int_0^\phi h_1 + \int_\phi^1 h_2 = \phi h_1 + (1 - \phi) h_2 \quad (3)$$

Let h_j^R be the relative human capital of group j : $h_1^R := \frac{\phi h_1}{h}$ and $h_2^R := \frac{(1-\phi)h_2}{h}$. We chose an aggregation of working hours that help us solve some critical functions in this model. Our aggregate working hours consider the relative human capital of each group as a weighting factor. Therefore, this measurement of working time captures the efficiency of the time devoted to work by each group. In the rest of the paper, we called this variable the “aggregate working time.” Let us define the aggregate working time, u , as:

$$u := u_1 h_1^R + u_2 h_2^R \quad (4)$$

From equation (2) and definition (4), we concluded that the growth rate of the average human capital is a function of the aggregate working time.

$$\hat{h} = \frac{\int_0^1 \dot{h}_j}{\int_0^1 h_j} = \frac{\phi \dot{h}_1 + (1 - \phi) \dot{h}_2}{\phi h_1 + (1 - \phi) h_2} = \delta(1 - u) \quad (5)$$

2.2.1 Firms

We assumed a large number of identical firms. These competitive firms use the intermediate commodity as input for producing final goods. The average human capital of the economy, $h_a(t)$, generates a positive externality on the production of the final good. This externality increases the productivity of all factors of production. In this decentralized economy, individuals cannot affect the average human capital, so $h_a(t)$ is given. Let A be the positive technology parameter.

$$Y = A \left(\int_0^1 z_j d_j \right) h_a^\gamma \quad (6)$$

Note that, as individual are equals within each group, we can write the production function as:

$$Y = A \left(\int_0^\phi z_1 d_j + \int_\phi^1 z_2 d_j \right) h_a^\gamma = A (\phi z_1 + (1 - \phi) z_2) h_a^\gamma$$

The definition of the aggregate working time, (4), tells us that the aggregate quantity of intermediate goods demanded by firms is equal to the aggregate amount of working hours times the average human capital.

$$z = uh \quad (7)$$

Firms maximize their profits by choosing the optimal quantity of intermediate goods. In equilibrium, intermediate goods are paid by their marginal productivity. Since all firms are identical, they will use the same proportion of intermediate goods produced by each group, and these proportions have to be averages:

$$w_t = A h_a^\gamma \quad (8)$$

Markets clear and all production is consumed. Let $c = \int_0^1 c_j d_j$ denote the aggregate consumption of the economy.

$$c_t = A z h_a^\gamma \quad (9)$$

2.2.2 Consumers

Consumers aim to maximize the discounted sum of instantaneous utilities over an infinity of periods. We assumed that a Constant Risk Relative Aversion function describes the preferences of consumers.

$$\int_0^\infty \frac{c_j^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \quad (10)$$

in which $\rho > 0$ is the subjective rate of discount and σ is the inverse of the elasticity of intertemporal substitution. For notational convenience, we defined $\theta \equiv (\delta, A, \rho, \gamma, \sigma)$, and $\theta \in \Theta$, where $\Theta = R_{++}^3 \times R_+^2$.

Households sell intermediate goods to firms and earn w_t as remuneration. This revenue is used to pay for consumption goods, manufactured by firms. The consumers' budget constraint is:

$$c_j(t) = w_t z_j(t) = w_t u_j h_j \quad (11)$$

The consumers face a dynamic optimization problem. They maximize their intertemporal utility subject to the budget constraint (11) and the human capital accumulation function (2). The optimal solution is obtained by setting up the present value Hamiltonian function:

$$H_{j,t} = \frac{(w_t u_j h_j)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} + \lambda_j \delta (1 - u_j) h_j \quad (12)$$

First-order conditions imply that:

$$(w_t u_j h_j)^{-\sigma} w_t e^{-\rho t} = \delta \lambda_j \quad (13)$$

$$\hat{\lambda}_j = -\delta \quad (14)$$

The Transversality Conditions:

$$\lim_{t \rightarrow \infty} \lambda_j h_j = 0 \quad (15)$$

From Arrow sufficient condition, a solution for these first-order conditions solves the optimal control problem if the Hamiltonian function is concave in the state variable (h_j) after the control variable (u_j) has been substituted out with their maximizing value, from FOC (13). In Appendix A, we showed that this Hamiltonian function calculated on the optimal control variable is linear in h_j , so the sufficient condition is satisfied.

Definition 1. (Equilibrium) Given the initial endowments of human capital $\{h_j(0)\}_{j=1,2}$ and population size $\{\phi, (1-\phi)\}$, a competitive equilibrium consists of a sequence of price $\{w_t\}$, aggregate quantity $\{h_a(t)\}$, and decision rules $\{c_j(t), u_j(t), h_j(t)\}_{j=1,2}^{t=0,\infty}$, such that:

1 - the households' decision rules $c_j(t)$, $u_j(t)$, $h_j(t)$ maximize utility function (10) subject to the constraints (2), (11);

2 - the firm's choice z_j maximizes profits (8);

3 - the price w_t is such that markets clear (9);

4 - and the equality $h_a(t) = h(t) \forall t$ holds.

Taking logarithms on both sides of the equation (13), differentiating with respect to time and using the equations (2) and (14), we got:

$$\hat{u}_j = \frac{1-\sigma}{\sigma} \hat{w}_t - \delta(1-u_j) + \frac{\delta-\rho}{\sigma} \quad (16)$$

In the equilibrium, from the equation (8), the growth rate of the price of intermediate goods depends on the average human capital growth rate and externality, $\hat{w}_t = \gamma \hat{h}_t$, so, using the dynamic path of the average human capital (5):

$$\hat{w}_t = \gamma \delta (1-u) \quad (17)$$

Using (17) on the dynamic path of the individual work time:

$$\hat{u}_j = \frac{(1-\sigma)\gamma\delta}{\sigma} (1-u) - \delta(1-u_j) + \frac{\delta-\rho}{\sigma} \quad (18)$$

The dynamic path of individual working time depends negatively on the amount of time dedicated to studying in each period and positively on the aggregate studying time. So individuals consider the average studying time of the economy when choosing their individual working time.

To find the dynamic of the aggregate working time, we differentiated equation (4) with respect to time and using the human capital accumulation (2) and the individual dynamic of working time (18), we found:

$$\hat{u} = \left(\frac{\gamma - \sigma(1+\gamma)}{\sigma} \right) \delta(1-u) + \frac{\delta-\rho}{\sigma} \quad (19)$$

The dynamic path of the aggregate working time depends only on the aggregate working time in period t . Taking logarithms on both sides of budget constraint of consumers (11), differentiating with respect to time and using the growth rate of the working time of individual j , we obtained the growth rate for individual consumption:

$$\hat{c}_j = \frac{\delta(1+\gamma) - \rho}{\sigma} - \frac{\gamma\delta}{\sigma} u \quad (20)$$

Note that the growth rate for individual consumption depends only on the aggregate working time. Thus, the dynamic path of individual consumption is equal for all agents. This implies that the aggregate consumption growth rate is equal to the individual consumption growth rate.

$$\hat{c} = \hat{c}_j \quad (21)$$

2.2.3 Balanced Growth Path

We focused our analysis on a particular solution of this model such that the growth rates of average human capital and average consumption are constant.

Definition 2. A balanced growth path is defined as a set of functions $\{h_j(t), c_j(t), u_j(t)\}_{j=1,2}$ that solve the optimal control problem and such that, h and c grows at a constant rate and u is constant over time.

