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Networks: a random walk in degree space

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Dissertation presented to the School of Arts, Sciences and Humanities of the University of São Paulo to obtain the title of Master of Science by the Graduate Program in Modeling of Complex Systems.

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ABSTRACT

AMPUERO, Fernanda. **Networks**: a random walk in degree space. 2018. 90 p. Dissertation (Master of Science) – School of Arts, Sciences and Humanities, University of São Paulo, São Paulo, 2018. Corrected version.

The present work aims to contribute to the study of networks by mapping the temporal evolution of the degree to a random walk in degree space. We analyzed how and when the degree approximates a pre-established value through a parallel with the first-passage problem of random walks. The mean time for the first-passage was calculated for the dynamical versions the Watts-Strogatz and Erdős-Rényi models. We also analyzed the degree variance for the random recursive tree and Barabási-Albert models.

Keywords: Networks. Random walk.

RESUMO

AMPUERO, Fernanda. **Redes**: um passeio aleatório no espaço dos graus. 2018. 90 p. Dissertação (Mestrado em Ciências) – Escola de Artes, Ciências e Humanidades, Universidade de São Paulo, São Paulo, 2018. Versão corrigida.

O presente trabalho visa contribuir com a pesquisa na área de redes através do mapeamento da evolução temporal do grau com um passeio aleatório no espaço do mesmo. Para tanto, foi feita uma análise de quando e como a quantidade de ligações do vértice se aproxima de um valor pré-estabelecido, mediante um paralelo com o problema da primeira passagem de passeios aleatórios. O tempo médio para a primeira passagem para as versões dinâmicas dos modelos Watts-Strogatz e Erdős-Rényi foram calculados. Além disso, foi realizado um estudo da variância do grau para os modelos da árvore recursiva aleatória e Barabási-Albert.

Palavras-chave: Redes. Passeio aleatório.

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1 Introduction

1.1 Motivation

According to (7, p.13), a complex system is “(...) a system in which large networks of components with no central control and simple rules of operation give rise to complex collective behavior, sophisticated information processing, and adaptation via learning or evolution”. In many fields of study, such as physics, biology, ecology and social sciences, the object of study can be a complex system and can, therefore, be represented as a network - *i.e.*, a collection of points — the components — joined in pairs by lines — the interactions between the components — (6). Some examples of complex systems that are studied through the science of networks are the World Wide Web, the Internet and neural systems (2, 7, 6).

Although a network is a simplified representation of a system that captures only the basics of connection patterns and a few other information, it is a powerful tool to understand interactions between parts of a system (6). The fact that, for example, the topology of social networks affects the spread of information shows the importance of knowing how a network is structured (6).

The study of the properties of networks led to the development of many mathematical, statistical and computational tools that can be used to analyze, model and understand how systems behave. Because these tools work with networks in their abstract form, they can in theory be applied to almost any system (6). Network models like Erdős-Rényi (8) and Watts-Strogatz (3) were attempts to represent real-world systems. As it would be pointed out later, these models are not able to represent some characteristics exhibited by the majority of systems we observe, such as the Internet or ecological networks (9, 6). Yet they helped constructing random graphs that are used, for example, for modeling networks in epidemiology and processes such as blood clotting and star formation (10, 11). Also, these models enlightened new research towards models that are more close to represent real-world networks such as the Barabási-Albert model (9).

Among the properties of networked systems that can be measured or modeled is the degree of a vertex, which is the quantity of edges attached to it. In many cases the vertices with the highest degrees in a network play important roles in the system. Also, it can be used to measure the density of a network, and thus give information as whether

the network is dense or sparse (6). The networks density is related to its resilience to random failures or attacks which, unfortunately, some networks are subject to, such as the Internet and supply chains (12). In other words, the science of networks is of great importance to the understanding of a wide variety of systems. And given the role that network properties play in this sense, it is sensible to study the properties of networks from different perspectives.

The theory of random walks has been applied to several studies in networks, such as electric currents, drug-target interactions and search algorithms, among many others (13, 14, 15). An extensively studied subject of random walks is the first-passage process, which is the occurrence of an event that depends on a variable reaching a threshold for the first time, such as the firing of a neuron or a chemical reaction (16). Some features and properties of networks have been studied under the perspective of a first-passage process, as for example the mean time for a random walker to reach a specified site in the network (17). However there is no study focused on estimating how long does it takes for a vertex to reach a specified degree.

Therefore, the present study aims to contribute to the study of vertex degree under an analytical perspective. We will map the temporal evolution of the degree to a random walk in degree space and analyze how and when the degree achieves a pre-established value for the first time through a parallel with the first-passage process. In addition, the degree variance of two different growing network models will be analyzed analytically and the results will be compared using computer simulations.

1.2 *Text organization*

The text presents a brief literature review on networks (chapter 2) and random walks (chapter 3). Chapter 2 comprises the network models used on this study, as well as its main properties. In chapter 3 we provide a study on the basics of random walks, which are the basis for developing the work presented in posterior sections. Chapters 4 and 5 comprise our study on, respectively, degree variance for the random recursive tree and Barabási-Albert models, and the first-passage process in degree space applied to the dynamical versions of the Erdős-Rényi and Watts-Strogatz models. The complete calculus for the equations shown in chapters 4 and 5 are presented as appendices.

2 About networks

The study of networks began in the middle of the 18th century, when in 1736 the mathematician Leonhard Euler offered a solution to a classic problem known as *the Königsberg bridge* (9). The city of Königsberg in Prussia (nowadays Kaliningrad, Russia) was set on both sides of a river, and included two islands. The two islands were connected between them and with the other two mainland portions of the city by seven bridges (figure 1). The problem was to find a path that would cross each of the seven bridges only once. Euler proved that there is no solution to this problem.

Figure 1 – A map of Königsberg in the eighteenth century, with its seven bridges highlighted (1).

The key to Euler's solution was to consider each island and land portion as a node and each bridge as an edge. With this he started a branch of mathematics known as graph theory, which is the basis for the knowledge on networks (9).

Networks are sets of elements (the vertices) and connections (the edges) that link the elements typically in pairs (1, 6) and despite this sounds simple, they are present everywhere (9). Almost anything can be elements and connections, for example people and friendships, or molecules and chemical reactions (1). Figure 2 shows the network representation of the bridges of Königsberg.

Complex systems in general can be represented as networks: its components are the elements and the interactions between them are the connections (6, 18). Some examples of complex systems that are studied through network science include the World Wide Web, the Internet, neural systems, genetic networks, airline transportation and many others (6, 9, 18). Figure 3 shows an example of a real network.

Figure 2 – Network representation of the bridges of Königsberg. Each land portion is represented as a node and the bridges are the links between the nodes.

Figure 3 – A Twitter network for a political hashtag in the USA produced by Truthy, a free tool for analyzing how information spreads on Twitter. Available at: <https://plus.maths.org/content/probing-dark-web/>.

The systems behavior is consequence of the pattern of connections between components (6), therefore the structure of a network has direct implication on it (6, 19). That is why studying the properties of networks is so important. Over the past few years many tools for analyzing, modeling and understanding networks have been developed (6). These tools help to answer questions such as why rumors spread so quickly, or what types of events can cause an once-stable ecological community to fall apart (7).

Designing models that reproduce the properties of real networks is not an easy task. There is an apparent randomness in many real networks (20) and until the first half of the 20th century graph theory was mainly about regular graphs. It was in 1959 that the mathematicians Paul Erdős and Alfréd Rényi began formally the study of random networks in graph theory (20). Basically, a random network is generated by randomly

placing links between vertices. And although reproducing real networks with random graphs was never Erdős and Rényi's intent, it seemed to be a very straightforward way to mimic networks that are too complex to be captured in simple terms (20, 9).

The random network theory of Erdős and Rényi created several paradigms present in the minds of those who were scientifically thinking about networks in the 1960's; complexity and randomness were basically synonyms (9). However, it is somehow evident that real-world networks must have different organizing principles from the random network model (9). It seems a little odd that society, for example, would be organized the way it is if the bonds that build this network were simply and completely random. Still, the work of Erdős and Rényi was of extreme importance to the development of subsequent work in graph theory (9). Since then researchers have been trying to understand and reproduce behavioral patterns of real networks.

In 1967 the psychologist Stanley Milgram drew attention to the idea of *small world*. Milgram's experiment on social networks showed that a person can reach to anyone else in only a few steps (21). The experiment consisted in giving letters to a few hundred people randomly selected. The letters were to be sent to a specific person. It was very unlikely that they knew who was supposed to receive the letter, so they should send it to someone they did know and who they thought to be someone closer to the target. The result revealed that the letter would pass by, in average, six people until it arrived to its final destination. When Milgram published his results he omitted the fact that only 18 of the initial 300 letters actually arrived the final destination. But by the time someone realized that Milgram's experiment was not universally applicable, the idea of *six degrees of separation* was already disseminated (21).

There is a paradox in Milgram's experiment. At the same time that people are only a few steps away distant from each other, many of someone's friend are also friends with each other. This brings to light that the world is actually highly clustered (21). Mark Granovetter's paper published in 1973, *The Strength of Weak Ties* (22), presents this idea that society is structured into highly connected clusters, with a few connection between these clusters keeping them from being isolated (9). It was only 25 years later that the researchers Duncan J. Watts and Steven Strogatz came up with a model that produces real small-world networks with high clustering (3).

Watts and Strogatz were able to combine in one model the Erdős-Rényi worldview of randomness with Granovetter's clustered society (9). With this model they were able to

Figure 4 – National Highway System of the United States. Available at: <https://en.wikipedia.org/>

reproduce some real small-world networks, such as the power grid of the western United States, neural network of the worm *Caenorhabditis elegans* and the collaboration graph of film actors (3). Still, the model was not suitable for reproducing large complex networks (9).

Networks such as the World Wide Web (WWW), food webs or chemical reactions happening in body cells present hubs, which are nodes with an extraordinarily large number of links. And neither the Erdős-Rényi or Watts-Strogatz model are able to explain the emergence of hubs (9). While studying the WWW, Albert-László Barabási and his coworkers found that the distribution of links on various Webpages does not follow a peaked distribution. Instead, it follows what is called a *power-law* distribution (23). That means that there is a large number of barely connected Webpages along with a smaller number of Webpages with a huge amount of links.

An easy way to visualize the difference between a network with a peaked distribution and one with a power-law distribution is to compare a road map with an airline routing map (figures 4 and 5, respectively). In the first one, major cities have at least one link to the highway system and there are no cities with hundreds of links to highways, while the other one has a majority of airports with at most a few links and a few airports with an anomalously high number of links (9).

In 1999, Albert-László Barabási and Réka Albert proposed a new model with two important new features: preferential attachment and growth (23, 9). With these, the Barabási-Albert model was able to reproduce the power-law scaling observed in networks

Figure 5 – US Airways route map. Available at: <http://thempfa.org/>

with complex topology, such as the WWW (23). Preferential attachment means that a given node is more likely to connect with nodes that have more links than others. Growth accounts for the addition of new nodes to the network for each given period of time. The previous models, Erdős-Rényi and Watts-Strogatz, both assumed that the network has a fixed number of nodes and that every node has the same probability of connecting.

The scale-free model opened a door to an avalanche of discoveries. Following Barabási and Albert's work several researchers around the world worked to incorporate new processes to the scale-free model, such as aging (nodes stop acquiring links after some time), internal links, rewiring (3), removal of nodes and links and many others. Today, there is a rich and consistent theory of network growth and evolution and how it impacts on the topology and stability of complex systems (9).

2.1 Network properties

Behind the study of networks there are tools used to describe and analyze its properties and characteristics. These tools come from graph theory, a branch of mathematics that studies pairwise relations between objects (1). Because the topology of a network is directly related to its properties, the quantitative data provided by these measures allows us to compare different models and understand the rules of interaction that generate different networks (9, 6, 18).

As mentioned in the previous section, networks are collections of vertices (or nodes) and edges (or links). The size of a network refers to the number of nodes N . Models such as Erdős-Rényi and Watts-Strogatz have a fixed number of vertices, while other models such

as Barabási-Albert can have an increasing number of vertices with time (2). These three models just cited will be presented ahead, in sections 2.2.2, 2.2.3 and 2.2.4, respectively.

Nodes can be connected in pairs by one or more edges. If each pair of nodes within a graph has only a single edge connecting that pair of nodes, then it is called a *simple network* or *simple graph*. But if at least one pair of nodes within the graph has more than one edge between that pair of nodes, the network is called a *multigraph* (6). There is also the possibility that a vertex connects to itself, creating a *self-loop*. Also, the vertices of a graph can be connected to each other through directed or undirected edges resulting in *directed* or *undirected* graphs, respectively (2). Figure 6 shows an example of each. All models considered in the present study generate undirected graphs.

Figure 6 – Examples of undirected and directed graphs. (A) Undirected graph, with vertices connected through undirected edges. (B) Directed graph, with vertices connected through directed edges.

Different models have different rules for nodes to connecting to each other. In the Erdős-Rényi model (8), pairs of vertices are randomly chosen and connected via undirected edges with the same probability. The Barabási-Albert model (23), on the other hand, also selects pairs of vertices at random, but it selects preferentially those vertices that have higher numbers of edges. Due to the randomness in this process, these networks are called *random networks*. It is important to say that a random network is actually an ensemble of networks (2). That is to say, that a particular network observed is just one sample of a statistical ensemble of all possible realizations (2).

The rules for connecting vertices dictates the degree of each vertex. In its turn, the amount of connections of a vertex has direct implication on the *path length* and the *clustering coefficient* (2). The first one is the distance between two nodes, which is the smallest number of steps connecting them. The second characterizes the density of the network, also defined as the network probability that two network neighbors of a vertex are also neighbors of each other (6). These two properties have been briefly mentioned in section 2.

It is possible to measure which is the most connected vertex in a graph (*degree centrality*), or if vertices are all connected with each other or not (*giant* and *small components*) (6, 18, 2). But since they all somehow depend on the quantity of edges attached to a vertex (*degree*), a brief topic on *degree* and *degree distribution* is presented next.

2.1.1 Degree and degree distribution

The number of edges connected to a given vertex in a graph is its *degree* (6). In an undirected graph every edge has two ends; therefore, if there are l edges, then there is a total of $M = 2l$ ends of edges. Consider an undirected graph of N vertices where the degree of a vertex i is denoted by k_i . Then,

$$M = \sum_{i=1}^N k_i, \quad (2.1)$$

and the mean degree of a vertex, denoted by $\bar{k}$, is

$$\bar{k} = \frac{1}{N} \sum_{i=1}^N k_i. \quad (2.2)$$

Therefore, by combining equations (2.1) and (2.2), it follows that

$$\bar{k} = \frac{M}{N}. \quad (2.3)$$

Usually, the degrees of vertices are statistically distributed, and whether a vertex is attached or not to another one depends on the rules of interaction of the network (2). These rules dictate how the network is constructed by specifying, for example, if the number of vertices is fixed or varies with time, what is the probability of a vertex being chosen at random and what is the probability of a pair of chosen vertices to connect to each other (2).

By definition, the degree distribution $P(k)$ is the fraction of nodes in the network with degree k (2), expressed as

$$P(k) = \frac{\langle N(k) \rangle}{N}, \quad (2.4)$$

where $N(k)$ is the number of vertices with degree k . It tells whether all the vertices in the network have roughly the same number of connections or not. In sections 2.2.2 to 2.2.4 we

will see that some networks have a degree distribution that follows a Poisson distribution, whilst others follow a power-law distribution where a large number of vertices have a few connections and a few number of vertices have lots of connections.

2.2 Networks used in this study

Over the years, many different models have been developed, each new model bringing new theories and data regarding how to build networks that explain what is observed in real-world networks. Four different network models were used in this study. The following subsections provide brief information on each model. The details of each model are presented in sections 4 and 5.

2.2.1 Random recursive tree model

In graph theory, a tree is a connected, undirected network that contains no closed loops (6), also denoted as acyclic graph (24). It opposes to cyclic graphs, in which an edge set forms a path such that the first node of the path corresponds to the last. Figure 7 shows an example of an acyclic and a cyclic graphs. Tree graphs have special application in data storage, searching and communication (24). In nature, tree graphs represent, for example, networks formed by rivers (6). Sometimes trees appear as a class of small subgraphs within a graph.

Figure 7 – Left: an acyclic graph. Right: a cyclic graph. Notice that there are closed loops in the cyclic graph (2).

Since trees have no closed loops, there is only one path between any pair of vertices. This property simplifies certain kinds of calculations, which is the main reason for sometimes using trees as a basic model of a network, like measures based on the shortest path (6, 24).

The random recursive tree model is a simple growing connected graph without cycles where new incoming vertices attach to each other randomly (without any preference) (2). Its recursiveness lies in the repetition of adding a new vertex and attaching it to other vertex at random at each time step.