Since the aggregate working time is constant on a Balanced Growth Path (BGP), we found the equilibrium value u on a BGP by setting $\hat{u} = 0$ in equation (19):

$$u^* = \frac{\delta(1+\gamma)(1-\sigma) - \rho}{\delta(\gamma - \sigma(1+\gamma))} \quad (22)$$

The BGP equilibrium of the aggregate amount of working time is the same found in Lucas's homogeneous agent model (1988) in the absence of physical capital. As mentioned in Benhabib and Perli (1994), the problem with equation (22) is that there is no guarantee that u^* lies on the unit interval. Therefore, it is essential to analyze the necessary conditions for this to be true. These conditions consist of some restrictions on parameter values. When parameters lie in one of the two subsets below, then $u^* \in [0, 1]$.¹

$$\Theta_1 := \left\{ \theta \in \Theta : (1-\sigma)(1+\gamma) < \frac{\rho}{\delta} \leq 1 \right\} \quad (23)$$

$$\Theta_2 = \left\{ \theta \in \Theta \mid \delta \leq \rho < \delta(1+\gamma) \text{ and } \sigma < 1 - \frac{\rho}{\delta(1+\gamma)} \right\} \quad (24)$$

The surface Θ_3 separates these two surfaces:

$$\Theta_3 = \left\{ \theta \in \Theta \mid \delta = \rho \text{ and } \sigma = 1 - \frac{\rho}{\delta(1+\gamma)} \right\} \quad (25)$$

If $\theta \in \Theta_3$, then the aggregate time dedicated to work is null, $u^* = 0$. However, $u^* = 0$ cannot be an equilibrium because it does not satisfy the Transversality Condition of Consumer's Problem. To demonstrate that we solve a dynamic optimization problem allowing corner solutions in Appendix C.²

From equations (9) and (7), we observed that the aggregate consumption growth rate is proportional to the average human capital growth rate on the Balanced Growth Path. In the absence of the externality of human capital, average consumption and human capital grow at the same rate.

$$\hat{c}_t = (1+\gamma)\hat{h}_t \quad (26)$$

On a Balanced Growth Path, the average human capital grows at the rate given below:

$$\hat{h}_t = \frac{\delta - \rho}{\sigma(1+\gamma) - \gamma} \quad (27)$$

¹See Appendix B for demonstration.

²Note that in order to $u^* = 0$, we must have $u_j^* = 0 \forall j$. But this is not a consumer equilibrium.

The differential equation of the aggregate amount of time dedicated to work, (19), is a Riccati equation and, therefore, we know their solutions. This Riccati equation has one non-degenerate solution, $u(t)$, and two degenerate solutions: 0 and u^* . The non-degenerate solution is given below:

$$u(t) = u^* \left(1 - \left(1 - \frac{u^*}{u(0)} \right) \frac{1}{\exp \left(\frac{\delta(1+\gamma)(1-\sigma)-\rho}{\sigma} t \right)} \right)^{-1}$$

To simplify the notation, we define the following new constant:

$$\psi := \frac{\delta(1+\gamma)(1-\sigma)-\rho}{\sigma}$$

The non-degenerate solution is written as:

$$u(t) = u^* \left(1 - \frac{1 - \frac{u^*}{u_0}}{\exp(\psi t)} \right)^{-1} \quad (28)$$

The first degenerate solution, $u(t) = 0$, is not an equilibrium, because the Transversality Condition is not satisfied.³ The second degenerate solution, $u(t) = u^*$, shows that the equilibrium amount of time devoted to work on a Balanced Growth Path is a solution to this differential equation.

To study the stability of the Balanced Growth Path, we first analyzed the behavior of the function $u(t)$ over time. We had to assure that even when the Balanced Growth Path is stable, the solution $u(t)$ is always on the unit interval.

Lemma 1. *The solution $u(t)$ is a monotonic function.*

Proof. See Appendix I. □

Since $u(0)$ and u^* are on the unit interval, this lemma assures us that, if the Balanced Growth Path is stable, $u(t)$ is always on the unit interval. Given Lemma 1, our strategy for analyzing the stability of the BGP was to verify the behavior of solution $u(t)$ in the long term. We focused on trajectories in which $u(0) \leq u^*$, for which we characterized all solution paths.⁴

Proposition 1. *The stability of the Balanced Growth Path equilibrium is such that:*

- (i) if $\theta \in \Theta_1$, the Balanced Growth Path is unstable;
- (ii) if $\theta \in \Theta_2$, the Balanced Growth Path is globally stable.

Proof. See the Appendix J. □

³See Appendix C: Corner Solution

⁴When $u(0) > u^*$ and $\theta \in \Theta_1$, the domain of the function $u(t)$ is not $(0, \infty)$.

As shown in the proof of Proposition 1, if $\theta \in \Theta_1$ and $u(0) < u^*$, function $u(t)$ converges to the degenerate solution zero. However, all these solution paths are not an economic solution, because the Transversality Condition is not satisfied along all those solution paths.⁵

From the solution of the differential equation of the aggregate efficient working hours (28), we characterized the solutions path of the average human capital and individual consumption:

$$c_j(t) = \frac{c_j(0)}{\left(\frac{1}{u^*} (u^* - u(0) (1 - e^{\psi t}))\right)^{\frac{\gamma}{\gamma - \sigma(1 + \gamma)}}} \exp \left\{ \frac{\delta(1 + \gamma) - \rho}{\sigma} t \right\} \quad (29)$$

2.2.4 The Dynamics of Inequality

After the characterization of the Balanced Growth Path and its conditions for stability, we focused on studying the dynamics of inequality that emerge along dynamic paths that converge to an equilibrium. For that, we solved the differential equation of individual working of time, (18), using the solution of the aggregate working time, (28). This differential equation has an exact solution below:

$$u_j(t) = \frac{e^{\psi t}}{\frac{1}{u(0)} + \left(\frac{1}{u_j(0)} - \frac{1}{u(0)} \right) \left(1 - \frac{u(0)}{u^*} (1 - e^{\psi t}) \right)^{\frac{(1 - \sigma)\gamma}{(\gamma - \sigma(1 + \gamma))}} - \frac{1}{u^*} (1 - e^{\psi t})} \quad (30)$$

The individual working time depends on the initial individual and aggregate allocation of time to work, $u_j(0)$ and $u(0)$, and the equilibrium aggregate working time, u^* . The behavior of individual working time, (30), depends on the set of parameters. First, we studied the behavior of function (30) on parameter set Θ_2 , in which the Balanced Growth Path is globally stable.

Proposition 2. *If $\theta \in \Theta_2$, then:*

- (i) *at the initial time, all agents devote the same amount of time to work, $u_j(0) = u(0)$;*
- (ii) *on the Balanced Growth Path, all agents devote the same amount of time to work, $u_j^* = u^*$.*

Proof. See the Appendix J. □

Proposition 2 implies that, along dynamic paths that converge to the Balanced Growth Path, individual working time at every instant of time is equal between individuals regardless of their initial human capital. According to Proposition 2, the individual amount of time devoted to work simplifies to:

$$u_j(t) = \frac{e^{\psi t}}{\frac{1}{u(0)} - \frac{1}{u^*} (1 - e^{\psi t})} = u(t) \quad (31)$$

⁵See Appendix D for demonstration.

From solution (31), we found the individual human capital path:

$$h_j(t) = h_j(0) \frac{e^{\delta t}}{\left(1 - \frac{u(0)}{u^*} (1 - e^{\psi t})\right)^{\frac{\sigma}{(\gamma - \sigma)(1 + \gamma)}}} \quad (32)$$

To study the dynamics of inequality, we could utilize different measurements of inequality. We analyzed the simplest one, the ratio of the human capital of the two groups considered in this paper, but the result would be the same if another measurement of inequality were used.

Corollary 1. *If $\theta \in \Theta_2$, inequality is permanent and equal to initial inequality.*

Proof. By Proposition 2, we know that $u_j(0) = u(0)$. Taking the ratio of human capital of individuals in groups 2 and 1, one sees that the ratio of human capital is constant over time. $\square$

If $\theta \in \Theta_2$, the ratio of human capital is constant over time and equal to the initial human capital ratio. Therefore, inequality is constant over time.

$$\frac{h_2(t)}{h_1(t)} = \frac{h_2(0)}{h_1(0)} \quad (33)$$

Moreover, the heterogeneity in this model have a sufficient structure to ensure that the sum of all consumers behave as if it were a single consumer, and so it allows a representative consumer. As $u_j(t) = u(t)$ does not change with individuals, the consumer's budget constraint can be written as:

$$c_j(t) = f(w_t, u(t))h_j(t) \quad (34)$$

The linearity of consumption function, (29), in $c_j(0)$, combined with the linearity of the budget constraint, (34), in $h_j(t)$ allows nice aggregation properties, therefore this model admits representative consumer, and the budget constraint of the average consumer is:

$$c(t) = (\phi c_1(t) + (1 - \phi)c_2(t)) = f(w_t, u(t))(\phi h_1(t) + (1 - \phi)h_2(t)) = f(w_t, u(t))h(t)$$