2.2.2 The Erdős-Rényi model

Paul Erdős and Alfréd Rényi were two mathematicians who proposed in the 1950's that networks could be generated under a random process. Although scientists from a variety of fields would discover years later that most real-world networks are generated under very specific rules, the work of Erdős and Rényi was of extreme importance to the development of subsequent research (9).

In the Erdős-Rényi (ER) model, the network has a fixed number of vertices N , and each possible pairs of vertices is linked with probability p . The resulting graph has small-world property, which means that two randomly chosen nodes can be connected by very short paths (25). Figure 8 shows an example of this type of graph.

Figure 8 – Erdős-Rényi random graph model with 100 vertices and $p = 0.002$ generated by Python.

By pairing nodes randomly, it is quite possible that some nodes will have more links than others, and some will have fewer. However, if the network is large enough, then most of the nodes will have approximately the same number of links (9).

Random graphs of ER type have a binomial degree distribution, which can be expressed as

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k} . \quad (2.5)$$

Equation (2.5) represents the probability of a given vertex being connected to exactly k other vertices (6). The variable p is the independent probability of a vertex being connected to each of the $N-1$ other vertices, and $p^k(1-p)^{N-1-k}$ is the probability of being connected to a particular k other vertices and not to any of the others. The binomial represents the $\binom{N-1}{k}$ ways to choose those k other vertices. If $N \gg 1$ and $p \ll 1$, then the binomial distribution converges to a Poisson distribution (6),

$$P(k) = e^{-\bar{k}} \frac{\bar{k}^k}{k!} , \quad (2.6)$$

where $\bar{k}$ is the average degree $\bar{k} = \sum_{k=0}^{\infty} kP(k)$ (2). The ER model's degree distribution follows a Poisson distribution, as derived and proved by Erdős' student Béla Bollobás (9, 26).

2.2.3 The Watts-Strogatz model

For Duncan J. Watts and Steven Strogatz, networks can present three different types of connection topology: completely ordered, completely random, or somewhere in between. Erdős and Rényi had already proposed a completely random model. So Watts and Strogatz came up with the idea of combining the order of regular graphs with the randomness of the ER model (9, 3).

In order to achieve their idea, the network starts as a regular graph: a ring lattice which has a fixed number N of vertices and with k_0 edges per vertex. Each edge is then rewired at random with a given probability p . This process allows a transition between regularity ($p = 0$) and disorder ($p = 1$) (3), as shown in figure 9. The result is that shortcuts between distant vertices emerge, which is responsible for the property of small-world. Also, for a certain range of p , it can generate random graphs with a high degree of clustering, something that did not happen in the ER model (3, 18).

However, the networks generated by the Watts-Strogatz (WS) model have a degree distribution that follows a Poisson distribution (6). And most real-world networks follow a power-law distribution, as some authors would discover later (9). Despite that, the WS

Figure 9 – Rewiring procedure in the Watts-Strogatz model (3). For $p = 0$ the network shows a regular topology, but as p increases disorder is introduced generating a random network.

model was able to fit some real small-world networks, such as neural network of the worm *Caenorhabditis elegans*, the collaboration graph of film actors, among others (3).

2.2.4 The Barabási-Albert model

Back in the time when the Erdős-Rényi model was proposed, there were no data on large networks to test the predictions of the model for real networks (9). After the development of computer science and data acquisition, such topological information became available, and different softwares can now be used to generate graphs that represent networks topology as shown in figure 10.

Figure 10 – Left: undirected Erdős-Rényi network with 50 vertices and connection probability 0.1. Available at: (<http://mathinsight.org/>). Right: semi-random internet map with 2,000 nodes (4).

The two very different images presented in figure 10 shows that the Erdős-Rényi model cannot represent properly large real-world networks, such as the Internet. The

Figure 11 – Barabási-Albert graph consisting of 1000 nodes (5).

Barabási-Albert (BA) model (figure 11) is based on the fact that networks expand continuously by the addition of new vertices, and that new vertices attach preferentially to vertices that have higher degrees (23). That is why networks based on this model are also called preferential networks.

Growth and preferential attachment are mechanisms common to a number of complex systems, such as the World-Wide Web, the Internet, social networks and biological systems. These features are responsible for the emergence of hubs and the power-law scaling observed in networks generated by the BA model (23). Older nodes in the network have more time to acquire links. Also, since they will probably have more links than the recently added nodes, they will have more chances of earning new links. This implies that older nodes will be more likely to become hubs (23). As a result, the network will have lots of nodes with few links and a few nodes with a large quantity of links.

The degree distribution in the BA model follows a power-law distribution (23), expressed mathematically as

$$P(k) = C k^{-\alpha} \quad (2.7)$$

for $\alpha > 0$; C is the normalization constant (6). And this is the degree distribution seen in most real-world networks (23, 6).

The BA model had captured an important feature of real-world networks, although it did not represent these networks in every single detail. Still, it raised new questions like how could latecomer nodes become hubs in a world in which only the "rich gets richer", and brought the science of network closer to understanding the architecture of complexity (9).

3 About random walks

A random walk is a sequence of displacements of a randomly-moving object that wanders away from where it started in any direction, like a molecule traveling in a liquid or the path of a foraging animal (27). This corresponds to the stochastic process formed by the successive summation of independent, identically distributed random variables (or displacements) (27). Because of this general definition, random walks can be applied to a wide range of scientific fields. They have been used to describe configurational properties of polymers, the motion of microorganisms on surfaces, stock price behavior, among others (28).

The classical theory of random walks studies its qualitative behavior, asking questions like what is the mean displacement of a random walk, or what is the probability that a random walk returns to its starting point (29). These topics have been widely explored and there is plenty of literature available. Despite that, we present in the following subsections a brief introduction to the logics and calculus for three characteristics of random walks: the mean displacement, the recurrence and the first passage process. The following content corresponds to the essential knowledge on random walks that helped tracing a parallel between the behavior of a random walk and the temporal evolution of a degree (section 4.2).

3.1 Mean displacement of random walks

The average displacement of a random walk is usually referred to as the root mean square displacement, and it is represented by $X_{\text{RMS}} = \sqrt{\langle \vec{R}^2 \rangle}$, where the notation $\langle \dots \rangle$ accounts for the mean value (of many realizations) and $\vec{R}$ is a vector that represents the sum of all individual displacements. Given that a random walk is a set of random steps in different directions, it is possible to calculate the average displacement of a random walk after a considered number of steps.

Consider a random walk of t steps in d dimensions with fixed step length $r = \|\vec{r}_i\|$ for $1 \leq i \leq t$, with equal probability of moving in any direction, where $\vec{r}_i = (x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(d)})$ is the displacement vector of the i^{th} path. The total displacement is given by $\vec{R}$, where $\vec{R} = \vec{r}_1 + \vec{r}_2 + \dots + \vec{r}_t = \sum_{i=1}^t \vec{r}_i$, as shown in figure 12. The X_{RMS} for this example is obtained

Figure 12 – Random walk in d dimensions with fixed step length $||\vec{r}_1|| = ||\vec{r}_2|| = \dots = ||\vec{r}_t||$ and average displacement $\vec{R}$. The red dot marks the starting point.

by calculating $\langle \vec{R}^2 \rangle$, which is the mean for the sum of all individual displacements $\vec{r}_i$ squared, as shown below,

$$\langle \vec{R}^2 \rangle = \left\langle \left(\sum_{i=1}^t \vec{r}_i \right)^2 \right\rangle. \quad (3.1)$$

Expanding the sum on equation (3.1) results in the expression

$$\langle \vec{R}^2 \rangle = \left\langle \sum_{i=1}^t \sum_{j=1}^t \vec{r}_i \cdot \vec{r}_j \right\rangle = \left\langle \sum_{i=1}^t \sum_{j=1}^t \left[x_i^{(1)} x_j^{(1)} + \dots + x_i^{(d)} x_j^{(d)} \right] \right\rangle \quad (3.2)$$

or

$$\langle \vec{R}^2 \rangle = \left\langle \sum_{i=1}^t \sum_{j=1}^t \sum_{\alpha=1}^d x_i^{(\alpha)} x_j^{(\alpha)} \right\rangle = \left\langle \sum_{i=1}^t \sum_{\alpha=1}^d \left(x_i^{(\alpha)} \right)^2 \right\rangle + \left\langle \sum_{\substack{i,j \\ i \neq j}}^d \sum_{\alpha=1}^d x_i^{(\alpha)} x_j^{(\alpha)} \right\rangle. \quad (3.3)$$

For random walks the steps are not correlated, so the joint probability of the independent events x and y is $P(x, y) = P(x)P(y)$. Given that $\langle xy \rangle = \sum_x \sum_y xy P(x, y)$, then $\langle xy \rangle = \langle x \rangle \langle y \rangle$. Therefore, the second term on the right hand side of equation (3.3) can be rewritten as

$$\left\langle \sum_{\substack{i,j \\ i \neq j}}^d \sum_{\alpha=1}^d x_i^{(\alpha)} x_j^{(\alpha)} \right\rangle = \sum_{\substack{i,j \\ i \neq j}}^d \sum_{\alpha=1}^d \langle x_i^{(\alpha)} \rangle \langle x_j^{(\alpha)} \rangle. \quad (3.4)$$

Because $\langle x_i^{(\alpha)} \rangle = 0$ for every $1 \leq i \leq t$ and $1 \leq \alpha \leq d$, equation (3.4) is equal to 0. Since all $\vec{r}$ have the same length and $\sum_{\alpha=1}^d \left(x_i^{(\alpha)} \right)^2 = r^2$, equation (3.3) becomes

$$\langle \vec{R}^2 \rangle = \left\langle \sum_{i=1}^t r_i^2 \right\rangle = r^2 t, \quad (3.5)$$

and therefore the X_{RMS} in this example is

$$X_{\text{RMS}} = \sqrt{\langle \vec{R}^2 \rangle} = r\sqrt{t}. \quad (3.6)$$

This result shows that the displacement does not increase linearly with the number of steps, but only with the square root of the number of steps.

3.2 Recurrence of random walks

If a random walk returns to a given point, it is said that it is a recurrent random walk. Considering that the space is isotropic, a random walk can be delimited by a d -dimensional sphere of radius $r \sim \sqrt{t}$, which is the mean displacement of the walk after a time t , as shown in the previous section. The density ρ of points visited inside that sphere is then given by the *number of points visited over sphere's volume*. Dimensional analysis gives that the volume V for a d -dimensional sphere of radius r is $V \sim r^d$. Considering that after a time t the walker visited t points inside that sphere, then (16)

$$\rho \sim \frac{t}{(\sqrt{t})^d} . \quad (3.7)$$

In other words, the density of a random walk is a function of its dimension d and t .

The density (3.7) shows that $\rho \sim t^{1-\frac{d}{2}}$ and that for $d < 2$, it diverges, which means that certainly at some instant the walker will return to a given point — i.e., it has a recurrent behavior. Therefore, according to the density (3.7), a one dimensional random walk is recurrent. On the other hand, for $d > 2$, ρ approaches zero as t increases. Hence, its return is uncertain — i.e., it is a transient random walk. For $d = 2$, the equation gives that ρ is constant, and to conclude if a two dimensional random walk is recurrent or not, it is necessary to look at this particular case from the perspective of a first-passage process. Section 3.3 discusses the first-passage process and shows that a two dimensional random walk is recurrent.

3.3 First passage process

An important question for the study of random walks is what is the probability that a random walk first reaches a specified site at a specified time. This phenomenon is called the first-passage process. A neuron that fires only when a fluctuating voltage level first reaches a specified level, or the act of buying or selling orders when a stock price first reaches a threshold are examples of the application of the first-passage process (16).

There is an extensive literature showing how to estimate the probability that a random walk first reaches a specified site at a specified time (16, 10). Still, to facilitate the understanding of the parallel built between a random walk and variations of the degree

of a certain vertex, in this section we present the calculus involved in estimating the first-passage probability.

To calculate the probability that a random walk first reaches a specified site at a specified time, we can consider that time is continuous and investigate $F(\vec{r}, t)$, which is the probability density of the walker being at a point $\vec{r}$ at time t for the first time, and $P(\vec{r}, t)$, which is the occupation probability of the walker being at a point $\vec{r}$ at instant t . Figure 13 illustrates that for the walker to be at $\vec{r}$ at instant t — situation (A) — it can first reach this point at time t' ($\leq t$) — situation (B) —, which occurs with probability $F(\vec{r}, t')$, continue the walk and then return to $\vec{r}$ spending $t - t'$ — situation (C). This last event occurs with probability $P(0, t - t')$ (16), where the isotropy of the space is assumed.

Figure 13 – Probability of a random walk reaching a specified point $\vec{r}$ at instant t , illustrated at (A). The walker can first reach the point $\vec{r}$ at time t' ($\leq t$) (B), continue the walk and then return to $\vec{r}$ spending $t - t'$ (C).

Because space is isotropic, the probability density of returning to $\vec{r}$, starting from $\vec{r}$, equals the probability density of returning to $\vec{0}$ starting from the origin and spending the same amount of time — so we can say that, once the walker arrives at $\vec{r}$ at t' , the point $\vec{r}$ becomes the “new origin”. Summing over t' from 0 to t leads to

$$P(\vec{r}, t) = \int_0^t dt' F(\vec{r}, t') P(0, t - t') + e^{-2dt} \delta_{\vec{r}, 0}, \quad (3.8)$$

as presented in (10). The last term on the right hand side of equation (3.8) is the initial condition, where it was assumed that the walk starts at $\vec{r} = 0$ for $t = 0$. The exponential represents the probability that there is no hopping by time t , and d is the spatial dimension. This term is necessary because equation (3.8) represents a continuous-time random walk with a unit hopping rate between neighboring sites (30).

Since the integral on equation (3.8) is a convolution product, the Laplace transform can be used to solve the equation. For any function $G(x, t)$, the notation used for its Laplace transform is $G^L(\vec{x}, s) = \int_0^\infty e^{-st} G(\vec{x}, t) dt$. The Laplace transform for the occupation and first-passage probabilities are, respectively,

$$P^L(\vec{r}, s) = \int_0^\infty e^{-st} P(\vec{r}, t) dt \quad \text{and} \quad F^L(\vec{r}, s) = \int_0^\infty e^{-st} F(\vec{r}, t) dt. \quad (3.9)$$

Performing the Laplace transform of equation (3.8) leads to the algebraic equation

$$F^L(\vec{r}, s) = \frac{P^L(\vec{r}, s) - \frac{\delta_{\vec{r},0}}{s+2d}}{P^L(0, s)}, \quad (3.10)$$

as presented by (10), which shows that the probability density of the walker being at a point $\vec{r}$ at t for the first time is determined by the corresponding Laplace transform of the probability distribution $P(\vec{r}, t)$ (16). Now, if we consider only the case where $\vec{r} = 0$, then

$$F^L(0, s) = 1 - \frac{1}{(s + 2d)P^L(0, s)} = \int_0^\infty F(0, t) e^{-st} dt, \quad (3.11)$$

and for $s \rightarrow 0$ this last result implies $\int_0^\infty F(0, t) dt$, which is the return probability, since it is the sum of all the probabilities of being at $\vec{r} = 0$ for the first time at some moment.

To complete this section on random walks, it is necessary to calculate if a two-dimensional random walk is recurrent or transient. In order to do that, one must calculate $P^L(0, s)$ for $d = 2$ and $s \rightarrow 0$. This result is the Laplace transform of $P(0, t) = e^{-2Ddt} \prod_{j=1}^d I_0(2Dt)$, which is the probability of being at the origin at $t \neq 0$, and the calculus that show this result are presented in appendix C. The variable D is the probability of moving in any direction.

For long time, $P(0, t) = e^{-2Ddt} \prod_{j=1}^d I_0(2Dt)$ asymptotically goes to $\frac{1}{(4\pi D t)^{d/2}}$ (31).

Appendix D shows that the Laplace transform of $P(0, t)$ for $s \rightarrow 0$ and $d = 2$ results in

$$P^L(0, s) \simeq -\frac{1}{4\pi} \ln s, \quad (3.12)$$

and using the result from (3.12) in equation (3.11) gives

$$F^L(0, s) \simeq 1 + \frac{\pi}{\ln s} \quad (3.13)$$

when $s \sim 0$. This means that a random walk in two dimensions is also recurrent ($F^L(0, s \sim 0) \sim \int_0^\infty F(0, t) dt \sim 1$) (10).

4 Studies on degree variance for growing networks

There are many approaches for studying degree such as degree centrality, eigenvector centrality or transitivity. A simple approach for measuring the heterogeneity of vertices in a graph is to calculate the degree variance (32). The idea here is to calculate the degree variance of a given vertex in two different growing network models and analyze how the number of edges attached to it varies along time. In both models studied (random recursive tree and Barabási-Albert model), the networks are undirected.