Let us now consider the set of parameters Θ_1 . On this set of parameters, the Balanced Growth Path is unstable, and all solutions are not an economic equilibrium, except in the case of $u(0) = u^*$. Nonetheless, we can study the dynamics of inequality on BGP equilibrium because even when $\theta \in \Theta_1$, if the economy starts at the BGP, it remains in this equilibrium. Assuming that the economy starts on the Balanced Growth Path, then $u(0) = u^*$, the individual working time, (30), reduces to:

$$u_j(t) = \frac{1}{\frac{1}{u^*} + \left(\frac{1}{u_j(0)} - \frac{1}{u^*}\right) e^{\delta u^* t}} \quad (35)$$

Proposition 3. *If $\theta \in \Theta_1$ and $u(0) = u^*$, then:*

(i) at the beginning of time, all agents dedicate the same amount of time to work, $u_j(0) = u^$.*

Proof. See Appendix J. □

Again, the individual working time is equal in every period regardless of the initial human capital. According to Proposition 3, the individual working time path, equation (35), simplifies to:

$$u_j(t) = u^* \quad (36)$$

Therefore, the individual allocation of time to work is constant over time, and all individuals accumulate human capital at the same rate. The individual human capital path is:

$$h_j(t) = h_j(0) \exp\left(\frac{\rho - \delta}{\gamma - \sigma(1 + \gamma)} t\right) \quad (37)$$

Corollary 2. *If $\theta \in \Theta_1$ and $u(0) = u^*$, inequality is permanent and equal to initial inequality.*

Proof. By Proposition 3, we know that $u_j(0) = u(0)$. Taking the ratio of human capital of individuals in groups 2 and 1, one sees that the ratio of human capital is constant over time. □

Again, the initial inequality is permanent over time. To arrive at this, we calculated the ratio between the human capital of the two groups, using equation (37):

$$\frac{h_2(t)}{h_1(t)} = \frac{h_2(0) \exp\left(\frac{\rho - \delta}{\gamma - \sigma(1 + \gamma)} t\right)}{h_1(0) \exp\left(\frac{\rho - \delta}{\gamma - \sigma(1 + \gamma)} t\right)} = \frac{h_2(0)}{h_1(0)} \quad (38)$$

Therefore, on solutions that converge to the Balanced Growth Path, inequality is constant over time. The group of individuals that was endowed with higher human capital will be permanently richer than the other group.

2.3 The Central Planner Problem

Suppose there is a central planner concerned with the social welfare of the economy. We assumed a particularly simple class of welfare functions that takes a linear form, in which μ_j is the welfare weights that the central planner assigns to the utility of each agent. The central planner chooses how much each agent should consume and work to maximize the intertemporal social welfare function.

$$\max_{c_j, u_j, h_j} \int_0^1 \mu_j \left(\int_0^\infty \frac{c_j^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt \right) dj \quad (39)$$

The central planner faces a resource constraint in which total consumption must be equal to the output. Contrary to the decentralized model, in the centralized model, the central planner knows and affects the average human capital. Therefore, the central planner internalizes the externality of average human capital, and this implies that maximization of social welfare is subject to the constraint $h_a(t) = h(t)$ for all t .

$$A \left(\int_0^1 u_j h_j dj \right) \left(\int_0^1 h_j dj \right)^\gamma \geq \int_0^1 c_j dj \quad (40)$$

Besides that, the central planner also faces a human capital accumulation constraint for each individual, equation (2). Thus, the optimal control problem has mixed constraints. In such a case, we must set a Lagrangian function or a generalized Hamiltonian function $\mathcal{L}_t$:⁶

$$\mathcal{L}_t = \int_0^1 \mu_j \left(\frac{c_j^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} \right) dj + \int_0^1 \lambda_{j,t}^A \delta (1 - u_j) h_j dj + \lambda_t^B \left(A \left(\int_0^1 u_j h_j dj \right) \left(\int_0^1 h_j dj \right)^\gamma - \int_0^1 c_j dj \right) \quad (41)$$

The first-order conditions for this problem are thus:

$$\mu_j c_j^{-\sigma} \exp(-\rho t) = \lambda_t^B \quad (42)$$

$$\lambda_t^B A h^\gamma = \lambda_{j,t}^A \delta \quad (43)$$

$$\hat{\lambda}_{j,t}^A = -\delta (\gamma u + 1) \quad (44)$$

The Transversality Condition:

$$\lim_{t \rightarrow \infty} \int_0^1 \lambda_{j,t}^A h_j dj = 0 \quad (45)$$

Note from the first order conditions (FOC) that individual working time u_j vanishes. Therefore, it was not possible to find solutions for an optimal individual allocation of working time. We could only solve the model for the average allocation of working time, u . In the absence of FOC for u_j , we also could not apply the sufficiency theorems, so we could not assure that the so-

⁶ Note that this Lagrangian is equivalent to:

$$\begin{aligned} \mathcal{L}_t = & \phi \left(\mu_1 \frac{c_1^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} \right) + (1 - \phi) \left(\mu_2 \frac{c_2^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} \right) + \\ & \lambda_{1,t}^A \phi (\delta (1 - u_1) h_1) + \lambda_{2,t}^A (1 - \phi) (\delta (1 - u_2) h_2) \\ & + \lambda_t^B (A (\phi u_1 h_1 + (1 - \phi) u_2 h_2) (\phi h_1 + (1 - \phi) h_2)^\gamma - \phi c_1 - (1 - \phi) c_2) \end{aligned}$$

lutions from first-order conditions maximized the generalized Hamiltonian function. However, since we could solve the problem for the average working time, we showed that this solution satisfied the Arrow sufficient condition and, therefore, maximized the Hamiltonian function. This is the procedure that we followed throughout this section.

Although we could not solve the central planner problem for the individual working time, we found from (43) that, on the margins, the individual allocation of time must be equally valuable in its two uses - production and human capital accumulation. Moreover, condition (42) implies that the co-state variable associated with the human capital is equal for all agents, $\lambda_{j,t}^A = \lambda_t^A$.

From differentiating equations (43) and using equations (44) and (5), we found that the Hamiltonian multiplier associated with the resource constraint grows at a constant rate.

$$\hat{\lambda}_t^B = -\delta(1 + \gamma) \quad (46)$$

We obtained the growth rate of individual consumption by differentiating equation (42) and by using (46). As the Hamiltonian multiplier, λ_t^B grows at a constant rate, individual consumption will also grow at a constant rate.

$$\hat{c}_j = \frac{\delta(1 + \gamma) - \rho}{\sigma} \quad (47)$$

Note that, the growth rate of individual consumption is constant across agents and, therefore, equals the growth rate of the average consumption, $\hat{c}_j = \hat{c}$. To find the aggregate efficient working time path $\hat{u}$, we must differentiate the resource constraint over time and replace $\hat{c}$ by its equilibrium path, (47), and $\hat{h}$ by its definition, (5).

$$\hat{u} = \frac{\delta(1 + \gamma)(1 - \sigma) - \rho}{\sigma} - (1 + \gamma)\delta u \quad (48)$$

2.3.1 Balanced Growth Path

As stated in section 3.2.3, our goal is studying a particular solution for system $u(t), c(t), h(t)$. Let us define the Balanced Growth Path of the Central Planner problem as:

Definition 3. A balanced growth path is defined as a set of functions $\{h_j(t), c_j(t), u_j(t)\}_{j=1,2}$ that solve the central planner control problem and such that h and c grow a constant rate and u is constant over time.

We found the aggregate efficient working time in the BGP by setting, $\hat{u}_t = 0$ in the equation (48).

$$u_C^* = \frac{\rho - \delta(1 + \gamma)(1 - \sigma)}{\delta\sigma(1 + \gamma)} \quad (49)$$

The aggregate working time on the BGP is equal to the solution of the central planner's problem in Lucas (1988) in the absence of physical capital. On Appendix E, we show that u^* satisfies the Arrow sufficient condition for a problem with mixed constraints. Therefore, we can assure that u^* maximizes the generalized Hamiltonian function.

Equilibrium u_C^* lies on the unit interval in the subset below:⁷

$$\Theta_4 = \left\{ \theta \in \Theta \mid \rho \leq \delta(1 + \gamma) \text{ and } \sigma > 1 - \frac{\rho}{\delta(1 + \gamma)} \right\} \quad (50)$$

To compare the solutions for the centralized and decentralized models, we must restrict the analysis to the parameter space in which both equilibrium allocations of time, u^* and u_C^* , are in the unit interval. We showed in Appendix G that $\Theta_1 \subseteq \Theta_4$. Therefore, we focused our analyzes on the set of parameters Θ_1 .

Proposition 4. *If $\theta \in \Theta_1$, then, in the equilibrium, individuals choose to allocate more time to work than the Central Planner would choose, $u_C^* < u^*$.*

Proof. See Appendix J. □

According to Proposition 4, the central planner chooses to allocate less time to work, u_C^* , than the agents would individually choose u^* . This implies that the human capital growth rate on the BGP of a centralized problem is higher than on the BGP of decentralized problem.

Once again, the differential equation of the aggregate amount of time dedicated to work, (48), is a Riccati equation and we know its solution. As in decentralized problem, this Riccati equation has one non-degenerate solution, $u_C(t)$, and two degenerate solutions: 0 and u_C^* . The non-degenerate solution is:

$$u_C(t) = u_C^* \left(1 - \left(1 - \frac{u_C^*}{u(0)} \right) \frac{1}{\exp(\psi t)} \right)^{-1} \quad (51)$$

Proposition 5. *The Balanced Growth Path of the Central Planner problem is unstable.*

Proof. See Appendix J. □

The Balanced Growth Path of the centralized problem is unstable, and the function $u(t)$ converges to the degenerate solution zero. However, all these mathematical solutions are not an economic solution, because the Transversality Condition is not satisfied along those solution paths.⁸

Although we cannot recover the dynamic path of the individual working time chosen by the central planner, we know from the definition of the aggregate working time, (4), that on the Balanced Growth Path the relationship between the individual working time is:

$$\dot{u}_1 h_1^R + \dot{u}_2 h_2^R = \delta \left((u_1)^2 h_1^R + (u_2)^2 h_2^R - (u^*)^2 \right) \quad (52)$$

⁷See Appendix F

⁸See Appendix H: Non-economic solutions - Central Planner Problem

2.4 Uzawa's Human Capital Accumulation

To analyze the robustness of the results found in previous sections, we studied the possibility of the human capital accumulation function having diminishing returns in studying time, as in Uzawa (1965).

$$\dot{h}_j = \delta (1 - u_j)^\eta h_j \quad (53)$$

We assumed the preference of consumers is logarithmic, which means that $\sigma = 1$:

$$\int_0^\infty \ln(c_j) e^{-\rho t} dt \quad (54)$$

Consumers maximize their intertemporal utility subject to the budget constraint (11) and the human capital accumulation function (53). The optimal solution is obtained by setting up the present-value Hamiltonian function:

$$H_{j,t} = \ln(w_t u_j h_j) e^{-\rho t} + \lambda_j \delta (1 - u_j)^\eta h_j \quad (55)$$

The first-order conditions are:

$$\frac{1}{u_j} = \lambda_j \eta \delta (1 - u_j)^{\eta-1} h_j \quad (56)$$

$$\hat{\lambda}_j = -\frac{1}{\lambda_j h_j} e^{-\rho t} - \delta (1 - u_j)^\eta \quad (57)$$

The Transversality Condition:

$$\lim_{t \rightarrow \infty} \lambda_j h_j = 0 \quad (58)$$

From (56) and (57), we find:

$$\hat{\lambda}_j = -\delta (1 - u_j)^\eta \left(1 + \frac{\eta u_j}{1 - u_j} \right) \quad (59)$$

Taking logarithms on both sides of the equation (56), differentiating with respect to time and using the equations (53) and (59), we got the individual working time growth rate:

$$\hat{u}_j = \frac{\delta (1 - u_j)^\eta \eta u_j - \rho (1 - u_j)}{(1 - \eta u_j)} \quad (60)$$

The individual working time path depends only on the individual allocation of working time in period t . Therefore, if there is a Balanced Growth Path with average human capital and consumption growing at constant rates, the individual working time does not increase. Consequently, we set $\hat{u}_j = 0$:

$$(1 - u_j^*)^{\eta-1} u_j^* = \frac{\rho}{\delta \eta} \quad (61)$$

From equation (61), we find that the equilibrium individual amount of time dedicated to work u_j^* is equal for all agents. Let us log-linearize the differential equation $\dot{u}_j$ around the steady state u_j^* :

$$\frac{\partial \dot{u}_j}{\partial u_j} \Big|_{u_j^*} = \rho > 0 \quad (62)$$

Therefore, the Balanced Growth Path is unstable. Despite the instability of the BGP, we could analyze the dynamics of inequality in such equilibrium. Therefore, let us assume that the economy starts on the Balanced Growth Path $u_j(0) = u_j^*$. However, in such case, the individual human capital grows at the same rate, and the inequality is constant.

2.5 Conclusion

This paper aimed at exploring the distributive dynamics implicit in the Lucas model. The analysis led to one main conclusion. In the absence of physical capital, the inherent inequality in the Lucas model is constant over time. Agents choose to allocate the same amount of time to studying every period. Therefore, the gap between the human capital of individuals from different groups does not change over time, and inequality remains constant at its initial level.

Introduce a government concerned with education would be an interesting extension. We observed that the presence of an externality of human capital on the production function causes a sub-accumulation of human capital, which justifies a governmental intervention. Therefore, it is interesting to study the optimal educational policy and its impacts on the dynamics of inequality. An interesting question is if an education policy focuses on increasing the amount of time dedicated to studying reduce the inequality.

Appendix:

A: Arrow sufficient condition for the decentralized problem

From the first-order condition (13):

$$u_j = (\delta \lambda_j e^{\rho t})^{-\frac{1}{\sigma}} w_t^{\frac{1-\sigma}{\sigma}} h_j^{-1} \quad (63)$$

Replacing the equation (63) on the Hamiltonian, we find that:

$$H_{j,t}^* = (\lambda_j \delta)^{\frac{\sigma-1}{\sigma}} e^{-\frac{\rho}{\sigma} t} w_t^{\frac{1-\sigma}{\sigma}} \left(\frac{1}{1-\sigma} w_t^\sigma - 1 \right) + \lambda_j \delta h_j \quad (64)$$

From Arrow sufficient condition, we must have $\lambda_j \geq 0$. The Hamiltonian $H_{j,t}^*$ is linear in h_j , so it is concave in h_j , and the sufficient condition is satisfied.

B: Set of Parameters

The value of u^* lies on the unit interval, when the two inequality below is satisfied. For the amount of time u^* be less or equal to one, we need:

$$\frac{\rho - \delta}{\delta (\gamma - \sigma (1 + \gamma))} \geq 0 \quad (65)$$

For the value of u^* be positive, we need:

$$\frac{\rho - \delta}{\delta (\gamma - \sigma (1 + \gamma))} < 1 \quad (66)$$

The combination of these two inequalities generates the following sets of parameters:

Case 1: Let us assume that $\rho - \delta \leq 0$ and $\gamma - \sigma (1 + \gamma) < 0$, then condition (65) is satisfied. For condition (66) be satisfied it is necessary that $(1 - \sigma) (1 + \gamma) < \frac{\rho}{\delta}$. Let us define Θ_1 as the parameter set where the value of the parameters satisfy these conditions:

$$\Theta_1 := \left\{ \theta \in \Theta \mid (1 - \sigma) (1 + \gamma) < \frac{\rho}{\delta} \leq 1 \right\}$$

Note that when $\theta \in \Theta_1$, then $\psi := \frac{\delta(1+\gamma)(1-\sigma)-\rho}{\sigma} < 0$

Case 2: Let us assume that $\rho - \delta \geq 0$ and $\gamma - \sigma (1 + \gamma) > 0$, then the restriction (65) is satisfied. For inequality (66) be satisfied, it is necessary that $\sigma < 1 - \frac{\rho}{\delta(1+\gamma)}$. We also need an additional restriction to assure that σ remains positive. Otherwise, the parameter set would be empty. The extra restriction is $\rho < \delta (1 + \gamma)$. Let us define Θ_2 as the parameter set where the value of the parameters satisfy these conditions:

$$\Theta_2 := \left\{ \theta \in \Theta \mid \delta \leq \rho < \delta (1 + \gamma) \text{ and } \sigma < 1 - \frac{\rho}{\delta (1 + \gamma)} \right\}$$

Note that when $\theta \in \Theta_2$, then $\psi := \frac{\delta(1+\gamma)(1-\sigma)-\rho}{\sigma} > 0$