The probability that a given vertex s has degree k at time t is denoted by $p(k, s, t)$. This probability is governed by the rules of interaction of the system, and plays an important role on the description of the systems' dynamics. From this probability, it is possible to obtain the mean degree and the mean squared degree for vertex s at time t — respectively $\bar{k}(s, t)$ and $\bar{k}^2(s, t)$ (both defined in the following sections) — which are fundamental expressions to calculate the degree variance for any model since $\text{var}(k) = \bar{k}^2(s, t) - \bar{k}^2(s, t)$.

Sections 4.1 and 4.2 present, respectively, the results for the analysis of degree variance for the random recursive tree model and the Barabási-Albert model. Final considerations are presented in section 4.3.

4.1 The random recursive tree model

The random recursive tree model is a simple growing connected graph without cycles where new incoming vertices attach to each other randomly (without any preference). Its dynamics has the following rules: at each time step, a new vertex is added to the graph and it is attached with equal probability to a randomly selected old vertex; the growth starts from two doubly connected vertices, $s = 1$ and $s = 2$, at time $t = 2$. This initial condition is merely a technical convenience, and it is not expected to affect the properties of the network after a sufficiently long time (2). Also, for $t > 2$, a recently added vertex $s = t$ can only have one connection, and this gives the boundary condition $p(k, s, t = s) = \delta_{k,1}$ for $t > 2$, where the Kroenecker delta $\delta_{k,j}$ is a function that makes this probability equal to 1 for $k = j$ and zero for $k \neq j$.

The probability that a vertex s has degree k at time $t + 1$ is given by

$$p(k, s, t + 1) = \frac{1}{t}p(k - 1, s, t) + \left(1 - \frac{1}{t}\right)p(k, s, t). \quad (4.1)$$

There are two possibilities for a vertex to have degree k at instant $t + 1$: (a) the vertex had degree $k - 1$ at the previous instant t and earned a new connection with probability $1/t$; or (b) the vertex already had degree k at the previous instant t and earned no connections.

The average degree of vertex s at time t is described by

$$\bar{k}(s, t) = \sum_{k=1}^{\infty} k p(k, s, t). \quad (4.2)$$

For $t \gg 1$, combining equations (4.1) and (4.2) results in

$$\bar{k}(s, t) \simeq 1 + \ln \left(\frac{t}{s} \right), \quad (4.3)$$

as shown in appendix A. Also from equations (4.1) and (4.2) results the equation that describes the average squared degree for individual vertices (calculus presented also in appendix A),

$$\overline{k^2}(s, t) \simeq \ln^2 \left(\frac{t}{s} \right) + 3 \ln \left(\frac{t}{s} \right) + 1. \quad (4.4)$$

Given that $\text{var}(k) = \overline{k^2}(s, t) - \bar{k}^2(s, t)$, the variance for this first model is

$$\text{var}(k) = \overline{k^2} - \bar{k}^2 \simeq \ln \left(\frac{t}{s} \right). \quad (4.5)$$

The results for equations (4.3), (4.4) and (4.5) are asymptotic for long time.

A Python code (appendix E) was written in order to simulate the evolution of the network and determine the mean variance from multiple realizations. The code was built following the steps below:

1. Initial settings: define the maximum number of vertices, which matches the maximum time for the dynamics; choose a vertex to analyze; create lists for summing the degree and the degree squared.
2. For the simulation:
 - a) create lists for appending time steps and the degree of the chosen vertex;
 - b) create a loop to: add a new vertex at each time step, select a vertex at random and append the actual degree of the chosen vertex to the degree's list depending on if it has earned a new connection or not.
 - c) Sum the degrees and degrees' squared.
3. Repeat steps 2a to 2c as many times as needed in order to improve the statistics.
4. With the lists that contain the values of the sum of the degrees and the sum of the degrees squared, calculate the variance.

Figure 14 shows the graph of the mean variance for 1000 realizations and $s = 3$ (this vertex was chosen in order to follow its evolution for a long time) and t up to 10^5 . The two curves in the graph represent the analytical and numerical results for this simulation, and they show that, in a network without preferential connections, the variance degree does not increase linearly with time, as expected from (4.5).

Figure 14 – Mean degree variance for $s = 3$ for 1000 realizations and t up to 10^5 — random recursive tree model. Blue curve: analytical results; red curve: numerical results.

4.2 The Barabási-Albert model

The second model is the Barabási-Albert network (23), in which a vertex attaches preferentially to vertices with higher degrees. Its dynamics has the following rules: the growth starts from two doubly connected vertices, $s = 1$ and $s = 2$, at time $t = 2$, and at each time step a new vertex is added to the network with a single link. Again, the initial condition is merely a technical convenience, and it is not expected to affect the properties of the network after a long time. With this convenient choice, the total degree of the system at time t is $2t$. Also, like the previous model, the boundary condition is $p(k, s, t = s) = \delta_{k,1}$ for $t > 2$.

The difference from the previous model is that here the attachment of a new vertex is proportional to its degree, so the probability that a vertex of degree k gets a new connection is $\frac{k}{2t}$ at time t — recall from the previous paragraph that the total degree of

the network at time t is $2t$. The probability that a vertex s has degree k at time $t + 1$ is given by

$$p(k, s, t + 1) = \frac{k - 1}{2t} p(k - 1, s, t) + \left(1 - \frac{k}{2t}\right) p(k, s, t). \quad (4.6)$$

There are two possibilities for a vertex to have degree k at instant $t + 1$: (a) the vertex had degree $k - 1$ at the previous instant t and earned a new connection with probability $\frac{k-1}{2t}$; or (b) the vertex already had degree k at the previous instant t and earned no connections. Like the previous model, the average degree of vertex s at time t is described as

$$\bar{k}(s, t) = \sum_{k=1}^{\infty} k p(k, s, t). \quad (4.7)$$

For $t \gg 1$, combining equations (4.6) and (4.7) results in the following expression for the average degree for individual vertices,

$$\bar{k}(s, t) \simeq \left(\frac{t}{s}\right)^{\frac{1}{2}}, \quad (4.8)$$

as shown in appendix B. Also from (4.6) and (4.7) results the equation that describes the average squared degree for individual vertices (calculus presented also in appendix B), which is

$$\overline{k^2}(s, t) \simeq \frac{2t}{s} - \left(\frac{t}{s}\right)^{\frac{1}{2}}. \quad (4.9)$$

Therefore, the variance for the Barabási-Albert model is

$$\text{var}(k) = \overline{k^2} - \bar{k}^2 \simeq \frac{t}{s} - \left(\frac{t}{s}\right)^{\frac{1}{2}}. \quad (4.10)$$

A Python code (appendix F) was written in order to simulate the evolution of the network and determine the mean variance from multiple realizations. The code was built following the steps below:

1. Initial settings: define the maximum number of vertices, which matches the maximum time for the dynamics; choose a vertex to analyze; create lists for summing the degree and the degree squared.
2. For the simulation:
 - a) create lists for appending time steps and the degree of the chosen vertex;
 - b) create a loop to: add a new vertex at each time step, select a vertex with probability $\frac{k}{2t}$ and append the actual degree of the chosen vertex to the degree's list depending on if it has earned a new connection or not.

- c) Sum the degrees and degrees' squared.
3. Repeat steps 2a to 2c as many times as needed in order to improve the statistics.
4. With the lists that contain the values of the sum of the degrees and the sum of the degrees squared, calculate the variance.

Figure 15 shows the graph of the mean variance for 1000 realizations and $s = 3$ (this vertex was chosen in order to follow its evolution for a long time) and t up to 10^5 . The two curves in the graph represent the analytical and numerical results for this simulation. What the graph shows is that, in a network with preferential connections, the degree variance increases linearly with time for long time, as expected from (4.10).

Figure 15 – Mean degree variance for $s = 3$ for 1000 realizations and t up to 10^5 — Barabási-Albert model. Blue curve: analytical results; red curve: numerical results.

4.3 Final considerations

In this study, we calculated analytically the degree variance in two different growing networks: the random recursive tree model and the Barabási-Albert model. The validity of the formulas obtained was supported by a computer simulation of the graphs. The results showed that the degree variance depends on the rules of interaction between the nodes in the graph, more directly, the probability of a vertex changing its degree.

The equations for the mean degree of a given vertex s ($\bar{k}(s, t)$) for both models had already been published by (2), but the complete calculus for the mean degree, mean degree squared and variance are presented here for the first time. This study raises the possibility of a new challenge: to suggest possible rules of interaction for a network based on its degree variance.

5 First-passage process in degree space

Random walks have been applied to several studies on networks for a wide variety of subjects, such as electric currents, drug-target interactions and search algorithms, among many others (13, 14, 15). Some features and properties of networks have been studied under the perspective of a first-passage process, as for example the mean time for a random walker to reach a specified site in the network and the centrality of a node (17, 33). Here, we propose a study of the first-passage process in degree space to estimate the mean time for a vertex to reach a specified degree. The first-passage process in degree space has been done before by (34), not to estimate the mean time but as an approach to study the time-dependent degree distribution of some networks.

Considering that vertices in general gain or loose connections, we can look at the degree space as a random walk in one dimension. Like a walker that randomly moves left or right, a vertex can win or loose edges (figure 16). Therefore, we can use equation (3.8) as a guide to build an equation for calculating the first-passage and occupation probabilities (respectively $F(\vec{r}, t)$ and $P(\vec{r}, t)$) in degree space. Hence, following the logic presented in section 3.3, it is possible to calculate the mean time for a vertex to reach a specified degree.

Figure 16 – Representation of the gain or loss of edges of a vertex with degree $k_0 = 3$ as a random walk in one dimension.

We calculated the mean time for the first-passage in degree space for two different network models: Erdős-Rényi and Watts-Strogatz. In purpose of illustrating the main idea of the work, we start with a dynamical version of the Erdős-Rényi model — which is analytically more accessible. Then, we move on to the dynamical version of the Watts-Strogatz model. In the following sections, we present the results for each model.

5.1 Time-dependent Erdős-Rényi model

In the original Erdős-Rényi (ER) model, the network has a fixed number N of vertices, and each $\frac{N(N-1)}{2}$ possible pairs of vertices are connected with probability p . This

study used a dynamics for the ER model where the network starts with a fixed number of vertices N and no edges. At each time step, two vertices are randomly chosen and connected with probability p . The initial condition is an empty network, that is to say, all vertices have degree zero.

For this dynamical version, $\frac{N(N-1)}{2}$ pairs of vertices will have been chosen after $t = \frac{N(N-1)}{2}$ discrete time steps. Statistically, in average, each possible pair of vertices would have been chosen once, just like with the original ER model. Hence, it is possible to say that for $t = \frac{N(N-1)}{2}$, this dynamical model recovers the original ER model that allows self-loops.

Because the probability of a vertex earning a new connection does not depend on time, and because this study focuses on the situation where a given vertex starts with degree $k_0 = 0$, the initial condition is $p(k, s, t = 0) = \delta_{k,k_0}$. Then, the probability of a given vertex s having degree k at time $t + 1$ is described by the following equation:

$$p(k, s, t + 1) = \omega_{ER}(k|k-2) p(k-2, s, t) + \omega_{ER}(k|k-1) p(k-1, s, t) + \omega_{ER}(k|k) p(k, s, t), \quad (5.1)$$

where $\omega_{ER}(k|m)$ is the time-independent conditional probability of a vertex changing its degree from m to k . For the present case, the equations for $\omega_{ER}(k|m)$ are the following:

$$\omega_{ER}(k|k-2) = \frac{1}{N^2} p, \quad (5.2)$$

$$\omega_{ER}(k|k-1) = \frac{2}{N} \left(1 - \frac{1}{N}\right) p \quad \text{and} \quad (5.3)$$

$$\begin{aligned} \omega_{ER}(k|k) &= \underbrace{\left(1 - \frac{1}{N}\right)^2 p}_A + \underbrace{\left(1 - \frac{1}{N}\right)^2 (1-p)}_B + \underbrace{\frac{1}{N^2} (1-p)}_C + \underbrace{\frac{2}{N} \left(1 - \frac{1}{N}\right) (1-p)}_D \\ & \quad (5.4) \end{aligned}$$

$$= 1 - \frac{2p}{N} + \frac{p}{N^2}. \quad (5.5)$$

Each $\omega_{ER}(k|m)$, for $m \neq k$, is the probability of a vertex s being chosen once or twice — which happens respectively with probability $\frac{2}{N} \left(1 - \frac{1}{N}\right)$ and $\frac{1}{N^2}$ — times the probability p of connecting to the other chosen vertex. For $m = k$, $\omega_{ER}(k|k)$ is a sum of probabilities, as described by equation (5.4), where:

- A: vertex s is not chosen at all and the connection between the other chosen vertices happens;

- B: vertex s is not chosen at all and the connection between the other chosen vertices does not happen;
- C: vertex s is chosen twice and the connection does not happen;
- D: vertex s is chosen once and the connection does not happen.

This is the same time-dependent version of the ER model used by (34), though here we corrected each time-independent conditional probability by considering that the variable p can assume any value between 0 and 1 — in (34), the authors assumed that $p = 1$.

The degree distribution for the entire network is $P(k, t) = \frac{1}{N} \sum_{s=1}^N p(k, s, t)$, and since for this model the vertices are statistically indistinguishable, then $p(k, s, t) = P(k, t)$. The equation for the first-passage problem for this model, constructed under the same logic for the equation (3.8), is expressed as

$$P(k, t | k_0, 0) = \sum_{t'=0}^t F(k, t' | k_0, 0) P(k, t | k, t'). \quad (5.6)$$

Here, unlike equation (3.8), there is no extra term ($e^{-2dt} \delta_{\vec{r}, 0}$, in equation (3.8)). In equation (5.6), since time is discrete, $t = 0$ implies

$$\begin{aligned} P(k, 0 | k_0, 0) &= F(k, 0 | k_0, 0) P(k, 0 | k, 0) \\ &= F(k, 0 | k_0, 0) \\ &= \delta_{k, k_0} \end{aligned} \quad (5.7)$$

The term on the left hand side of equation (5.6) represents the probability distribution of a vertex having degree k at time t given that it had degree k_0 at time t_0 — in this case, $t_0 = 0$. Analogously, $F(k, t' | k_0, 0)$ is the probability of having degree k for the first time at t' and $P(k, t | k, t')$ is the probability of not earning new edges during the remaining time $t - t'$. The model is invariant under temporal translations, therefore $P(k, t | k, t') = P(k, t - t' | k, 0)$, *i.e.*, the occurrence of an event depends only on the difference of time, and not on a particular instant.

The probability density $P(k, t | k, t')$ is the probability of a vertex having degree k at time t' , and having the same degree at time t — which, given the peculiarity of the dynamical ER, is the same of simply saying that the vertex did not earn new connections during the time interval $t - t'$ —, which is represented by $\omega_{ER}(k|k)$ (equation (5.5)) and can be simplified to

$$P(k, t | k, t') = \left[1 - \frac{2}{N}p + \frac{1}{N^2}p \right]^{t-t'}. \quad (5.8)$$

Equation (5.6) is then rewritten as

$$P(k, t | k_0, 0) = \sum_{t'=0}^t F(k, t' | k_0, 0) q^{(t-t')}, \quad (5.9)$$

where the notation

$$q = 1 - \frac{2}{N}p + \frac{1}{N^2}p^2 \quad (5.10)$$

was used. Given that equation (5.6) is a convolution product, the characteristic function

$$G^Z(z) = \sum_{t=0}^{\infty} z^t g(t) \quad (5.11)$$

can be used to solve it. Multiplying both sides of the equation (5.9) by z^t and summing over t from 0 to ∞ , results in

$$\sum_{t=0}^{\infty} z^t P(k, t | k_0, 0) = \sum_{t'=0}^{\infty} z^{t'} \left[\sum_{t=t'}^{\infty} F(k, t | k_0, 0) q^{(t-t')} \right]. \quad (5.12)$$

Next, changing the order of the sums and applying a change of variables ($t - t' = u$) gives

$$\begin{aligned} P^Z(k | k_0, z) &= \sum_{t'=0}^{\infty} z^{t'} F(k, t' | k_0, 0) \sum_{u=0}^{\infty} (zq)^u \\ &= F^Z(k | k_0, z) \frac{1}{1 - zq}. \end{aligned} \quad (5.13)$$

Therefore, the characteristic function for the probability of a vertex achieving a certain k for the first time, given that it started from k_0 , is given by

$$F^Z(k | k_0, z) = P^Z(k | k_0, z)(1 - zq). \quad (5.14)$$

The probability of a vertex achieving degree k starting from degree k_0 (in any time) can be expressed as $\sum_{t=0}^{\infty} F(k, t | k_0, 0)$. Furthermore, from (5.14), one has

$$\begin{aligned} \sum_{t=0}^{\infty} F(k, t | k_0, 0) &= \lim_{z \rightarrow 1} \sum_{t=0}^{\infty} z^t F(k, t | k_0, 0) \\ &= F^Z(k | k_0, z = 1) \\ &= P^Z(k | k_0, z = 1)(1 - q). \end{aligned} \quad (5.15)$$

Appendix E shows that

$$P^Z(k | k_0, z = 1) = \frac{N}{2p} \left[1 + \frac{(-1)^\Delta}{(2N - 1)^{\Delta+1}} \right], \quad (5.16)$$

where $\Delta = k - k_0$. With this result, and recalling equation (5.10), it is possible to evaluate the probability of a vertex achieving degree k , starting from k_0 , as

$$\sum_{t=0}^{\infty} F(k, t | k_0, 0) = \left(1 - \frac{1}{2N}\right) \left(1 + \frac{(-1)^\Delta}{(2N-1)^{\Delta+1}}\right). \quad (5.17)$$

Note from equation (5.17) that, for $\Delta = 0$, $\sum_{t=0}^{\infty} F(k, t | k_0, 0) = 1$, as expected, since this is simply the probability of a vertex having degree k given that it already has degree k . Also, for large networks, where $N \rightarrow \infty$, $\sum_{t=0}^{\infty} F(k, t | k_0, 0)$ tends to 1 for any Δ . All the calculus that lead to equation (5.17) is presented at appendix E.