C: Corner Solutions

In section 2.2, we study interior solutions of the consumer's dynamic optimization problem. However, the solution could be a corner solution. To investigate the presence of corner solution, let us add some inequality constraints on the interval of the control variable u_j . In this case, we must apply the Kuhn-Tucker Conditions.

$$\max_{c_j, u_j, h_j} \int_0^\infty \frac{c_j^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} dt$$

subject to:

$$c_j = w_t u_j h_j$$

$$\dot{h}_j = \delta (1 - u_j) h_j$$

$$0 \leq u_j \leq 1$$

Besides, for these Kuhn-Tucker conditions to be necessary, a constraint qualification must be satisfied. The Theorem of Arrow, Hurwicz, and Uzawa establishes the constraint qualifications that must be satisfied to the solution of this problem be optimal. Note that, all constraints above are linear in the control variable u_j , therefore the Theorem of Arrow, Hurwicz, and Uzawa is satisfied.⁹

We now augment the Hamiltonian into a Lagrangian function:

$$\mathcal{L}_j = \frac{(w_t u_j h_j)^{1-\sigma} - 1}{1 - \sigma} e^{-\rho t} + \lambda_j^A(t) (\delta (1 - u_j) h_j) + \lambda_j^B(t) (1 - u_j)$$

The Kuhn-Tucker first-order conditions are:

$$\frac{\partial \mathcal{L}_j}{\partial u_j} = (w_t u_j h_j)^{-\sigma} w_t h_j e^{-\rho t} - \lambda_j^A(t) \delta h_j - \lambda_j^B(t) \leq 0 ; u_j \geq 0 ; u_j \frac{\partial \mathcal{L}_j}{\partial u_j} = 0 \quad (67)$$

$$\frac{\partial \mathcal{L}_j}{\partial \lambda_j^B} = 1 - u_j \geq 0 ; \lambda_j^B \geq 0 ; \lambda_j^B(t) (1 - u_j) = 0 \quad (68)$$

$$\dot{h}_j = \delta (1 - u_j) h_j \quad (69)$$

$$- (w_t u_j)^{1-\sigma} h_j e^{-\rho t} - \lambda_j^A \delta (1 - u_j) = \dot{\lambda}_j^A \quad (70)$$

⁹See Chiang (1999).

$$\lim_{t \rightarrow \infty} \lambda_j^A h_j = 0 \quad (71)$$

We will demonstrate that $u_j = 0$ cannot be an equilibrium. Let us take $u_j = 0$, then from FOC (68), $\lambda_j^B = 0$, from FOC (70), $\dot{\lambda}_j^A = -\delta$ and from FOC (69), $\dot{h}_j = \delta$. We solve these differential equations and find:

$$\lambda_j^A(t) = \lambda_j^A(0) e^{-\delta t} \quad (72)$$

$$h_j(t) = h_j(0) e^{\delta t} \quad (73)$$

Using (72) and (73), the limit of the multiplication of the variables of co-state and state is, therefore:

$$\lim_{t \rightarrow \infty} \lambda_j^A(t) h_j(t) = \lim_{t \rightarrow \infty} \lambda_j^A(0) h_j(0) \quad (74)$$

For the Transversality condition (71) be valid, we must have $\lambda_j^A(0) = 0$. However, it implies that the vector of the co-state variable is null: $(\lambda(0), \lambda_j^A(t), \lambda_j^B) = (0, 0, 0)$, which contradicts the Maximum Principle and cannot be an equilibrium.¹⁰

We argue that $u_j = 1$ is a corner solution to this problem because it respects all of the Kuhn-Tucker Conditions. Let us calculate the FOCs when $u_j = 1$:

$$(w_t h_j)^{1-\sigma} e^{-\rho t} - \lambda_j^A(t) \delta h_j - \lambda_j^B(t) = 0 \quad (75)$$

$$\lambda_j^B \geq 0 \quad (76)$$

$$\hat{h}_j = 0 \quad (77)$$

$$-(w_t)^{1-\sigma} h_j e^{-\rho t} = \dot{\lambda}_j^A \quad (78)$$

The equations (75) and (76) implies:

$$-\lambda_j^A(t) \delta h_j \geq -(w_t h_j)^{1-\sigma} e^{-\rho t} \quad (79)$$

From (79) and (78), we get:

$$\dot{\lambda}_j^A \leq -\delta h_j^{1+\sigma} \quad (80)$$

From (77), the human capital is constant over time. Let us assume that holds with equality, so we can solve the differential equation ():

$$\lambda_j^A = \lambda_j^A(0) e^{-\delta h_j^{1+\sigma} t} \quad (81)$$

¹⁰See Seierstad and Sydsaeter (1987).

Therefore, we know that the Transversality condition:

$$\lim_{t \rightarrow \infty} (\lambda_j^A h_j) \leq \lim_{t \rightarrow \infty} \left(\lambda_j^A (0) e^{-\delta h_j^{1+\sigma} t} h_j \right),$$

in which h_j is constant over time.

$$\lim_{t \rightarrow \infty} (\lambda_j^A h_j) \leq \lim_{t \rightarrow \infty} h_j \left(\lambda_j^A (0) e^{\delta h_j^{1+\sigma} t} \right) = 0$$

Let us argue that $\lim_{t \rightarrow \infty} (\lambda_j^A h_j) \geq 0$ cannot be negative. From Arrow sufficient condition, we must have $\lambda_j \geq 0$. From the initial condition, $h_j(0) > 0$, and from FOC (77), we know that human capital is never negative. Therefore, $\lim_{t \rightarrow \infty} (\lambda_j^A h_j) = 0$.

D: Non-economic solutions

Suppose that $\theta \in \Theta_1$ and $u(0) < u^*$. Let us assume, for contradiction, that the Transversality Condition holds for all individuals: $\lim_{t \rightarrow \infty} \lambda_j(t) h_j(t) = 0 \forall j$. If $\lim_{t \rightarrow \infty} \lambda_j(t) h_j(t) = 0 \forall j$, then $\int_0^1 \lim_{t \rightarrow \infty} \lambda_j(t) h_j(t) = 0$, which can be re-written as:

$$\phi \lim_{t \rightarrow \infty} \lambda_1(t) h_1(t) + (1 - \phi) \lim_{t \rightarrow \infty} \lambda_2(t) h_2(t) = 0 \quad (82)$$

From FOC (15), we know that the growth rate of the co-state variable is constant over time and the solution of this differential equation is:

$$\lambda_j(t) = \lambda_j(0) e^{-\delta t} \quad (83)$$

Let us argue that $\lambda_j(0) \neq 0 \forall j$. From equation (83), if $\lambda_j(0) = 0$, then $\lambda_j(t) = 0 \forall t$, which contradicts the Maximum Principle.

Therefore, the co-state variable paths are multiplicative. Let us take one of them, $\lambda(t)$:

$$\lambda(t) = \lambda(0) e^{-\delta t} \quad (84)$$

Using (83) and (84), we get:

$$\lambda_j(t) = \frac{\lambda_j(0)}{\lambda(0)} \lambda(t) \quad (85)$$

Replacing (85) into the Transversality condition:

$$\lim_{t \rightarrow \infty} \lambda_j(t) h_j(t) = \lim_{t \rightarrow \infty} \frac{\lambda_j(0)}{\lambda(0)} \lambda(t) h_j(t) = 0 \quad (86)$$

As already shown, $\frac{\lambda_j(0)}{\lambda(0)} \neq 0$. Moreover, $\frac{\lambda_j(0)}{\lambda(0)}$ is constant over time, therefore using the properties of limits:

$$\frac{\lambda_1(0)}{\lambda(0)} \lim_{t \rightarrow \infty} \lambda(t) h_j(t) = 0,$$

which implies that:

$$\lim_{t \rightarrow \infty} \lambda(t) h_j(t) = 0 \quad \forall j \quad (87)$$

From (82) and (87), we get:

$$\phi \lim_{t \rightarrow \infty} \lambda(t) h_j(t) + (1 - \phi) \lim_{t \rightarrow \infty} \lambda(t) h_2(t) = 0$$

Using the properties of limits:

$$\lim_{t \rightarrow \infty} \lambda(t) (\phi h_1(t) + (1 - \phi) h_2(t)) = \lim_{t \rightarrow \infty} \lambda(t) h(t) = 0 \quad (88)$$

So, if the Transversality condition holds for all individuals, $\lim_{t \rightarrow \infty} \lambda_j(t) h_j(t) = 0$, then $\lim_{t \rightarrow \infty} \lambda(t) h(t) = 0$.