The mean time $\langle t \rangle$ to achieve a certain degree k for the first time (starting from k_0 at time $t_0 = 0$) is given by

$$\langle t \rangle = \sum_{t=0}^{\infty} t F(k, t | k_0, 0). \quad (5.18)$$

By calculating the derivative of the characteristic function $F^Z(k | k_0, z) = \sum_{t=0}^{\infty} z^t F(k, t | k_0, 0)$ in z and multiplying both sides of the resulting equation by z follows that

$$z \frac{\partial}{\partial z} F^Z(k | k_0, z) = \sum_{t=0}^{\infty} t z^t F(k, t | k_0, 0). \quad (5.19)$$

Notice that the case where $z \rightarrow 1$ is precisely the mean time for the first-passage in degree space. Therefore, the path for finding $\langle t \rangle$ is to obtain the partial derivative of $F^Z(k, z)$ from equation (5.14), which is given by

$$\lim_{z \rightarrow 1} z \frac{\partial}{\partial z} F^Z(k | k_0, z) = \lim_{z \rightarrow 1} \frac{\partial}{\partial z} [P^Z(k | k_0, z)(1 - zq)], \quad (5.20)$$

and take $z \rightarrow 1$. This results in

$$\langle t \rangle = \lim_{z \rightarrow 1} \frac{\partial}{\partial z} [P^Z(k | k_0, z)(1 - zq)]. \quad (5.21)$$

From the equation (E.12) obtained in appendix E, it is possible to calculate the asymptotic expression for $\langle t \rangle$ when $N \gg 1$, which is

$$\langle t \rangle \simeq \frac{N\Delta}{2p}, \quad (5.22)$$

where $\Delta = k - k_0$ is always non-negative. Equation (5.22) shows that the mean time for the first-passage in degree space for this dynamic version of the Erdős-Rényi model is linear with the network's size N .

Appendix H shows the complete calculation for this last result, which has been compared to a computer simulation. A code built in Python (appendix I) simulates the

connection between pairs of nodes for this model and calculates the mean time it takes for a given vertex to reach a pre-established degree. The code was built based on the steps below:

1. Initial settings: define a maximum number of vertices; choose a vertex; define the initial and final degrees; create lists for appending network size and mean time.
2. For the simulation:
 - a) start an empty list for appending time steps and set initial time to zero;
 - b) create a loop to add an edge to the chosen vertex based on the probability of choosing the specified vertex and the probability of earning a new connection until it reaches the pre-established degree.
3. Repeat steps 2a and 2b as many times as needed in order to improve the statistics.
4. Calculate the mean time and append the result to the mean time list and the network size to its corresponding list.
5. Repeat steps 2 to 4 for each network size.

The graph presented in Figure 17 shows the mean time for the first-passage process for different sizes of networks and compared to the analytical prediction (5.22), which is valid for large network size.

Figure 17 – Mean time for first-passage for $\Delta = 4$ and 10^4 realizations — dynamical Erdős-Rényi model. Blue dots: numerical results; red curve: analytical results.

We also calculated the degree variance for the dynamical Erdős-Rényi model. From the equation that describes the probability of a given vertex s having k edges at time $t + 1$ — equation (5.1) — we obtained that

$$var(k) = \frac{2pt}{N} \left(1 + \frac{1}{N} \right). \quad (5.23)$$

This result shows that the degree variance increases linearly with time. All the calculus to get to this result is presented at appendix M.

5.2 Time-dependent Watts-Strogatz model

The original Watts-Strogatz (WS) model is also a network with a fixed number N of vertices, but unlike the ER model all vertices are connected to their next neighbors. It represents an intermediate network between a regular and a random network (3). Each edge end is rewired with probability p , so vertices can gain or loose edges. Just like the ER model, the present study used a dynamical version of the WS model, which is the same used by (34).

For this study, the network has a fixed number N of vertices and the initial condition is that each vertex has degree k_0 by having a single link to its k_0 neighbors. Therefore, the entire network has $\frac{k_0 N}{2}$ edges and total degree $M = k_0 N$. At each time step, an edge end is chosen at random with uniform probability $\frac{1}{M}$ and reconnected with probability p . Therefore, the probability of a given vertex s having k edges at time $t + 1$ is described by

$$p(k, s, t + 1) = \omega(k|k - 1) p(k - 1, s, t) + \omega(k|k) p(k, s, t) + \omega(k|k + 1) p(k + 1, s, t), \quad (5.24)$$

where $\omega(k|m)$ is the time-independent conditional probability of a vertex changing its degree from m to k . For the present case, the equations for $\omega(k|m)$ are the following:

$$\omega(k|k - 1) = \frac{p}{N} \left(1 - \frac{k - 1}{M} \right), \quad (5.25)$$

$$\omega(k|k + 1) = \frac{k + 1}{M} p \left(1 - \frac{1}{N} \right) \quad \text{and} \quad (5.26)$$

$$\omega(k|k) = \underbrace{\frac{kp}{MN}}_A + \underbrace{\frac{k}{M}(1 - p)}_B + \underbrace{\left(1 - \frac{k}{M} \right) p \left(1 - \frac{1}{N} \right)}_C + \underbrace{\left(1 - \frac{k}{M} \right) (1 - p)}_D. \quad (5.27)$$

The conditional probability in equation (5.25) corresponds to the probability that a vertex s had degree $k - 1$ at time t and an edge end (not connected to s) is chosen with

probability $1 - \frac{k-1}{M}$, rewired with probability p and connected to s with probability $1/N$. In equation (5.26), $\omega(k|k+1)$ corresponds to the probability that vertex s had degree $k+1$ at time t and an edge end connected to it is chosen with probability $\frac{k+1}{M}$, rewired with probability p and connected to any vertex other than s with probability $1 - 1/N$. As for equation (5.27), the conditional probability is the sum of the probabilities that vertex s had degree k at time t and:

- A: an edge end connected to s is chosen with probability $\frac{k}{M}$, rewires with probability p and connects again to s with probability $\frac{1}{N}$;
- B: an edge end connected to s is chosen with probability $\frac{k}{M}$ and does not rewire with probability $1 - p$;
- C: an edge end not connected to s is chosen with probability $1 - \frac{k}{M}$, rewires with probability p and connects to a other than s with probability $1 - \frac{1}{N}$;
- D: an edge end not connected to s is chosen with probability $1 - \frac{k}{M}$ and does not rewire with probability $1 - p$.

From equation (5.24), it is possible to obtain the stationary degree distribution for this dynamical version of the WS model, which is a Poisson distribution,

$$P_{Poisson}(k) = \frac{e^{-k_0} k_0^k}{k!}, \quad (5.28)$$

as shown in appendix J.

The equation for the first-passage problem, constructed under the same logic of the equation (3.8), is expressed as

$$P(k, t | k_0, 0) = \sum_{t'=0}^t F(k, t' | k_0, 0) P(k, t | k, t'). \quad (5.29)$$

It is the same equation for the ER model (equation (5.6)), with the difference that here vertices can gain or loose edges. The first factor on the right hand side of the equation represents the probability of having k edges for the first time at t' given that it started with k_0 edges. The second factor on the right hand side of the equation represents the probability of having k edges at t given that it started with degree k at time t' , independently of earning or loosing edges during the time interval $t - t'$. Note that this system is also time-translationally invariant, like the ER model. Therefore, equation (5.29) can be rewritten as

$$P(k | k_0, t - 0) = \sum_{t'=0}^t F(k | k_0, t' - 0) P(k | k, t - t'), \quad (5.30)$$

and since it is a convolution product in time variable, the characteristic function

$$G^Z(k, z) = \sum_{t=0}^{\infty} z^t G(k, t) \quad (5.31)$$

can be directly applied, resulting in

$$P^Z(k | k_0, z) = F^Z(k | k_0, z) P^Z(k | k, z). \quad (5.32)$$

Just like in the previous model, the mean time to achieve degree k (starting at degree k_0 at time 0) is given by

$$\langle t \rangle = \sum_{t=0}^{\infty} t F(k | k_0, t - 0). \quad (5.33)$$

In section 5.1, we have shown that

$$\langle t \rangle = \lim_{z \rightarrow 1} z \frac{\partial}{\partial z} F^Z(k | k_0, z) = \lim_{z \rightarrow 1} \sum_{t=0}^{\infty} t z^t F(k | k_0, t - 0), \quad (5.34)$$

and from equation (5.32) and the characteristic function (5.31), it follows that

$$\sum_{t=0}^{\infty} z^t F(k | k_0, t - 0) = F^Z(k | k_0, z) = \frac{P^Z(k | k_0, z)}{P^Z(k | k, z)}. \quad (5.35)$$

Therefore, the first step towards finding the mean time for the first-passage is to calculate $F^Z(k | k_0, z)$ and then take the partial derivative.

The characteristic function for the probability of a vertex achieving a certain degree k , given that it started from k_0 , can be calculated through equation (5.35), as shown in appendix K, and is given by

$$F^Z(k | k_0, z) \simeq \frac{1 + \alpha \Omega(k_0, k_0)}{1 + \alpha \Omega(k, k_0)} - \left[k! \sum_{m=0}^{\Delta-1} \frac{k_0^{-(k_0+m+1)}}{(\Delta-1-m)! (m+k_0+1)} \sum_{n=0}^m \frac{(-k_0)^n}{n!} \right] \alpha e^{k_0} \Theta(\Delta-1), \quad (5.36)$$

where $\alpha = \frac{M(z^{-1}-1)}{p}$, $\Delta = k - k_0$ and

$$1 + \alpha [\Omega(k, k_0)] = \int_K^1 d\xi \xi^{k-1} e^{-k_0 \beta (\xi-1)} (1-\xi)^\alpha (k - k_0 \beta \xi) \Big|_{K=0}, \quad (5.37)$$

being $\beta = \frac{z^{-1}-1}{p} + 1$. The unit step function ($\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$) was introduced to consider the cases where $k \leq k_0$ and $k > k_0$ in a single formula.

To calculate the mean time, $F^Z(k | k_0, z)$ will have to be derived once, so the important terms in the expanded form of the first term on the right hand side of equation (5.36) are the linear ones — all the other terms will go to zero when $z \rightarrow 1$ (or $\alpha \rightarrow 0$)

after deriving $F^Z(k | k_0, z)$ once. An expansion of the first term on the right hand side of equation (5.36) gives

$$\frac{1 + \alpha \Omega(k_0, k_0)}{1 + \alpha \Omega(k, k_0)} \simeq 1 - \alpha [\Omega(k, k_0) - \Omega(k_0, k_0)] . \quad (5.38)$$

Equation (5.38) can now be derived once to calculate that the mean time for the first-passage in degree space for this dynamic of the Watts-Strogatz model, which depends on k and k_0 separately (and not on its difference $\Delta = k - k_0$ only) as shown below:

$$\langle t \rangle = \frac{M}{p} \left[\Omega(k, k_0) - \Omega(k_0, k_0) + k! e^{k_0} \sum_{m=0}^{\Delta-1} \frac{k_0^{-(k_0+m+1)}}{k_0 + m + 1} \sum_{n=0}^m \frac{(-k_0)^n}{n!} \Theta(\Delta - 1) \right] . \quad (5.39)$$

This analytical result has been compared to a computer simulation. A code built in Python (apenddix L) simulates the connection between pairs of nodes for this model and calculates the mean time it takes for a given vertex to reach a pre-established degree. The code was built based on the steps below:

1. Initial settings: define a maximum number of vertices; define the initial and final degrees; create lists for appending network size and mean time.
2. For the simulation:
 - a) start an empty list for appending time steps and set initial time to zero;
 - b) define the total number of edges based on the network size;
 - c) create a loop to add or subtract an edge from the chosen vertex based on the rewiring probability and the probability of choosing the specified vertex until it reaches the pre-established degree.
3. Repeat steps 2a to 2c as many times as needed in order to improve the statistics.
4. Calculate the mean-time and append the result to the mean time list and the network size to its corresponding list.
5. Repeat steps 2 to 4 for each network size.

The graph presented in Figure 18 shows the mean time for the first-passage process for different sizes of networks and compared to the analytical prediction (5.39), which is valid for large network size. The mean time was calculated for a vertex starting with degree $k_0 = 2$ and reaching degree $k = 4$.

Simulations for different starting and finishing degrees show that the mean time for the first-passage depends on k and k_0 separately, as expected from the analytical equation

Figure 18 – Mean time for first-passage for $k_0 = 2$, $k = 4$, $p = 1$ and 10^5 realizations — dynamical Watts-Strogatz model. Blue dots: numerical results; red curve: analytical results.

(5.39), and not only on the difference between k and k_0 as in the dynamical ER model. For $N = 9 \times 10^5$, for example, $\langle t \rangle = 7,17 \times 10^5$ for $k_0 = 4$ and $k = 2$, and $\langle t \rangle = 6,37 \times 10^5$ for $k_0 = 2$ and $k = 4$. We also calculated the degree variance for the dynamical Watts-Strogatz model. From the equation that describes the probability of a given vertex s having k edges at time $t + 1$ — equation (5.1) — we obtained that

$$\text{var}(k) = k_0 \left(1 - \frac{1}{N}\right) \left(1 - e^{-\frac{2pt}{M}}\right). \quad (5.40)$$

This result shows that, opposite to what happens with the dynamical ER model, the degree variance does not increase linearly with time. All the calculus to get to this result is presented at appendix N.

5.3 Final considerations

In the present study, we formulated a dynamical version of the Erdős-Rényi and Watts-Strogatz model in order to calculate the mean time for the first-passage in degree space. The validity of the formulas obtained was supported by a computer simulation of the graphs. The results showed that the probability of a vertex achieving a certain degree k is 1 for $N \rightarrow \infty$, and that the mean time depends on the difference between the starting and finishing degrees in the ER model, while in the WS model it depends on the values

of the starting and finishing degrees separately. Also, for the Watts-Strogatz model, it showed that it takes longer to a vertex to decrease its degree than to increase.

From a mathematical perspective, both models are analytically accessible, though the Watts-Strogatz model was trickier to develop. This is a novel result that brings an idea on how to calculate the mean time for the first-passage in degree space for networks in general.