Now, let us calculate the $\lim_{t \rightarrow \infty} \lambda(t) h(t)$, using the average human capital path (??) and (84):

$$\lim_{t \rightarrow \infty} \lambda(0) e^{-\delta t} h(0) \left(\frac{u^*}{u(0) \left(1 - \left(1 - \frac{u^*}{u(0)} \right) e^{-\psi t} \right)} \right)^{\frac{\sigma}{\gamma - \sigma(1 + \gamma)}} e^{\frac{\rho - \delta}{\gamma - \sigma(1 + \gamma)} t} \quad (89)$$

Recall that $\psi = \frac{\delta(1 + \gamma)(1 - \sigma) - \rho}{\sigma}$:

$$\lim_{t \rightarrow \infty} \lambda(0) h(0) (u^*)^{\frac{\sigma}{\gamma - \sigma(1 + \gamma)}} \left(u(0) - u(0) \left(1 - \frac{u^*}{u(0)} \right) e^{-\frac{\delta(1 + \gamma)(1 - \sigma) - \rho}{\sigma} t} \right)^{\frac{-\sigma}{\gamma - \sigma(1 + \gamma)}} e^{\frac{\rho - \delta(1 + \gamma)(1 - \sigma)}{\gamma - \sigma(1 + \gamma)} t} =$$

$$= \lim_{t \rightarrow \infty} \lambda(0) h(0) (u^*)^{\frac{\sigma}{\gamma - \sigma(1 + \gamma)}} \left(u(0) e^{\frac{\delta(1 + \gamma)(1 - \sigma) - \rho}{\sigma} t} - u(0) \left(1 - \frac{u^*}{u(0)} \right) \right)^{\frac{\sigma}{\sigma(1 + \gamma) - \gamma}}$$

If $\theta \in \Theta_1$, then $\psi = \frac{\delta(1 + \gamma)(1 - \sigma) - \rho}{\sigma} < 0$:

$$\lim_{t \rightarrow \infty} \lambda(t) h(t) = \lambda(0) h(0) (u^*)^{\frac{\sigma}{\gamma - \sigma(1 + \gamma)}} (u^* - u(0))^{\frac{\sigma}{\sigma(1 + \gamma) - \gamma}} \neq 0 \quad (90)$$

E: Arrow sufficient condition for the Central Planner Problem

When the optimal control problem involves mixed constraint, the standard Arrow sufficient condition need some modification. We follow the Arrow sufficient condition for mixed constraints given by Seierstad and Sydsaeter (1987).

The Lagrangian of the Central Planner problem is given by (41). To apply Arrow sufficient condition we need to analyze the Hamiltonian of the central planner problem. The Hamiltonian is given below:

$$H_t = \int_0^1 \mu_j \frac{c_j^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dj + \int_0^1 \lambda_{j,t}^A \delta (1 - u_j) h_j dj$$

As already shown, the first order condition (43), the multiplier $\lambda_{j,t}^A$ is equal for all individual so that we can write the Hamiltonian as:

$$H_t = \int_0^1 \mu_j \frac{c_j^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dj + \lambda_t^A \delta \left(\int_0^1 h_j dj - \int_0^1 u_j h_j dj \right)$$

From our definition, $\int_0^1 h_j dj = h$ and $\int_0^1 u_j h_j dj = uh$. Therefore, the Hamiltonian can be re-written as:

$$H_t = \int_0^1 \mu_j \frac{c_j^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dj + \lambda_t^A \delta (h - uh) \quad (91)$$

For the Arrow sufficient condition hold, we must prove that the Hamiltonian $\hat{H}(h_j^*(t), c_j^*(t), u_j^*(t), \lambda(t))$ calculated in optimal control variable is concave on $h(t)$. Now we must replace the first-order condition of central planner problem on the Hamiltonian. For the control variable consumption, we use equation (42). However, the control variable u_j vanishes on the first-order condition (43). Therefore, we cannot apply the Arrow sufficient condition for all solutions to this problem, but we can apply for Balanced Growth Path solution. In such case, the Hamiltonian calculated on optimal control variable $\hat{H}$ is:

$$\hat{H}_t = \mu_j \frac{\left(\left(\frac{Ah^\gamma \mu_j}{\lambda_{j,t}^B \delta} e^{-\rho t} \right)^{\frac{1}{\sigma}} \right)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} + \lambda_t^A \delta \left(1 - \left(\frac{\rho - \delta(1+\gamma)(1-\sigma)}{\delta\sigma(1+\gamma)} \right) \right) h \quad (92)$$

To show that this Hamiltonian is concave in h , let us calculate the derivatives with respect to h :

$$\frac{\partial \hat{H}}{\partial h} = \frac{1}{\sigma} \left(\frac{A}{\lambda_{j,t}^B \delta} \right)^{\frac{1-\sigma}{\sigma}} (\mu_j e^{-\rho t})^{\frac{1}{\sigma}} \gamma h^{\frac{\gamma-\sigma(1+\gamma)}{\sigma}} + \delta \lambda^B \left(1 - \left(\frac{\rho - \delta(1+\gamma)(1-\sigma)}{\sigma(1+\gamma)} \right) \right) \quad (93)$$

As $\frac{\rho - \delta(1+\gamma)(1-\sigma)}{\delta\sigma(1+\gamma)} \leq 1$, then $\frac{\partial \hat{H}}{\partial h} > 0$.

The second-order derivative:

$$\frac{\partial^2 \hat{H}}{\partial h^2} = \frac{\gamma - \sigma(1+\gamma)}{\sigma} \frac{1}{\sigma} \left(\frac{A}{\lambda_{j,t}^B \delta} \right)^{\frac{1-\sigma}{\sigma}} (\mu_j e^{-\rho t})^{\frac{1}{\sigma}} \gamma h^{\frac{\gamma-\sigma(1+\gamma)}{\sigma} - 1} \quad (94)$$

If $\theta \in \Theta_1$, then $\frac{\partial^2 \hat{H}}{\partial h^2} < 0$.

F: Set of parameters - Central Planner Problem:

The equilibrium value of the aggregate working time on the BGP can be written as:

$$u_c^* = \frac{\rho - \delta(1 + \gamma)}{\sigma(1 + \gamma)\delta} + 1$$

The value of u_c^* lies on the unit interval, when the two inequality below is satisfied. For the amount of time u_c^* be positive, we need:

$$\frac{\rho - \delta(1 + \gamma)}{\sigma(1 + \gamma)\delta} > -1 \quad (95)$$

For the amount of working time u_c^* be less or equal to one, we must require:

$$\frac{\rho - \delta(1 + \gamma)}{\sigma(1 + \gamma)\delta} \leq 0 \quad (96)$$

The inequality (95) implies that $\sigma > 1 - \frac{\rho}{(1+\gamma)\delta}$. As the denominator of (96) is always positive, to this inequality holds we must have $\rho \leq \delta(1 + \gamma)$. We define Θ_4 as the set of parameters that these inequalities hold:

$$\Theta_4 = \left\{ \theta \in \Theta \mid \rho \leq \delta(1 + \gamma) \text{ and } \sigma > 1 - \frac{\rho}{\delta(1 + \gamma)} \right\}$$

G: Set of parameters for the centralized and the decentralized problem

The value of the aggregate working time in equilibrium lies in the unit interval, if and only if four inequalities below holds:

$$\frac{\rho - \delta(1 + \gamma)(1 - \sigma)}{\delta(\sigma(1 + \gamma) - \gamma)} > 0 \quad (97)$$

$$\frac{\rho - \delta(1 + \gamma)(1 - \sigma)}{\delta(\sigma(1 + \gamma) - \gamma)} \leq 1 \quad (98)$$

$$\frac{\rho - \delta(1 + \gamma)(1 - \sigma)}{\delta\sigma(1 + \gamma)} \geq 0 \quad (99)$$

$$\frac{\rho - \delta(1 + \gamma)(1 - \sigma)}{\delta\sigma(1 + \gamma)} \leq 1 \quad (100)$$

As already shown, (99) and (100) form the set of parameters Θ_4 and the inequalities (97) and (98) form the set of parameters Θ_1 and Θ_2 . These four inequalities together imply that two sets could happen:

CASE 1:

- (1) by (99): $\rho - \delta (1 + \gamma) (1 - \sigma) \geq 0$; and
- (2) by (100): $\rho - \delta (1 + \gamma) (1 - \sigma) \leq \delta \sigma (1 + \gamma)$; and
- (3) by (97): (3.1) $\sigma (1 + \gamma) > \gamma$ and (3.2) $\rho - \delta (1 + \gamma) (1 - \sigma) > 0$; and
- (4) by (98): $\rho \leq \delta$

Inequalities (1) and (3.2) imply that:

$$(5) \rho - \delta (1 + \gamma) (1 - \sigma) > 0.$$

Inequalities (2) and (5) imply that:

$$(6) \sigma (1 + \gamma) > \frac{\delta(1+\gamma)-\rho}{\delta}$$

Therefore, the inequalities (3.1), (4) and (6) must hold. Comparing inequalities (3.1) and (6), we see that one of them is stronger. Let us suppose that (3.1) is stronger, so $\frac{\delta(1+\gamma)-\rho}{\delta} < \gamma$, which implies that $\delta < \rho$. But this is absurd, because (4) does not hold. Therefore, the inequalities (97), (98), (99) and (100) imply:

$$(6) \sigma (1 + \gamma) > \frac{\delta(1+\gamma)-\rho}{\delta} \text{ and}$$

$$(4) \rho \leq \delta.$$

Combining (6) and (4), we have: $(1 - \sigma) (1 + \gamma) < \frac{\rho}{\delta} \leq 1$. Note that, these inequalities form the set of parameters Θ_1 .

Now we need to demonstrate that inequalities (6) and (4) imply the inequalities (97), (98), (99) and (100). If $(1 - \sigma) (1 + \gamma) < \frac{\rho}{\delta}$, then $\sigma (1 + \gamma) - \gamma > 1 - \frac{\rho}{\delta}$. But $\frac{\rho}{\delta} \leq 1$, then $1 - \frac{\rho}{\delta} \geq 0$. Therefore, $\rho - \delta (1 - \sigma) (1 + \gamma) > 0$, which implies that (??) and (??).

Let us suppose for contradiction that (98) does not hold. In such case, $\rho - \delta (1 - \sigma) (1 + \gamma) > \delta (\sigma (1 + \gamma) - \gamma)$, which implies $\rho - \delta (1 + \gamma) > -\delta \gamma$ or $\rho > \delta$, absurd.

Now, let us assume for contradiction that (100) does not hold. In such case, $\rho - \delta (1 - \sigma) (1 + \gamma) > \delta \sigma (1 + \gamma)$, which implies $\rho - \delta (1 + \gamma) > 0$ and $\frac{\rho}{\delta} > 1 + \gamma \geq 1$, absurd.

The inequalities (97), (98), (99) and (100) are equivalent to: $(1 - \sigma) (1 + \gamma) < \frac{\rho}{\delta} \leq 1$. The intersection of Θ_1 and Θ_4 is Θ_1 . Therefore, $\Theta_1 \subseteq \Theta_4$.

CASE 2:

- (1) by (99): $\rho - \delta (1 + \gamma) (1 - \sigma) \geq 0$; and
- (2) by (100): $\rho - \delta (1 + \gamma) (1 - \sigma) \leq \delta \sigma (1 + \gamma)$; and
- (7) by (97): (7.1) $\sigma (1 + \gamma) < \gamma$ and (7.2) $\rho - \delta (1 + \gamma) (1 - \sigma) < 0$; and
- (8) by (98): $\rho \geq \delta$

From (1) and (7.2), we find that in Case 2, the set is empty.

H: Non-economic solution of the Central Planner Problem

Suppose that $\theta \in \Theta_4$. Let us assume, for contradiction that the Transversality condition of the Central Planner problem holds:

$$\lim_{t \rightarrow \infty} \int_0^\infty \lambda_{j,t}^A h_j dj = 0$$

The first-order condition (43) implies that the co-state variable associated with the human capital is equal for all agents, $\lambda_{j,t}^A = \lambda_t^A$, so:

$$\lim_{t \rightarrow \infty} \int_0^\infty \lambda_{j,t}^A h_j dj = \lim_{t \rightarrow \infty} \lambda_t^A \int_0^1 h_j dj$$

From the definition of the average human capital $\int_0^1 h_j dj = h(t)$, which implies that:

$$\lim_{t \rightarrow \infty} \int_0^\infty \lambda_{j,t}^A h_j dj = \lim_{t \rightarrow \infty} \lambda_t^A h(t) \quad (101)$$

Using the function $u(t)$, (51), we find the average human capital solution path:

$$h(t) = h(0) e^{\delta t} \left(1 - \frac{u(0)}{u_C^*} (1 - e^{\psi t}) \right)^{\frac{1}{1+\gamma}} \quad (102)$$

From FOC (44), we see that the growth rate of the co-state variable depends only on the aggregate working time. Let us replace the solution of $u(t)$ on (44) and solve this differential:

$$\lambda(t) = \lambda(0) e^{-\delta t} \left(1 - \frac{u_0}{u_C^*} (1 - e^{\psi t}) \right)^{\frac{\gamma}{1+\gamma}} \quad (103)$$

Replacing the solutions (102) and (103) in (101), we get:

$$\lim_{t \rightarrow \infty} \lambda_t^A h(t) = \lim_{t \rightarrow \infty} \lambda(0) h(0) \left(1 - \frac{u(0)}{u_C^*} (1 - e^{\psi t}) \right) \quad (104)$$

Recall that when $\theta \in \Theta_4$, then $\psi < 0$, therefore this limits is equal to:

$$\lim_{t \rightarrow \infty} \lambda_t^A h(t) = \lambda(0) h(0) \left(1 - \frac{u_0}{u_C^*} \right) \neq 0 \quad (105)$$

which is absurd.

I: Proof of Lemmas

Proof of Lemma 1

To study the monotonicity of the non-degenerate solution $u(t)$, we calculate the first-order derivative of $u(t)$ with respect to time:

$$\frac{\partial u(t)}{\partial t} = -u(0) u^* \psi e^{\psi t} \frac{u(0) - u^*}{(u^* - u(0)(1 - e^{\psi t}))^2} \quad (106)$$

The behavior of the derivative of the function $u(t)$ depends on the difference between $u(0)$ and u^* and the sign of ψ . First, let us consider the set of parameters Θ_1 . If $\theta \in \Theta_1$, then $\psi < 0$. If $u(0) < u^*$, then $\frac{\partial u(t)}{\partial t} \leq 0$ and $u(t)$ is monotonically non-increasing. On the other hand, if $u(0) > u^*$, then $\frac{\partial u(t)}{\partial t} \geq 0$ and $u(t)$ is monotonically non-decreasing.

Now, let us consider the set of parameters Θ_2 . If $\theta \in \Theta_2$, then $\psi > 0$. If $u(0) < u^*$, then $\frac{\partial u(t)}{\partial t} > 0$ and $u(t)$ is monotonically increasing. On the other hand, if $u(0) > u^*$, then $\frac{\partial u(t)}{\partial t} < 0$ and $u(t)$ is monotonically decreasing.

J: Proof of Propositions:

Proof of Proposition 1

The differential equation (19) is a Riccati equation with two degenerate solution and the non-degenerate solutions given by (28). In section 3.2.3, we showed that the degenerate zero solution is not an equilibrium, the second degenerate solution is the BGP equilibrium solution. To show that the second degenerate solution is stable, we will argue that, for some parameter values, the non-degenerate solution $u(t)$ converges to the second solution as time goes to infinity. Recall that the aggregate time devoted to work on the BGP is given by:

$$u^* = \frac{\delta(1+\gamma)(1-\sigma) - \rho}{\delta(\gamma - \sigma(1+\gamma))}$$

Let $u(0) > 0$, for any $u(0) \neq u^*$, taking the limit of the $u(t)$ as time goes to infinity:

$$\lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} u^* \left(1 - \frac{1 - \frac{u^*}{u_0}}{\exp(\psi t)} \right)^{-1}$$

As u^* is constant:

$$\lim_{t \rightarrow \infty} u(t) = u^* \lim_{t \rightarrow \infty} \left(1 - \frac{1 - \frac{u^*}{u_0}}{\exp(\psi t)} \right)^{-1}$$

First, let $\theta \in \Theta_1$, which implies that $\psi < 0$. Let us consider the case $u(0) < u^*$, then:

$$\lim_{t \rightarrow \infty} \left(1 - \frac{1 - \frac{u^*}{u_0}}{\exp(\psi t)} \right)^{-1} = 0$$

Therefore, the solution $u(t)$ converges to the degenerate zero solution.

$$\lim_{t \rightarrow \infty} u(t) = 0$$

Let us consider now the case $u(0) > u^*$. In such case, as shown in the proof of Lemma 2.1, $u(t)$ is monotonically increasing. It implies that $u(t)$ does not converge to u^* . Therefore, the solution u^* is unstable when $\theta \in \Theta_1$.