Bibliography

- 1 NEWMAN, M. E. J.; BARABÁSI, A.-L.; WATTS, D. J. *The structure and dynamics of networks*. [S.l.]: Princeton University Press, 2011.
- 2 DOROGOVTSSEV, S. N.; MENDES, J. F. F. *Evolution of networks: From biological nets to the Internet and WWW*. [S.l.]: Clarendon Press, Oxford, 2013.
- 3 WATTS, D. J.; STROGATZ, S. H. Collective dynamics of ‘small-world’ networks. *Nature*, Nature Publishing Group, v. 393, n. 6684, p. 440, 1998.
- 4 MAHADEVAN, P. et al. Orbis: rescaling degree correlations to generate annotated internet topologies. In: ACM. *ACM SIGCOMM Computer Communication Review*. [S.l.], 2007. v. 37, n. 4, p. 325–336.
- 5 SHIINO, T. Phylodynamic analysis of a viral infection network. *Frontiers in microbiology*, Frontiers Media SA, v. 3, 2012.
- 6 NEWMAN, M. E. J. *Networks: an introduction*. [S.l.]: Oxford University Press, 2010.
- 7 MITCHELL, M. *Complexity: A guided tour*. [S.l.]: Oxford University Press, 2009.
- 8 ERDŐS, P.; RÉNYI, A. On random graphs, i. *Publicationes Mathematicae (Debrecen)*, v. 6, p. 290–297, 1959.
- 9 BARABÁSI, A.-L. *Linked: The new science of networks*. [S.l.]: Perseus Publishing, 2003.
- 10 KRAPIVSKY, P. L.; REDNER, S.; BEN-NAIM, E. *A kinetic view of statistical physics*. [S.l.]: Cambridge University Press, 2010.
- 11 NEWMAN, M. E. J.; STROGATZ, S. H.; WATTS, D. J. Random graphs with arbitrary degree distributions and their applications. *Physical Review (Letters, E)*, v. 64, n. 2, p. 026118, 2001.
- 12 KIM, H.; ANDERSON, R. An experimental evaluation of robustness of networks. *IEEE Systems Journal*, IEEE, v. 7, n. 2, p. 179–188, 2013.
- 13 NASH-WILLIAMS, C. S. J. Random walk and electric currents in networks. In: CAMBRIDGE UNIVERSITY PRESS. *Mathematical Proceedings of the Cambridge Philosophical Society*. [S.l.], 1959. v. 55, n. 2, p. 181–194.
- 14 CHEN, X.; LIU, M.-X.; YAN, G.-Y. Drug–target interaction prediction by random walk on the heterogeneous network. *Molecular BioSystems*, Royal Society of Chemistry, v. 8, n. 7, p. 1970–1978, 2012.
- 15 BISNIK, N.; ABOUZEID, A. Modeling and analysis of random walk search algorithms in p2p networks. In: IEEE. *Hot topics in peer-to-peer systems, 2005. HOT-P2P 2005. Second International Workshop on*. [S.l.], 2005. p. 95–103.
- 16 REDNER, S. *A guide to first-passage processes*. [S.l.]: Cambridge University Press, 2001.

- 17 TEJEDOR, V.; BÉNICHOU, O.; VOITURIEZ, R. Global mean first-passage times of random walks on complex networks. *Physical Review (Letters, E)*, v. 80, n. 6, p. 065104, 2009.
- 18 STEEN, M. V. Graph theory and complex networks: An introduction. *Amsterdam, Netherlands: ISBN 978-90-815406-1-2*, v. 144, 2010.
- 19 STROGATZ, S. H. Exploring complex networks. *Nature*, Nature Publishing Group, v. 410, n. 6825, p. 268, 2001.
- 20 BARABÁSI, A.-L. *Network science*. [S.l.]: Cambridge university press, 2016.
- 21 WATTS, D. J. *Six degrees: The science of a connected age*. [S.l.]: WW Norton & Company, 2004.
- 22 GRANOVETTER, M. The strength of weak ties. *American Journal of Sociology*, v. 78, p. 1360, 1973.
- 23 BARABÁSI, A.-L.; ALBERT, R. Emergence of scaling in random networks. *Science*, v. 286, n. 5439, p. 509–512, 1999.
- 24 WEST, D. B. *Introduction to graph theory*. [S.l.]: Prentice Hall Upper Saddle River, 2001. v. 2.
- 25 EASLEY, D.; KLEINBERG, J. *Networks, crowds, and markets: Reasoning about a highly connected world*. [S.l.]: Cambridge University Press, 2010.
- 26 BOLLOBÁS, B. Random graphs. In: *Modern Graph Theory*. [S.l.]: Springer, 1998. p. 215–252.
- 27 LAWLER, G. F.; LIMIC, V. *Random walk: a modern introduction*. [S.l.]: Cambridge University Press, 2010. v. 123.
- 28 MONTROLL, E. W.; WEISS, G. H. Random walks on lattices. ii. *Journal of Mathematical Physics*, AIP, v. 6, n. 2, p. 167–181, 1965.
- 29 LOVÁSZ, L. Random walks on graphs: A survey. *Combinatorics, Paul Erdős in Eighty*, v. 2, 1993.
- 30 REDNER, S. *Errata: A Kinetic View of Statistical Physics*. 2013. Accessed January 8th, 2018. Disponível em: <http://physics.bu.edu/~redner/projects/kinetics/errata.pdf>.
- 31 SINGH, S.; PATHRIA, R. Privman-fisher hypothesis on finite systems: Verification in the case of the spherical model of ferromagnetism. *Physical Review (Letters, B)*, v. 31, n. 7, p. 4483, 1985.
- 32 SNIJDERS, T. A. The degree variance: an index of graph heterogeneity. *Social networks*, Elsevier, v. 3, n. 3, p. 163–174, 1981.
- 33 NOH, J. D.; RIEGER, H. Random walks on complex networks. *Physical Review Letters*, v. 92, n. 11, p. 118701, 2004.
- 34 CASA GRANDE, H. L.; COTACALLAPA, M.; HASE, M. O. Random walk in degree space and the time-dependent Watts-Strogatz model. *Physical Review E*, v. 95, n. 1, p. 012321, 2017.

Appendix

APPENDIX A – Calculating $\bar{k}$ and $\overline{k^2}$ for the random recursive tree model

(i) Solving the equation for $\bar{k}(s, t)$

From the equations

$$p(k, s, t+1) = \frac{1}{t} p(k-1, s, t) + \left(1 - \frac{1}{t}\right) p(k, s, t) \quad (\text{A.1})$$

and

$$\bar{k}(s, t) = \sum_{k=0}^{\infty} k p(k, s, t), \quad (\text{A.2})$$

it is possible to obtain the equation that gives the mean degree of a vertex s at time t as demonstrated below.

Multiplying both sides of the equation (A.1) by k and summing in k from $k = 0$ to ∞ results in

$$\sum_{k=0}^{\infty} k p(k, s, t+1) = \frac{1}{t} \sum_{k=0}^{\infty} k p(k-1, s, t) + \left(1 - \frac{1}{t}\right) \sum_{k=0}^{\infty} k p(k, s, t) \quad (\text{A.3})$$

or

$$\bar{k}(s, t+1) = \frac{1}{t} \sum_{k=0}^{\infty} k p(k-1, s, t) + \left(1 - \frac{1}{t}\right) \bar{k}(s, t), \quad (\text{A.4})$$

and applying a change of variables ($k-1 = l$) to the term $\sum_{k=0}^{\infty} k p(k-1, s, t)$, it follows that

$$\sum_{l=-1}^{\infty} (l+1) p(l, s, t) = \sum_{l=-1}^{\infty} l p(l, s, t) + \sum_{l=-1}^{\infty} p(l, s, t). \quad (\text{A.5})$$

Considering that there are no negative degrees, then the first value for the sum above is zero, and since

$$\sum_{l=0}^{\infty} l p(l, s, t) = \bar{k}(s, t) \quad \text{and} \quad \sum_{l=0}^{\infty} p(l, s, t) = 1, \quad (\text{A.6})$$

then

$$\bar{k}(s, t+1) = \bar{k}(s, t) + \frac{1}{t}. \quad (\text{A.7})$$

Next, if the approximation $t \gg 1$ is used, then

$$\frac{\partial}{\partial t} \bar{k}(s, t) = \frac{1}{t}. \quad (\text{A.8})$$

Integrating the equation above gives

$$\int d\bar{k}(s, t) = \int \frac{1}{t} dt, \quad (\text{A.9})$$

which results in

$$\bar{k}(s, t) = \ln|t| + F(s), \quad (\text{A.10})$$

where $F(s)$ is a constant if integration that depends only on s , and $t > 0$. Given that $\bar{k}(s = t, t > 2) = 1$, it is possible to calculate $F(s)$,

$$1 = \ln s + F(s) \quad \text{or} \quad F(s) = 1 - \ln s, \quad (\text{A.11})$$

which leads to the final equation for $\bar{k}(s, t)$,

$$\bar{k}(s, t) \simeq 1 + \ln\left(\frac{t}{s}\right). \quad (\text{A.12})$$

(ii) Solving the equation for $\bar{k}^2(s, t)$

From the equations

$$p(k, s, t + 1) = \frac{1}{t} p(k - 1, s, t) + \left(1 - \frac{1}{t}\right) p(k, s, t) \quad (\text{A.13})$$

and

$$\bar{k}^2(s, t) = \sum_{k=0}^{\infty} k^2 p(k, s, t), \quad (\text{A.14})$$

it is possible to obtain the equation that gives the mean degree squared of a vertex s at time t as demonstrated below.

Multiplying both sides of the equation (A.13) by k^2 and summing in k from $k = 0$ to ∞ leads to

$$\sum_{k=0}^{\infty} k^2 p(k, s, t + 1) = \frac{1}{t} \sum_{k=0}^{\infty} k^2 p(k - 1, s, t) + \left(1 - \frac{1}{t}\right) \sum_{k=0}^{\infty} k^2 p(k, s, t) \quad (\text{A.15})$$

or

$$\bar{k}^2(s, t + 1) = \frac{1}{t} \sum_{k=0}^{\infty} k^2 p(k - 1, s, t) + \left(1 - \frac{1}{t}\right) \bar{k}^2(s, t), \quad (\text{A.16})$$

and applying a change of variables ($k - 1 = l$) to the term $\sum_{k=0}^{\infty} k^2 p(k - 1, s, t)$, it follows that

$$\sum_{l=-1}^{\infty} (l + 1)^2 p(l, s, t) = \sum_{l=-1}^{\infty} l^2 p(l, s, t) + \sum_{l=-1}^{\infty} 2l p(l, s, t) + \sum_{l=-1}^{\infty} p(l, s, t). \quad (\text{A.17})$$

Again, there is no negative degree, so the first value on each sum above is zero. Given that

$$\bar{k}(s, t) = \sum_{k=0}^{\infty} k p(k, s, t), \quad \sum_{l=0}^{\infty} l^2 p(l, s, t) = \bar{k}^2(s, t) \quad \text{and} \quad \sum_{l=0}^{\infty} p(l, s, t) = 1, \quad (\text{A.18})$$

then

$$\sum_{k=1}^{\infty} k^2 p(k - 1, s, t) = \bar{k}^2(s, t) + 2\bar{k}(s, t) + 1, \quad (\text{A.19})$$

and, therefore,

$$\overline{k^2}(s, t+1) = \overline{k^2}(s, t) + \frac{1}{t} (1 + 2\overline{k}(s, t)) . \quad (\text{A.20})$$

If the approximation $t \gg 1$ is used, along with the result from (A.12), then

$$\frac{\partial}{\partial t} \overline{k^2}(s, t) = \frac{1}{t} \left[1 + 2 \left(1 + \ln \left(\frac{t}{s} \right) \right) \right] , \quad (\text{A.21})$$

and integrating equation (A.21) results in

$$\int d\overline{k^2}(s, t) = \int dt \frac{1}{t} \left(3 + 2\ln \left(\frac{t}{s} \right) \right) . \quad (\text{A.22})$$

Therefore,

$$\overline{k^2}(s, t) = 3\ln t + \ln^2 \left(\frac{t}{s} \right) + F(s) , \quad (\text{A.23})$$

where $F(s)$ is a constant of integration and $s, t > 0$.

Given that $\overline{k^2}(s = t, t > 2) = 1$, it is possible to calculate $F(s)$ by inserting this boundary condition:

$$1 = 3\ln s + \ln^2 \left(\frac{s}{s} \right) + F(s) \quad \text{or} \quad F(s) = 1 - 3\ln s , \quad (\text{A.24})$$

which leads to the final equation for $\overline{k^2}(s, t)$,

$$\overline{k^2}(s, t) \simeq \ln^2 \left(\frac{t}{s} \right) + 3\ln \left(\frac{t}{s} \right) + 1 . \quad (\text{A.25})$$

APPENDIX B – Calculating $\bar{k}$ and $\bar{k}^2$ for the Barabási-Albert model

(i) Solving the equation for $\bar{k}(s, t)$

From the equations

$$p(k, s, t + 1) = \frac{k - 1}{2t} p(k - 1, s, t) + \left(1 - \frac{k}{2t}\right) p(k, s, t), \quad (\text{B.1})$$

and

$$\bar{k}(s, t) = \sum_{k=0}^{\infty} k p(k, s, t), \quad (\text{B.2})$$

it is possible to obtain the equation that gives the mean degree of a vertex s at time t as demonstrated below.

Multiplying both sides of equation (B.1) by k and summing in k from $k = 0$ to ∞ leads to

$$\bar{k}(s, t + 1) = \frac{1}{2t} \sum_{k=0}^{\infty} k(k - 1) p(k - 1, s, t) + \sum_{k=0}^{\infty} k p(k - 1, s, t) - \frac{1}{2t} \sum_{k=0}^{\infty} k^2 p(k - 1, s, t) \quad (\text{B.3})$$

and applying a change of variables ($k - 1 = l$) to the term $\sum_{k=0}^{\infty} k(k - 1) p(k - 1, s, t)$, it follows that

$$\sum_{l=-1}^{\infty} (l + 1) l p(l, s, t) = \sum_{l=-1}^{\infty} l^2 p(l, s, t) + \sum_{l=-1}^{\infty} l p(l, s, t). \quad (\text{B.4})$$

Considering that there is no negative degree, then the first value of each sum is zero, and given that

$$\sum_{l=0}^{\infty} l p(l, s, t) = \bar{k}(s, t) \quad \text{and} \quad \sum_{l=0}^{\infty} l^2 p(l, s, t) = \bar{k}^2(s, t), \quad (\text{B.5})$$

then

$$\bar{k}(s, t + 1) = \frac{1}{2t} \bar{k}(s, t) + \bar{k}(s, t). \quad (\text{B.6})$$

If the approximation $t \gg 1$ is used, then

$$\int \frac{1}{\bar{k}(s, t)} d\bar{k}(s, t) = \int \frac{1}{2t} dt. \quad (\text{B.7})$$

Integrating the equation (B.7) results in

$$\ln|\bar{k}(s, t)| = \frac{1}{2} \ln|t| + \ln F(s), \quad (\text{B.8})$$

where $\ln F(s)$ is a constant of integration and $t > 0$. Therefore,

$$\bar{k}(s, t) = t^{\frac{1}{2}} F(s). \quad (\text{B.9})$$

Given that $\bar{k}(s, s) = 1$, then

$$F(s) = s^{-\frac{1}{2}}, \quad (\text{B.10})$$

which leads to the final equation for $\bar{k}(s, t)$,

$$\bar{k}(s, t) \simeq \left(\frac{t}{s}\right)^{\frac{1}{2}}. \quad (\text{B.11})$$

(ii) Solving the equation for $\bar{k}^2(s, t)$

From the equations

$$p(k, s, t+1) = \frac{k-1}{2t} p(k-1, s, t) + \left(1 - \frac{k}{2t}\right) p(k, s, t), \quad (\text{B.12})$$

and

$$\bar{k}^2(s, t) = \sum_{k=0}^{\infty} k^2 p(k, s, t), \quad (\text{B.13})$$

it is possible to obtain the equation that gives the mean degree of a vertex s at time t as demonstrated below.

Multiplying both sides of the equation (B.12) by k^2 and summing in k from $k = 0$ to ∞ leads to

$$\bar{k}^2(s, t+1) = \frac{1}{2t} \sum_{k=0}^{\infty} k^2(k-1) p(k-1, s, t) + \bar{k}^2(s, t) - \frac{1}{2t} \sum_{k=0}^{\infty} k^3 p(k, s, t) \quad (\text{B.14})$$

and applying a change of variables ($k-1 = l$) to the term $\sum_{k=0}^{\infty} k^2(k-1) p(k-1, s, t)$, it follows that

$$\sum_{l=-1}^{\infty} (l+1)^2 l p(l, s, t) = \sum_{l=-1}^{\infty} l^3 p(l, s, t) + \sum_{l=-1}^{\infty} 2l^2 p(l, s, t) + \sum_{l=-1}^{\infty} l p(l, s, t). \quad (\text{B.15})$$

Again, there is no negative degree, so the first term of each sum above is zero. Since

$$\sum_{l=0}^{\infty} l p(l, s, t) = \bar{k}(s, t) \quad \text{and} \quad \sum_{l=0}^{\infty} l^2 p(l, s, t) = \bar{k}^2(s, t), \quad (\text{B.16})$$

then

$$\bar{k}^2(s, t+1) = \bar{k}^2(s, t) + \frac{\bar{k}^2(s, t)}{t} + \frac{\bar{k}(s, t)}{2t}. \quad (\text{B.17})$$

If the approximation $t \gg 1$ is used, along with (B.11), then

$$\frac{\partial}{\partial t} \bar{k}^2(s, t) = \frac{\bar{k}^2(s, t)}{t} + \frac{1}{2\sqrt{ts}}. \quad (\text{B.18})$$

Next, assume that $\bar{k}^2(s, t) = u(t)v(t)$ without compromising the generality of the equation.