Now, let $\theta \in \Theta_2$, which implies $\psi > 0$:

$$\lim_{t \rightarrow \infty} \left(1 - \frac{1 - \frac{u^*}{u_0}}{\exp(\psi t)} \right)^{-1} = 1$$

Therefore,

$$\lim_{t \rightarrow \infty} u(t) = u^* = \frac{\delta(1+\gamma)(1-\sigma) - \rho}{\delta(\gamma - \sigma(1+\gamma))}$$

The solution $u(t)$ converges to the solution u^* ; therefore, the solution u^* is globally stable, since the only solution that does not converge to u^* is the zero solution. Moreover, by Lemma 2.1, we know that function $u(t)$ converge to the BGP equilibrium monotonically within in the range $(0, 1)$.

Proof of Proposition 2

Suppose that $\theta \in \Theta_2$. Let us argue that at the initial time, agents do not allocate a different amount of time to work, i.e., $u_j(0) = u(0)$ for some j . Let us assume that at the initial time agents do not choose not to work, $u(0) \neq 0$. To find the individual amount of time devoted to work on the Balanced Growth Path, we take the limit of the function $u_j(t)$ as time goes to infinity.

$$\lim_{t \rightarrow \infty} u_j(t) = \lim_{t \rightarrow \infty} \left(\frac{e^{\psi t}}{\frac{1}{u_0} + \left(\frac{1}{u_{j0}} - \frac{1}{u_0} \right) \left(1 - \frac{u_0}{u^*} (1 - e^{\psi t}) \right)^{\frac{(1-\sigma)\gamma}{(\gamma-\sigma(1+\gamma))}} - \frac{1}{u^*} (1 - e^{\psi t})} \right)$$

This limit involves an indeterminate form. We must apply the L'Hôpital Rule:

$$\lim_{t \rightarrow \infty} u_j(t) = \lim_{t \rightarrow \infty} \left(\frac{\psi e^{\psi t}}{\left(\frac{1}{u_{j0}} - \frac{1}{u_0} \right) \left(\frac{(1-\sigma)\gamma}{(\gamma-\sigma(1+\gamma))} \right) \left(1 - \frac{u_0}{u^*} (1 - e^{\psi t}) \right)^{\frac{\sigma}{(\gamma-\sigma(1+\gamma))}} \left(\frac{u_0}{u^*} \psi e^{\psi t} \right) + \frac{\psi e^{\psi t}}{u^*}} \right)$$

We can simplify this limit using the value of u^* given in (22):

$$\lim_{t \rightarrow \infty} u_j(t) = \lim_{t \rightarrow \infty} \left(\frac{\psi e^{\psi t}}{\left(\frac{u_0}{u_{j0}} - 1 \right) \left(\frac{(1-\sigma)\delta\gamma}{\delta(1+\gamma)(1-\sigma)-\rho} \right) \left(1 - \frac{u_0}{u^*} (1 - e^{\psi t}) \right)^{\frac{\sigma}{(\gamma-\sigma(1+\gamma))}} (\psi e^{\psi t}) + \frac{\psi e^{\psi t}}{u^*}} \right)$$

We also divide the numerator and the denominator by $\psi e^{\psi t}$:

$$\lim_{t \rightarrow \infty} u_j(t) = \lim_{t \rightarrow \infty} \left(\frac{1}{\left(\frac{u_0}{u_{j0}} - 1 \right) \left(\frac{(1-\sigma)\delta\gamma}{\delta(1+\gamma)(1-\sigma)-\rho} \right) \left(1 - \frac{u_0}{u^*} (1 - e^{\psi t}) \right)^{\frac{\sigma}{(\gamma-\sigma(1+\gamma))}} + \frac{1}{u^*}} \right)$$

If $u_j(0) \neq u(0) \exists j$, then $\lim_{t \rightarrow \infty} u_j(t) = 0$. This implies that in the aggregate amount of time devoted to work, u , is equal to zero. Although it is a mathematical equilibrium, it is not an

economic equilibrium, because it does not satisfy the Transversality Condition. Therefore, at the beginning of time agents must devote the same amount of time to work, $u_j(0) = u(0) \forall j$.

Considering that $u_j(0) = u(0) \forall j$, the limit of $u_j(t)$ is equal to the equilibrium aggregate amount of time devoted to work:

$$\lim_{t \rightarrow \infty} u_j(t) = \frac{\delta(1+\gamma)(1-\sigma) - \rho}{\delta(\gamma - \sigma(1+\gamma))} \quad (107)$$

It proves the part (ii) of the Proposition 2.

Proof of Proposition 3

Let us argue that, at the beginning of time, individuals choose $u_j(0) = u(0)$. For contradiction, we assume that a individual choose $u_j(0) \neq u(0)$. Let us analyze the behavior of $u_j(t)$ in the long-term:

$$\lim_{t \rightarrow \infty} u_j(t) = \lim_{t \rightarrow \infty} \frac{1}{\frac{1}{u^*} + \left(\frac{1}{u_j(0)} - \frac{1}{u^*} \right) e^{\delta u^* t}} = 0 \quad (108)$$

If $u_j(0) \neq u^*$, then $\lim_{t \rightarrow \infty} u_j(t) = 0$. This implies that in the aggregate amount of time devoted to work, $u(t)$, is equal to zero, which is absurd. Because, if the economy starts on the Balanced Growth Path, it remains on the equilibrium, therefore $u(t) = u^*$. Therefore, at the beginning of time, agents dedicate the same time to work: $u_j(0) = u^*$.

Proof of Proposition 4

Only on the subset Θ_1 , the aggregate working time of both the centralized and decentralized problem is in the unit interval. Therefore, we can just compare the two optimal working time allocations on the subset Θ_1 . Let us suppose for contradiction that $u_C^* \geq u^*$:

$$\frac{(1+\gamma)\delta(\sigma-1) + \rho}{\sigma(1+\gamma)\delta} \geq \frac{\delta(1+\gamma)(1-\sigma) - \rho}{\delta(\gamma - \sigma(1+\gamma))}$$

In Θ_1 , the inequality $(\gamma - \sigma(1+\gamma)) < 0$ holds, then:

$$((1+\gamma)\delta(\sigma-1) + \rho)\delta(\gamma - \sigma(1+\gamma)) < (\delta(1+\gamma)(1-\sigma) - \rho)\sigma(1+\gamma)\delta$$

It implies that:

$$\gamma\delta((1+\gamma)\delta(\sigma-1) + \rho) < 0 \quad (109)$$

In Θ_1 , we know that $((1+\gamma)\delta(\sigma-1) + \rho) > 0$. Therefore, for inequality (109) hold, we must have $\gamma\delta < 0$, which is an absurd. Therefore $u_C^* < u^*$, as we would like to demonstrate.

Proof of Proposition 5

Recall that the aggregate time dedicated to work on the BGP of the Central Planner Problem is:

$$u_C(t) = u_C^* \left(1 - \left(1 - \frac{u_C^*}{u(0)} \right) \frac{1}{\exp(\psi t)} \right)^{-1}$$

Taking the limit of $u_C(t)$ as time goes to infinity:

$$\lim_{t \rightarrow \infty} u_C(t) = \lim_{t \rightarrow \infty} u_C^* \left(1 - \left(1 - \frac{u_C^*}{u(0)} \right) \frac{1}{\exp(\psi t)} \right)^{-1}$$

Suppose that $\theta \in \Theta_4$, which implies that $\psi < 0$, then:

$$\lim_{t \rightarrow \infty} u_C^* \left(1 - \left(1 - \frac{u_C^*}{u(0)} \right) \frac{1}{\exp(\psi t)} \right)^{-1} = 0$$

Therefore, the solution $u_C(t)$ converges to the degenerate zero solution:

$$\lim_{t \rightarrow \infty} u_C(t) = 0$$

The solution $u_C(t)$ converge to the degenerate solution zero on the long-term, therefore the Balanced Growth Path is unstable.

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