The resulting differential equation above can then be solved through the method of variation of parameters

$$\frac{d}{dt}[uv] = \frac{uv}{t} + \frac{1}{2\sqrt{ts}} \quad (\text{B.19})$$

or

$$u \frac{d}{dt} v + v \left[\frac{d}{dt} u - \frac{u}{t} \right] - \frac{1}{2\sqrt{ts}} = 0 . \quad (\text{B.20})$$

Imposing $\frac{d}{dt} u - \frac{u}{t} = 0$, then

$$\frac{d}{dt} u = \frac{u}{t} , \quad (\text{B.21})$$

and integrating the equation leads to

$$\int \frac{1}{u} du = \int \frac{1}{t} dt , \quad (\text{B.22})$$

which results in

$$\ln|u| = \ln|t| + C_1 \quad \text{or} \quad u = t e^{C_1} . \quad (\text{B.23})$$

Next, from (B.20), $u \frac{d}{dt} v - \frac{1}{2\sqrt{ts}} = 0$, then

$$u \frac{d}{dt} v = \frac{1}{2\sqrt{ts}} . \quad (\text{B.24})$$

Using (B.23) and integrating the equation (B.24) leads to

$$\int dv = \int \frac{1}{\sqrt{s}} \frac{1}{2te^{C_1}\sqrt{t}} dt , \quad (\text{B.25})$$

which results in

$$v = \frac{1}{e^{C_1}\sqrt{s}} \left[-t^{-\frac{1}{2}} + C_2 \right] , \quad (\text{B.26})$$

where C_2 is also an integration constant. From the results obtained for $u(t)$ and $v(t)$ in, respectively, equations (B.23) and (B.26), it follows that

$$\overline{k^2}(s, t) = (t e^{C_1}) \left(\frac{1}{e^{C_1}\sqrt{s}} \right) \left(-t^{-\frac{1}{2}} + C_2 \right) \quad (\text{B.27})$$

so

$$\overline{k^2}(s, t) = \left(\frac{t}{s} \right)^{\frac{1}{2}} \left(C_2 t^{\frac{1}{2}} - 1 \right) . \quad (\text{B.28})$$

Given that $\overline{k}(s, s) = 1$, then

$$1 = \overline{k^2}(s, s) = \left(\frac{s}{s} \right)^{\frac{1}{2}} \left(C_2 s^{\frac{1}{2}} - 1 \right) , \quad (\text{B.29})$$

and therefore $C_2 = \frac{2}{s^{\frac{1}{2}}}$, which leads to the final equation for $\overline{k^2}(s, t)$,

$$\overline{k^2}(s, t) \simeq \frac{2t}{s} - \left(\frac{t}{s} \right)^{\frac{1}{2}} . \quad (\text{B.30})$$

APPENDIX C – Calculating $P(\vec{n}, t)$ for a random walk on a hypercubic lattice

Consider a d -dimensional random walk on a hypercubic lattice with fixed unitary step length $\vec{n}$, where $\vec{n} = (n_1, n_2, \dots, n_d)$, and equal probability (D) of moving in any direction. For $t = 0$ and starting point $\vec{n} = \vec{0}$, the initial condition is $P(\vec{n}, t = 0) = \delta_{\vec{n}, \vec{0}}$. The probability of the walker being at $\vec{n}$ at time $t + 1$ is described as

$$P(\vec{n}, t + 1) = D \sum_{\vec{\delta}} P(\vec{n} + \vec{\delta}, t) + (1 - 2dD)P(\vec{n}, t), \quad (\text{C.1})$$

where $\vec{\delta} \in \{(+1, 0, \dots, 0), (-1, 0, \dots, 0), \dots, (0, \dots, 0, +1), (0, \dots, 0, -1)\}$ — i.e., $\vec{\delta}$ is a vector that points to all nearest neighbor sites —, and $2d$ is the number of possible neighbors in a d -dimensional hypercubic lattice. The first term on the right hand side of the equation is the sum of all possible displacements in d dimensions considering that the walker is at some point $\vec{n} + \vec{\delta}$ at t and goes to $\vec{n}$ at $t + 1$. The second term on the right hand of the equation is the probability that the walk is already at the point $\vec{n}$ at t and continues there at $t + 1$.

The discrete Fourier transform for a d -dimensional vector is

$$\hat{P}(\vec{k}, t) = \prod_{j=1}^d \sum_{n_j=-\infty}^{\infty} e^{-i\vec{k} \cdot \vec{n}} P(\vec{n}, t), \quad (\text{C.2})$$

where the notation of the inner product $\vec{k} \cdot \vec{n} = k_1 n_1 + k_2 n_2 + \dots + k_d n_d$ was used. Multiplying both sides of the equation (C.1) by $e^{-i\vec{k} \cdot \vec{n}}$ and summing each component n_j of $\vec{n}$ from $-\infty$ to ∞ gives

$$\hat{P}(\vec{k}, t + 1) = D \underbrace{\prod_{j=1}^d \sum_{n_j=-\infty}^{\infty} e^{-i\vec{k} \cdot \vec{n}} \sum_{\vec{\delta}} P(\vec{n} + \vec{\delta}, t)}_A + (1 - 2dD)\hat{P}(\vec{k}, t). \quad (\text{C.3})$$

Applying a change of variables $\vec{m} = \vec{n} + \vec{\delta}$ to the term A leads to

$$\sum_{\vec{\delta}} e^{i\vec{k} \cdot \vec{\delta}} \left[\sum_{\vec{m}} e^{-i\vec{k} \cdot \vec{m}} P(\vec{m}, t) \right], \text{ or simply } \sum_{\vec{\delta}} e^{i\vec{k} \cdot \vec{\delta}} \hat{P}(\vec{k}, t), \quad (\text{C.4})$$

and because $\sum_{\vec{\delta}} e^{i\vec{k} \cdot \vec{\delta}} = e^{ik_1} + e^{-ik_1} + \dots + e^{ik_d} + e^{-ik_d} = \sum_{j=1}^d 2 \cos k_j$, it follows that

$$A = \sum_{j=1}^d 2 \cos k_j \hat{P}(\vec{k}, t). \quad (\text{C.5})$$

Now equation (C.3) can be written as

$$\hat{P}(\vec{k}, t+1) - \hat{P}(\vec{k}, t) = 2D \sum_{j=1}^d \cos k_j \hat{P}(\vec{k}, t) - 2dD \hat{P}(\vec{k}, t), \quad (\text{C.6})$$

which leads to

$$\hat{P}(\vec{k}, t+1) - \hat{P}(\vec{k}, t) = -2D \left(d - \sum_{j=1}^d \cos k_j \right) \hat{P}(\vec{k}, t). \quad (\text{C.7})$$

For $t \gg 1$, the difference on equation (C.7) can be converted into the derivative

$$\frac{\partial}{\partial t} \hat{P}(\vec{k}, t) = -2D \left(d - \sum_{j=1}^d \cos k_j \right) \hat{P}(\vec{k}, t) \quad (\text{C.8})$$

and

$$\int \frac{1}{\hat{P}(\vec{k}, t)} d\hat{P}(\vec{k}, t) = -2D \int \left(d - \sum_{j=1}^d \cos k_j \right) dt. \quad (\text{C.9})$$

Therefore,

$$\hat{P}(\vec{k}, t) = e^{-2Dt(d - \sum_{j=1}^d \cos k_j)} C(k), \quad (\text{C.10})$$

where $C(k)$ is a constant of integration. From the initial condition $\hat{P}(\vec{k}, t=0) = 1$, it is possible to calculate that $C(k) = 1$; therefore,

$$\hat{P}(\vec{k}, t) = e^{-2Ddt} e^{2Dt(\cos k_1 + \cos k_2 + \dots + \cos k_d)}. \quad (\text{C.11})$$

The second exponential in the equation above is the generating function for the modified Bessel function of the first kind I_n , represented as

$$e^{z \cos \theta} = \sum_{n=-\infty}^{\infty} I_n(z) e^{in\theta}. \quad (\text{C.12})$$

Applying this definition to equation (C.11) gives

$$\hat{P}(\vec{k}, t) = e^{-2Ddt} \prod_{j=1}^d \sum_{n_j=-\infty}^{\infty} I_{n_j}(2Dt) e^{i\vec{n} \cdot \vec{k}}, \quad (\text{C.13})$$

and from the equation (C.2) results that

$$P(\vec{n}, t) = e^{-2Ddt} \prod_{j=1}^d I_{n_j}(2Dt). \quad (\text{C.14})$$

APPENDIX D – Calculating $P^L(0, s)$ for $d = 2$ and small s

From appendix C, in a random walk on a hypercubic lattice, with fixed unitary step length and equal probability D of moving in any direction, the probability of being at the origin at $t \neq 0$ is $P(0, t) = e^{-2Ddt} [I_0(2Dt)]^d$. The Laplace transform of $P(0, t)$ is

$$P^L(0, s) = \int_0^\infty e^{-st} P(0, t) dt, \quad (\text{D.1})$$

and because for long time $P(0, t) = e^{-2Ddt} [I_0(2Dt)]^d$ asymptotically goes to $\frac{1}{(4\pi Dt)^{\frac{d}{2}}}$, it is convenient to assume $s < 1$ (since $s \rightarrow 0$ will be considered in the end), take the first derivative in s of (D.1), split it into two and rename it as

$$\varphi'(s) = - \int_0^{-\ln s} t e^{-st} P(0, t) dt - \int_{-\ln s}^\infty t e^{-st} P(0, t) dt. \quad (\text{D.2})$$

And if we assume that $s \sim 0$ (i.e., $|s| \ll 1$), then $P(0, t)$ on the second integral of (D.2) can be replaced by its asymptotic form.

(i) Upper bound for the first integral

The first integral of (D.2) is bounded above by

$$\left| - \int_0^{-\ln s} t e^{-st} P(0, t) dt \right| \leq \int_0^{-\ln s} t e^{-st} |P(0, t)| dt. \quad (\text{D.3})$$

Before proceeding, it is important to note that

$$\frac{d}{dt} P(0, t) = -2Dd e^{-2Ddt} I_0^d(2Dt) + e^{-2Ddt} d 2D I_0^{d-1}(2Dt) I_0'(2Dt) \quad (\text{D.4})$$

or

$$\frac{d}{dt} P(0, t) = -2Dd e^{-2Ddt} I_0^{d-1}(2Dt) [I_0(2Dt) - I_0'(2Dt)]. \quad (\text{D.5})$$

Since $I_0(z) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{z \cos \theta}$ and $I_0'(z) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{z \cos \theta} \cos \theta$, then

$$I_0(2Dt) - I_0'(2Dt) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{z \cos \theta} (1 - \cos \theta) \geq 0. \quad (\text{D.6})$$

Using this last result to analyze equation (D.5), we can conclude that $\frac{d}{dt} P(0, t) \leq 0$. Given that $P(0, t=0) = 1$, $P(0, t) \geq 0$, and that its derivative is less than or equal to 0, then $|P(0, t)| \leq 1$.

Taking the maximum value for $|P(0, t)| \leq 1$, then the inequality (D.3) results in

$$\left| - \int_0^{-\ln s} t e^{-st} P(0, t) dt \right| \leq \int_0^{-\ln s} t e^{-st} dt = e^{s \ln s} \left(\frac{\ln s}{s} - \frac{1}{s^2} \right) + \frac{1}{s^2}. \quad (\text{D.7})$$

From the Taylor series, $e^{s \ln s} = 1 + s \ln s + \frac{(s \ln s)^2}{2} + \mathcal{O}((s \ln s)^3)$ and from this result follows that the right hand side term on equation (D.7) is equal to

$$\frac{(\ln s)^2}{2} + \frac{s(\ln s)^3}{3} + \mathcal{O}(s^2(\ln s)^4). \quad (\text{D.8})$$

Taking the limit of $s \ln s$ when $s \rightarrow 0$ results in $\lim_{s \rightarrow 0} s \ln s = 0$, and so the dominant term on equation (D.8) is $\frac{(\ln s)^2}{2}$. Therefore,

$$\left| - \int_0^{-\ln s} t e^{-st} P(0, t) dt \right| \leq \frac{(\ln s)^2}{2} + \mathcal{O}(s(\ln s)^3). \quad (\text{D.9})$$

(ii) Solving the second integral

Because for long time $P(0, t) = e^{-2Dt} I_0(2Dt)$ asymptotically goes to $\frac{1}{(4\pi D t)^{\frac{d}{2}}}$, the second integral of (D.2) for $d = 2$ can be solved as follows:

$$- \int_{-\ln s}^{\infty} t e^{-st} P(0, t) dt \simeq - \int_{-\ln s}^{\infty} t e^{-st} \frac{1}{(4\pi D t)} dt = - \frac{1}{(4\pi D)} \frac{e^{s \ln s}}{s}. \quad (\text{D.10})$$

Again, from the Taylor series, $e^{s \ln s} = 1 + s \ln s + \mathcal{O}((s \ln s)^2)$, and

$$- \frac{1}{(4\pi D)} \frac{e^{s \ln s}}{s} = - \frac{1}{(4\pi D)} \frac{(1 + s \ln s + \mathcal{O}((s \ln s)^2))}{s}. \quad (\text{D.11})$$

Given that the dominant term is $\frac{1}{s}$, this integral results in

$$- \int_{-\ln s}^{\infty} t e^{-st} P(0, t) dt \simeq - \frac{1}{(4\pi D s)} + \mathcal{O}(\ln s). \quad (\text{D.12})$$

(iii) Comparing the results from (i) and (ii)

Comparing the results for the two integrals, (D.9) and (D.12), it shows that the first integral is dominated by $\frac{(\ln s)^2}{2}$ and the second integral has the term of order $\frac{1}{s}$. L'Hopital's rule gives that $\lim_{s \rightarrow 0} \frac{\frac{1}{s}}{\ln^2 s} = -\infty$, which means that the first integral barely contributes to the total of $\varphi'(s)$. Therefore,

$$\varphi'(s) \simeq - \frac{1}{(4\pi D s)} + \mathcal{O}((\ln s)^2). \quad (\text{D.13})$$

(iv) Calculating $P^L(0, s)$

Integrating both sides of equation (D.13) results in

$$\varphi(s) = -\frac{\ln s}{4\pi D} + \mathcal{O}(s(\ln s)^2) \quad (\text{D.14})$$

and, therefore, for $d = 2$,

$$P^L(0, s) \simeq -\frac{1}{4\pi D} \ln s. \quad (\text{D.15})$$

APPENDIX E – Calculating $F^Z(k, 1)$ for the Erdős-Rényi model

Equation (5.14) gives the characteristic function for the probability of a vertex achieving a certain degree k given that it started from k_0 , which is $F^Z(k | k_0, z) = P^Z(k | k_0, z)(1 - zq)$. This expression shows that it is necessary to first calculate $P^Z(k, z)$ in order to obtain the characteristic function of $F(k, t)$. The starting point is the equation that gives the probability of a given vertex s having degree k at time $t + 1$, presented by equation (5.1), which is expressed as

$$p(k, s, t + 1) = \omega_{ER}(k | k - 2) p(k - 2, s, t) + \omega_{ER}(k | k - 1) p(k - 1, s, t) + \omega_{ER}(k | k) p(k, s, t). \quad (\text{E.1})$$

Because all vertices are statistically equal, $p(k, s, t) = P(k, t)$, and equation (E.1) can be written as

$$P(k, t + 1) = \omega_{ER}(k | k - 2) P(k - 2, t) + \omega_{ER}(k | k - 1) P(k - 1, t) + \omega_{ER}(k | k) P(k, t). \quad (\text{E.2})$$

The next step is to apply the characteristic function using the notation $P^K(K, t) = \sum_{k=0}^{\infty} K^k P(k, t)$. Multiplying both sides of equation (E.2) by K^k and summing over k from 0 to ∞ gives

$$P^K(K, t + 1) = \left[\frac{1}{N^2} p K^2 + \frac{2}{N} \left(1 - \frac{1}{N} \right) p K + 1 - \frac{p}{N} \left(2 - \frac{1}{N} \right) \right] P^K(K, t). \quad (\text{E.3})$$

The expressions of each $\omega(k | m)$ are given in section 5.1, by equations (5.2), (5.3) and (5.5). The initial condition is $P(k, 0) = \delta_{k, k_0}$, considering that $k_0 = 0$, which leads equation (E.3) to be written as

$$P^K(K, t) = \left[p \left(\frac{K}{N} + 1 - \frac{1}{N} \right)^2 + (1 - p) \right]^t K^{k_0}. \quad (\text{E.4})$$

Now the characteristic function is applied again, this time using the notation $P^{KZ}(K, z) = \sum_{t=0}^{\infty} z^t P^K(K, t)$, resulting in

$$P^{KZ}(K, z) = \sum_{t=0}^{\infty} z^t \left[p \left(\frac{K}{N} + 1 - \frac{1}{N} \right)^2 + (1 - p) \right]^t K^{k_0}, \quad (\text{E.5})$$

and from the relation $(a + b)^t = \sum_{i=0}^t \binom{t}{i} a^i b^{t-i}$, we obtain

$$P^{KZ}(K, z) = \sum_{t=0}^{\infty} z^t K^{k_0} \sum_{i=0}^t \binom{t}{i} \left[p \left(\frac{K}{N} + 1 - \frac{1}{N} \right)^2 \right]^i (1 - p)^{t-i}. \quad (\text{E.6})$$

Changing the order of the sums and rearranging the terms results in

$$P^{KZ}(K, z) = \sum_{i=0}^{\infty} K^{k_0} \left[p \left(\frac{K}{N} + 1 - \frac{1}{N} \right)^2 \right]^i (1-p)^{-i} \sum_{t=i}^{\infty} \binom{t}{i} [z(1-p)]^t. \quad (\text{E.7})$$

To simplify the second sum on the right hand side of equation (E.7), the relation

$$(1-x)^{(-n-1)} = \sum_{j=0}^{\infty} \binom{n+j}{j} x^j \quad (\text{E.8})$$

was used along with a change of variables ($j = t - i$) as follows:

$$\begin{aligned} \sum_{t=i}^{\infty} \binom{t}{i} [z(1-p)]^t &= \sum_{j=0}^{\infty} \binom{j+i}{i} [z(1-p)]^{j+i} \\ &= [z(1-p)]^i \sum_{j=0}^{\infty} \binom{i+j}{j} [z(1-p)]^j, \end{aligned} \quad (\text{E.9})$$

and, therefore,

$$\sum_{t=i}^{\infty} \binom{t}{i} [z(1-p)]^t = [z(1-p)]^i [1 - z(1-p)]^{-i-1}. \quad (\text{E.10})$$

Replacing this last result in equation (E.7) gives

$$P^{KZ}(K, z) = \sum_{i=0}^{\infty} K^{k_0} \left[p \left(\frac{K}{N} + 1 - \frac{1}{N} \right)^2 \right]^i z^i [1 - z(1-p)]^{-i-1}. \quad (\text{E.11})$$

Now, considering the case where $z \rightarrow 1$, then

$$\lim_{z \rightarrow 1} P^{KZ}(K, z) = \frac{K^{k_0}}{p} \sum_{i=0}^{\infty} \left(\frac{K}{N} + 1 - \frac{1}{N} \right)^{2i}, \quad (\text{E.12})$$

and expanding the last term in the parenthesis results in

$$\lim_{z \rightarrow 1} P^{KZ}(K, z) = \frac{K^{k_0}}{p} \sum_{i=0}^{\infty} \sum_{j=0}^{2i} \binom{2i}{j} \left(\frac{K}{N} \right)^j \left(1 - \frac{1}{N} \right)^{2i-j}. \quad (\text{E.13})$$

From $P^{KZ}(K, z) = \sum_{k=0}^{\infty} K^k P^Z(k, z)$, it follows that the coefficient of K^k is precisely $P^Z(k, z)$. Rewriting equation (E.13) gives

$$\lim_{z \rightarrow 1} P^{KZ}(K, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{2i} \binom{2i}{j} \frac{1}{p} \frac{1}{N^j} \left(1 - \frac{1}{N} \right)^{2i-j} K^{j+k_0}. \quad (\text{E.14})$$

If $j + k_0 = k$, then the other terms on the right hand side of equation (E.13) represent the coefficient of K^k . The notation $\Delta = k - k_0$ facilitates the calculation of the coefficient of K^k .

For $\Delta \in E$, where $E = \{\Delta \in \mathbb{Z} | \Delta = 2y, y \in \mathbb{Z}\}$, i.e., even Δ , it follows that

$$\lim_{z \rightarrow 1} P^Z(k, z) = \sum_{i=\frac{\Delta}{2}}^{\infty} \sum_{j=0}^{2i} \binom{2i}{j} \frac{1}{p} \frac{1}{N^j} \left(1 - \frac{1}{N}\right)^{2i-j} \delta_{j,\Delta}, \quad (\text{E.15})$$

where the Kronecker delta has been introduced to extract only the part of equation (E.14) that is proportional to K^k . Also, $\sum_{i=0}^{\frac{\Delta}{2}-1}$ was excluded from the sum because there are no terms proportional to K^k for i in that interval (so it does not contribute to the sum). Now the coefficient of K^k on equation (E.14) is the the term

$$\lim_{z \rightarrow 1} P^Z(k, z) = \sum_{i=\frac{\Delta}{2}}^{\infty} \binom{2i}{\Delta} \frac{1}{p} \frac{1}{N^{\Delta}} \left(1 - \frac{1}{N}\right)^{2i-\Delta}, \quad (\text{E.16})$$

which can be rewritten as

$$\lim_{z \rightarrow 1} P^Z(k, z) = \frac{1}{p} \frac{1}{N^{\Delta}} \left(1 - \frac{1}{N}\right)^{-\Delta} \sum_{i=\frac{\Delta}{2}}^{\infty} \binom{2i}{\Delta} \left(1 - \frac{1}{N}\right)^{2i}. \quad (\text{E.17})$$

Applying a change of variables ($i = l + \frac{\Delta}{2}$) to the sum on the right hand side of equation (E.16) gives

$$\sum_{i=\frac{\Delta}{2}}^{\infty} \binom{2i}{\Delta} \left(1 - \frac{1}{N}\right)^{2i} = \sum_{l=0}^{\infty} \binom{2l+\Delta}{\Delta} \left(1 - \frac{1}{N}\right)^{2l+\Delta}, \quad (\text{E.18})$$

and applying this result to equation (E.17) results in

$$\lim_{z \rightarrow 1} P^Z(k, z) = \frac{1}{p} \frac{1}{N^{\Delta}} \sum_{l=0}^{\infty} \binom{2l+\Delta}{\Delta} \left(1 - \frac{1}{N}\right)^{2l}. \quad (\text{E.19})$$

From the relation

$$(1-x)^{-(n+1)} + (1+x)^{-(n+1)} = 2 \sum_{i=0}^{\infty} \binom{n+2i}{2i} x^{2i}, \quad (\text{E.20})$$

and from the equality $\binom{2l+\Delta}{\Delta} = \binom{2l+\Delta}{2l}$ results that

$$\lim_{z \rightarrow 1} P^Z(k, z) = \frac{1}{2pN^{\Delta}} \left[\left(1 - \left(1 - \frac{1}{N}\right)\right)^{-(\Delta+1)} + \left(1 + \left(1 - \frac{1}{N}\right)\right)^{-(\Delta+1)} \right]. \quad (\text{E.21})$$

Therefore, for $\Delta \in E$, the coefficient of K^k in equation (E.14) is

$$\lim_{z \rightarrow 1} P^Z(k, z) = \frac{N}{2p} \left[1 + \frac{1}{(2N-1)^{\Delta+1}} \right]. \quad (\text{E.22})$$

Now it is necessary to calculate the same coefficient for $\Delta \in O$, where $O = \{\Delta \in \mathbb{Z} | \Delta = 2y+1, y \in \mathbb{Z}\}$, i.e., odd Δ , and $z \rightarrow 1$:

$$\lim_{z \rightarrow 1} P^Z(k, z) = \sum_{i=\frac{\Delta+1}{2}}^{\infty} \sum_{j=0}^{2i} \binom{2i}{j} \frac{1}{p} \frac{1}{N^j} \left(1 - \frac{1}{N}\right)^{2i-j} \delta_{j,\Delta}. \quad (\text{E.23})$$

Applying the same steps used to calculate $P^Z(k, z)$ for $\Delta \in E$, the case $\Delta \in O$ and $z \rightarrow 1$ leads to

$$\lim_{z \rightarrow 1} P^Z(k, z) = \frac{N}{2p} \left[1 - \frac{1}{(2N-1)^{\Delta+1}} \right]. \quad (\text{E.24})$$

Therefore, from equations (E.22) and (E.24), the coefficient of K^k for any $\Delta \in \mathbb{Z}$ (regardless of Δ being even or odd) is

$$\lim_{z \rightarrow 1} P^Z(k, z) = \frac{N}{2p} \left[1 + \frac{(-1)^\Delta}{(2N-1)^{\Delta+1}} \right]. \quad (\text{E.25})$$

Finally, recall that $\lim_{z \rightarrow 1} F^Z(k, z) = \lim_{z \rightarrow 1} P^Z(k, z) (1 - zq)$. From equations (E.25) and (5.10), it follows that

$$\begin{aligned} \lim_{z \rightarrow 1} F^Z(k, z) &= \frac{N}{2p} \left[1 + \frac{(-1)^\Delta}{(2N-1)^{\Delta+1}} \right] (1 - q) \\ &= \left(1 - \frac{1}{2N} \right) \left(1 + \frac{(-1)^\Delta}{(2N-1)^{\Delta+1}} \right). \end{aligned} \quad (\text{E.26})$$

APPENDIX F – Python code for the random recursive tree model

The following code was implemented on Python based on the instructions provided on page 30. It generates an ensemble of graphs for the random recursive tree model. From the ensemble, the code calculates the variance of the degree of a given vertex.

```

I = int(input('number of graphs:'))
N = int(input('number of vertices:'))
s = int(input('choose vertex:'))

sum_k, sum_k2 = 0.0, 0.0
for i in range(I):
    t_list = [0, 1]
    if s == 0 or s == 1: k_list = [2, 2]
    else: k_list = [0, 0]
    for t in range(2, N):
        if s > t: k_list.append(0)
        if s == t or s < t :
            r = random.randint(0, t - 1)
            if r == s: k_list.append(k_list[t - 1] + 1)
            else: k_list.append(k_list[t - 1])
        t_list.append(t)
    kvec = np.array(k_list)
    tvec = np.array(t_list)
    sum_k += kvec
    sum_k2 += kvec**2
var_E = sum_k2/I - (sum_k/I)**2
var_A = np.log(tvec/s)

```

APPENDIX G – Python code for the Barabási-Albert model

The following code was implemented on Python based on the instructions provided on page 32. It generates an ensemble of graphs for the Barabási-Albert model. From the ensemble, the code calculates the variance of the degree of a given vertex.

```

I = int(input('number of graphs:'))
N = int(input('number of vertices:'))
s = int(input('choose vertex:'))

sum_k, sum_k2 = 0.0, 0.0
for i in range(I):
    t_list = [0, 1]
    if s == 0 or s == 1: k_list = [2, 2]
    else: k_list = [0, 0]
    for t in range(2, N):
        if s > t:
            degree_list.append(0)
        if s == t or s < t:
            p = float(degree_list[-1]) / (2 * (t - 1))
            r = random.uniform(0.0, 1.0)
            if p <= r: degree_list.append(degree_list[-1] + 1)
            else: degree_list.append(degree_list[-1])
        t_list.append(t)
    kvec = np.array(degree_list)
    tvec = np.array(t_list)
    sum_k += kvec
    sum_k2 += kvec**2
var_E = sum_k2 / I - (sum_k / I)**2
frac = 1.0 / s * tvec
var_A = frac * np.sqrt(frac)

```

APPENDIX H – Mean time for the first-passage - Erdős-Rényi model

Recall from equation (5.21) that $\langle t \rangle = \lim_{z \rightarrow 1} \partial_z [P^Z(k, z)(1 - zq)]$. Appendix E shows how to calculate $\lim_{z \rightarrow 1} P^Z(k, z)$ and a way of calculating $\lim_{z \rightarrow 1} \partial_z P^Z(k, z)$ is to start from the partial result

$$P^{KZ}(K, z) = \sum_{i=0}^{\infty} K^{k_0} \left[p \left(\frac{K}{N} + 1 - \frac{1}{N} \right)^2 \right]^i z^i [1 - z(1 - p)]^{-i-1}, \quad (\text{H.1})$$

which corresponds to equation (E.11).

Considering that $P^{KZ}(K, z) = \sum_{k=0}^{\infty} P^Z(k, z) K^k$, then the coefficient for K^k of $\partial_z P^{KZ}(K, z)$ is precisely $\lim_{z \rightarrow 1} \partial_z P^Z(k, z)$. In order to isolate the term K^k , equation (H.1) can be rewritten as

$$P^Z(K, z) = \sum_{i=0}^{\infty} z^i [1 - z(1 - p)]^{-i-1} p^i \left(1 - \frac{1}{N} \right)^{2i} \sum_{j=0}^{2i} \binom{2i}{j} \frac{1}{N^j} \left(1 - \frac{1}{N} \right)^{-j} K^{k_0+j}. \quad (\text{H.2})$$

At this point, it is convenient to take the partial derivative, which results in

$$\begin{aligned} \frac{\partial}{\partial z} P^{KZ}(K, z) &= \sum_{i=0}^{\infty} z^i (1 - z(1 - p))^{-i-1} (iz^{-1} + (i+1)(1 - p)[1 - z(1 - p)]^{-1}) p^i \\ &\quad \times \left(1 - \frac{1}{N} \right)^{2i} \sum_{j=0}^{2i} \binom{2i}{j} \frac{1}{N^j} \left(1 - \frac{1}{N} \right)^{-j} K^{k_0+j}. \end{aligned} \quad (\text{H.3})$$

Taking the case where $z \rightarrow 1$ follows that

$$\begin{aligned} \lim_{z \rightarrow 1} \frac{\partial}{\partial z} P^{KZ}(K, z) &= p^{-1} \sum_{i=0}^{\infty} (ip^{-1} + p^{-1} - 1) \\ &\quad \times \left(1 - \frac{1}{N} \right)^{2i} \sum_{j=0}^{2i} \binom{2i}{j} \frac{1}{N^j} \left(1 - \frac{1}{N} \right)^{-j} K^{k_0+j}. \end{aligned} \quad (\text{H.4})$$

Now the coefficient for K^k in equation (H.4) is the partial derivative of $\lim_{z \rightarrow 1} \partial_z P^Z(k, z)$, and applying it to equation (5.21), along with the notation $q = 1 - \frac{p}{N} \left(2 - \frac{1}{N} \right)$, it follows that

$$\begin{aligned} \langle t \rangle &= p^{-1} \sum_{i=0}^{\infty} \sum_{j=0}^{2i} \binom{2i}{j} \left(1 - \frac{1}{N} \right)^{2i-j} \frac{1}{N^j} \times \\ &\quad \times \left[\left(\frac{i}{p} + \frac{1}{p} - 1 \right) \left(\frac{2p}{N} - \frac{p}{N^2} \right) - \left(1 - \frac{2p}{N} + \frac{p}{N^2} \right) \right] \delta_{j,\Delta}, \end{aligned} \quad (\text{H.5})$$

where $\Delta = k - k_0$ and the Kronecker delta has been introduced to assure that we are extracting only the part of the equation (H.4) that is proportional to K^k . Equation (H.5) can now be divided into two cases: (i) even Δ , i.e., $\Delta \in E$ where $E = \{\Delta \in \mathbb{Z} | \Delta = 2y, y \in \mathbb{Z}\}$; and (ii) odd Δ , i.e., $\Delta \in O$, where $O = \{\Delta \in \mathbb{Z} | \Delta = 2y + 1, y \in \mathbb{Z}\}$.

Case (i): even Δ — For $\Delta \in E$ where $E = \{\Delta \in \mathbb{Z} | \Delta = 2y, y \in \mathbb{Z}\}$, it follows that

$$\langle t \rangle = p^{-1} \sum_{i=\frac{\Delta}{2}}^{\infty} \binom{2i}{\Delta} \left(1 - \frac{1}{N}\right)^{2i-\Delta} \frac{1}{N^\Delta} \left[\left(\frac{i}{p} + \frac{1}{p} - 1\right) \left(\frac{2p}{N} - \frac{p}{N^2}\right) - \left(1 - \frac{2p}{N} + \frac{p}{N^2}\right) \right]. \quad (\text{H.6})$$

Since $\sum_{i=0}^{\frac{\Delta}{2}-1}$ is zero, this part of the sum can be eliminated. Equation (H.6) is then simplified to

$$\begin{aligned} \langle t \rangle = p^{-1} \frac{1}{N^\Delta} \left(1 - \frac{1}{N}\right)^{-\Delta} & \left[\left(\frac{2}{N} - \frac{1}{N^2}\right) \sum_{i=\frac{\Delta}{2}}^{\infty} \binom{2i}{\Delta} \left(1 - \frac{1}{N}\right)^{2i} i - \right. \\ & \left. - \left(1 - \frac{1}{N}\right)^2 \sum_{i=\frac{\Delta}{2}}^{\infty} \binom{2i}{\Delta} \left(1 - \frac{1}{N}\right)^{2i} \right]. \quad (\text{H.7}) \end{aligned}$$

With a change of variables ($i = l + \frac{\Delta}{2}$), equation (H.7) becomes

$$\begin{aligned} \langle t \rangle = p^{-1} \frac{1}{N^\Delta} \left(1 - \frac{1}{N}\right)^{-\Delta} & \left[\left(\frac{2}{N} - \frac{1}{N^2}\right) \sum_{l=0}^{\infty} \binom{2l+\Delta}{\Delta} \left(1 - \frac{1}{N}\right)^{2l+\Delta} \left(l + \frac{\Delta}{2}\right) - \right. \\ & \left. - \left(1 - \frac{1}{N}\right)^2 \sum_{l=0}^{\infty} \binom{2l+\Delta}{\Delta} \left(1 - \frac{1}{N}\right)^{2l+\Delta} \right]. \quad (\text{H.8}) \end{aligned}$$

Using the relations

$$(1-x)^{-(n+1)} + (1+x)^{-(n+1)} = 2 \sum_{i=0}^{\infty} \binom{n+2i}{2i} x^{2i} \quad (\text{H.9})$$

and

$$x(n+1) \left[(1-x)^{-(n+2)} - (1+x)^{-(n+2)} \right] = 4 \sum_{i=0}^{\infty} \binom{n+2i}{2i} x^{2i+1} \quad (\text{H.10})$$

in equation (H.8) results in

$$\begin{aligned} \langle t \rangle = \frac{1}{p} & \left[\left(2 - \frac{1}{N}\right) \frac{1}{4} \left[(\Delta+1) \left(1 - \frac{1}{N}\right) N \left(1 - \frac{1}{(2N-1)^{\Delta+2}}\right) + \Delta \left(1 + \frac{1}{(2N-1)^{\Delta+1}}\right) \right] - \right. \\ & \left. - \frac{N}{2} \left(1 - \frac{1}{N}\right)^2 \left(1 + \frac{1}{(2N-1)^{\Delta+1}}\right) \right]. \quad (\text{H.11}) \end{aligned}$$

For $N \gg 1$, some elements of the expression above go to zero:

$$\langle t \rangle = \frac{1}{p} \left[\left(2 - \frac{1}{N} \right) \frac{1}{4} \left[(\Delta + 1) \left(1 - \frac{1}{N} \right) N \left(1 - \frac{1}{(2N-1)^{\Delta+2}} \right) + \Delta \left(1 + \frac{1}{(2N-1)^{\Delta+1}} \right) \right] - \frac{N}{2} \left(1 - \frac{1}{N} \right)^2 \left(1 + \frac{1}{(2N-1)^{\Delta+1}} \right) \right]. \quad (\text{H.12})$$

This reduces equation (H.12) to

$$\langle t \rangle \simeq \frac{N\Delta}{2p}. \quad (\text{H.13})$$

Case (ii): odd Δ — For $\Delta \in O$, where $O = \{\Delta \in \mathbb{Z} | \Delta = 2y + 1, y \in \mathbb{Z}\}$, it follows that

$$\langle t \rangle = p^{-1} \sum_{i=\frac{\Delta+1}{2}}^{\infty} \binom{2i}{\Delta} \left(1 - \frac{1}{N} \right)^{2i-\Delta} \frac{1}{N^{\Delta}} \left[\left(\frac{i}{p} + \frac{1}{p} - 1 \right) \left(\frac{2p}{N} - \frac{p}{N^2} \right) - \left(1 - \frac{2p}{N} + \frac{p}{N^2} \right) \right]. \quad (\text{H.14})$$

Again, since $\sum_{i=0}^{\frac{\Delta}{2}-1}$ is zero, this part of the sum can be eliminated. Equation (H.14) is then simplified to

$$\langle t \rangle = p^{-1} \frac{1}{N^{\Delta}} \left(1 - \frac{1}{N} \right)^{-\Delta} \left[\left(\frac{2}{N} - \frac{1}{N^2} \right) \sum_{i=\frac{\Delta+1}{2}}^{\infty} \binom{2i}{\Delta} \left(1 - \frac{1}{N} \right)^{2i} i - \left(1 - \frac{1}{N} \right)^2 \sum_{i=\frac{\Delta+1}{2}}^{\infty} \binom{2i}{\Delta} \left(1 - \frac{1}{N} \right)^{2i} \right]. \quad (\text{H.15})$$

With a change of variables ($i = l + \frac{\Delta+1}{2}$), equation (H.15) becomes

$$\langle t \rangle = p^{-1} \frac{1}{N^{\Delta}} \left(1 - \frac{1}{N} \right)^{-\Delta} \left[\left(\frac{2}{N} - \frac{1}{N^2} \right) \sum_{l=0}^{\infty} \binom{2l+\Delta+1}{\Delta} \left(1 - \frac{1}{N} \right)^{2l+\Delta+1} \left(l + \frac{\Delta+1}{2} \right) - \left(1 - \frac{1}{N} \right)^2 \sum_{l=0}^{\infty} \binom{2l+\Delta+1}{\Delta} \left(1 - \frac{1}{N} \right)^{2l+\Delta+1} \right]. \quad (\text{H.16})$$

From the relations

$$(1-x)^{-(n+1)} - (1+x)^{-(n+1)} = 2 \sum_{i=0}^{\infty} \binom{n+2i+1}{2i+1} x^{2i+1} \quad (\text{H.17})$$

and

$$(n+1) x \left[(1-x)^{-(n+2)} + (1+x)^{-(n+2)} \right] + (1+x)^{-(n+1)} - (1-x)^{-(n+1)} = 4 \sum_{i=0}^{\infty} \binom{n+2i+1}{2i+1} i x^{2i+1}, \quad (\text{H.18})$$

equation (H.16) results in

$$\begin{aligned} \langle t \rangle = \frac{N}{4p} & \left[\frac{1}{N} \left(2 - \frac{1}{N} \right) \left[\left(1 - \frac{1}{N} \right) (\Delta + 1) N \left(1 + \frac{1}{(2N-1)^{\Delta+2}} \right) - 1 + \frac{1}{(2N-1)^{\Delta+1}} \right. \right. \\ & \left. \left. + (\Delta + 1) \left(1 - \frac{1}{(2N-1)^{\Delta+1}} \right) \right] - 2 \left(1 - \frac{1}{N} \right)^2 \left(1 - \frac{1}{(2N-1)^{\Delta+1}} \right) \right]. \quad (\text{H.19}) \end{aligned}$$

Again, for $N \gg 1$, some elements of the expression above go to zero:

$$\begin{aligned} \langle t \rangle = \frac{N}{4p} & \left[\frac{1}{N} \left(2 - \overset{0}{\cancel{\frac{1}{N}}} \right) \left[\left(1 - \overset{0}{\cancel{\frac{1}{N}}} \right) (\Delta + 1) N \left(1 + \frac{1}{(2N-1)^{\Delta+2}} \right) \overset{0}{\cancel{-1}} + \overset{0}{\cancel{\frac{1}{(2N-1)^{\Delta+1}}}} \right. \right. \\ & \left. \left. + (\Delta + 1) \left(1 - \frac{1}{(2N-1)^{\Delta+1}} \right) \overset{0}{\cancel{\left(1 - \frac{1}{(2N-1)^{\Delta+1}} \right)}} \right] - 2 \left(1 - \overset{0}{\cancel{\frac{1}{N}}} \right)^2 \left(1 - \frac{1}{(2N-1)^{\Delta+1}} \right) \overset{0}{\cancel{\left(1 - \frac{1}{(2N-1)^{\Delta+1}} \right)}} \right], \quad (\text{H.20}) \end{aligned}$$

reducing equation (H.18) to

$$\langle t \rangle \simeq \frac{N\Delta}{2p}. \quad (\text{H.21})$$

From (H.13) and (H.21) we conclude that the asymptotic form for $\langle t \rangle$ is the same for Δ even or odd.

APPENDIX I – Python code for the $\langle t \rangle$ - Erdős-Rényi model

The following code was implemented on Python based on the instructions provided on page 40. It generates an ensemble of graphs to estimate the mean time for the first-passage in degree space for the Erdős-Rényi model.

```

N = int(input('max number of vertices:'))
s = int(input('choose vertex: '))
k = int(input('degree to reach: '))
I = int(input('number of graphs:'))

vertices, meant_list, error_list, N_list = [], [], [], []
for n in range(1, N+1):
    vertices.append(n)
    t_list = []
    for i in range(I):
        t = 0
        s_edges = []
        s_edges.append(0)
        if n >= s:
            while s_edges[1] < k:
                p = random.uniform(0.0, 1.0)
                if p < 0.5:
                    node1 = random.choice(vertices)
                    node2 = random.choice(vertices)
                    if node1 == s and node2 == s:
                        s_edges.append(s_edges[1] + 2)
                    if node1 == s and node2 != s or node2 == s and
node1 != s:
                        s_edges.append(s_edges[1] + 1)
                t += 1
            t_list.append(t)
    mean_time = float(sum(t_list)/I)
    meant_list.append(mean_time)

```


APPENDIX J – Degree distribution for the dynamical Watts-Strogatz model

From section 5.2, it follows that the probability of a given vertex s having degree k at time $t + 1$, presented by equation (5.24), is expressed as

$$p(k, s, t + 1) = \omega(k|k - 1) p(k - 1, s, t) + \omega(k|k + 1) p(k + 1, s, t) + \omega(k|k) p(k, s, t), \quad (\text{J.1})$$

Recall from section 5.2, equations (5.25), (5.26) and (5.27), that

$$\omega(k|k - 1) = \frac{p}{N} \left(1 - \frac{k - 1}{M} \right), \quad (\text{J.2})$$

$$\omega(k|k + 1) = \frac{k + 1}{M} p \left(1 - \frac{1}{N} \right) \quad \text{and} \quad (\text{J.3})$$

$$\omega(k|k) = \frac{kp}{MN} + \frac{k}{M} (1 - p) + \left(1 - \frac{k}{M} \right) p \left(1 - \frac{1}{N} \right) + \left(1 - \frac{k}{M} \right) (1 - p), \quad (\text{J.4})$$

where $M = k_0 N$. If $\omega(k|k - 1) = A(k - 1)$ and $\omega(k|k + 1) = B(k + 1)$, then $\omega(k|k) = 1 - A(k) - B(k)$. So equation (J.1) can be rewritten as

$$\begin{aligned} p(k, s, t + 1) - p(k, s, t) &= A(k - 1) p(k - 1, s, t) - A(k) p(k, s, t) + \\ &\quad + B(k + 1) p(k + 1, s, t) - B(k) p(k, s, t). \end{aligned} \quad (\text{J.5})$$

For $t \gg 1$, $p(k, s, t + 1) \simeq p(k, s, t)$. Also, in the stationary regime, where $t \rightarrow \infty$, the probabilities will no longer depend on time, so $p(k, s, t) = p(k, s)$. Therefore, equation (J.5) results in

$$\begin{aligned} 0 &= \frac{1}{N} p(k - 1, s) - \frac{k - 1}{MN} p(k - 1, s) - \frac{1}{N} p(k, s) + \\ &\quad + \frac{k + 1}{M} \left(1 - \frac{1}{N} \right) p(k + 1, s) - \frac{k}{M} \left(1 - \frac{1}{N} \right) p(k, s), \end{aligned} \quad (\text{J.6})$$

where the equality $P(k, s) = p(k, s)$ has been used, since all vertices are statically equal.

The next step is to apply the characteristic function using the notation $P^K(K, s) = \sum_{k=0}^{\infty} K^k P(k, s)$. Multiplying both sides of equation (J.6) by K^k and summing over k from 0 to ∞ gives

$$\begin{aligned} 0 &= \frac{1}{N} \sum_{k=0}^{\infty} K^k P(k - 1, s) - \frac{1}{MN} \sum_{k=0}^{\infty} K^k (k - 1) P(k - 1, s) - \frac{1}{N} \sum_{k=0}^{\infty} K^k P(k, s) + \\ &\quad + \frac{1}{M} \left(1 - \frac{1}{N} \right) \sum_{k=0}^{\infty} K^k (k + 1) P(k + 1, s) - \frac{1}{M} \left(1 - \frac{1}{N} \right) \sum_{k=0}^{\infty} K^k k P(k, s). \end{aligned} \quad (\text{J.7})$$

A change of variables ($k - 1 = l$ and $k + 1 = l$) gives

$$\sum_{k=0}^{\infty} K^k P(k-1, s) = \sum_{l=-1}^{\infty} K^{l+1} P(l, s) = K \sum_{l=0}^{\infty} K^l P(l, s) = K P^K(K, s), \quad (\text{J.8})$$

$$\sum_{k=0}^{\infty} K^k (k-1) P(k-1, s) = \sum_{l=-1}^{\infty} K^{l+1} l P(l, s) = K \sum_{l=0}^{\infty} K^l l P(l, s) = K^2 \partial_K P^K(K, s), \quad (\text{J.9})$$

$$\sum_{k=0}^{\infty} K^k k P(k, s) = K \partial_K P^K(K, s) \quad \text{and} \quad (\text{J.10})$$

$$\sum_{k=0}^{\infty} K^k (k+1) P(k+1, s) = \sum_{l=1}^{\infty} K^{l-1} l P(l, s) = K^{-1} \sum_{l=0}^{\infty} K^l l P(l, s) = \partial_K P^K(K, s). \quad (\text{J.11})$$

Replacing the results from equations (J.8) to (J.11) to equation (J.7) results in

$$\begin{aligned} 0 = \frac{1}{N} K P^K(K, s) - \frac{1}{MN} K^2 \partial_K P^K(K, s) - \frac{1}{N} P^K(K, s) + \frac{1}{NM} K \partial_K P^K(K, s) \\ + \frac{1}{M} \left(1 - \frac{1}{N}\right) \partial_K P^K(K, s) - \frac{1}{M} \left(1 - \frac{1}{N}\right) K \partial_K P^K(K, s), \end{aligned} \quad (\text{J.12})$$

which can be simplified to

$$\partial_K P^K(K, s) = \frac{M}{K + N - 1} P^K(K, s). \quad (\text{J.13})$$

Integrating the differential equation (J.13) gives that

$$\int \frac{1}{P^K(K, s)} dP^K(K, s) = M \int \frac{1}{K + N - 1} dK, \quad (\text{J.14})$$

and solving the integrals above follows that

$$\ln(P^K(K, s)) = M \ln(K + N - 1) + C, \quad (\text{J.15})$$

where C is an integration constant. Considering that $P^K(1, s) = 1$, it is possible to calculate that $C = \ln(N^{-M})$, which leads to the final equation for $P^K(K, s)$,

$$P^K(K, s) = \left(\frac{K + N - 1}{N} \right)^M. \quad (\text{J.16})$$

The result obtained in equation (J.16) can be rewritten as

$$\begin{aligned} P^K(K, s) &= \sum_{k=0}^M \binom{M}{k} \left(\frac{K}{N} \right)^k \left(1 - \frac{1}{N} \right)^{M-k} \\ &= \sum_{k=0}^M K^k \binom{M}{k} \left(\frac{1}{N} \right)^k \left(1 - \frac{1}{N} \right)^{M-k}, \end{aligned} \quad (\text{J.17})$$

where the coefficient of K^k is actually $P(k, s)$ and represents a binomial distribution:

$$P(k, s) = \binom{M}{k} \left(\frac{1}{N}\right)^k \left(1 - \frac{1}{N}\right)^{M-k}. \quad (\text{J.18})$$

When $M \rightarrow \infty$, $1/N \rightarrow 0$ and $M \times 1/N = \lambda$, where λ is a constant — in this case $\lambda = k_0$, since $M = k_0 N$. So, in the limit where $M \rightarrow \infty$, the binomial becomes

$$\begin{aligned} \binom{M}{k} \left(\frac{1}{N}\right)^k \left(1 - \frac{1}{N}\right)^{M-k} &\simeq \lim_{M \rightarrow \infty} \frac{M(M-1)(M-2)\dots(M-k+1)}{k!} \left(\frac{k_0}{M}\right)^k \left(1 - \frac{1}{N}\right)^{M-k} \\ &\simeq \lim_{M \rightarrow \infty} \frac{M^k + \mathcal{O}(M^{k-1})}{k!} \frac{k_0^k}{M^k} \left(1 - \frac{k_0}{M}\right)^{M-k} \\ &\simeq \lim_{M \rightarrow \infty} \frac{k_0^k}{k!} \left(1 - \frac{k_0}{M}\right)^{M-k}. \end{aligned} \quad (\text{J.19})$$

Since

$$\lim_{M \rightarrow \infty} \left(1 - \frac{k_0}{M}\right)^M = e^{-k_0} \quad (\text{J.20})$$

and

$$\lim_{M \rightarrow \infty} \left(1 - \frac{k_0}{M}\right)^{-k} = 1, \quad (\text{J.21})$$

the binomial distribution (J.18) turns out to be a Poisson distribution, expressed as

$$P_{\text{Poisson}}(k, s) = \frac{e^{-k_0} k_0^k}{k!}. \quad (\text{J.22})$$

APPENDIX K – Calculating $F^Z(k | k_0, z)$ for the Watts-Strogatz model

Equation (5.35) gives the characteristic function for the probability of a vertex achieving a certain degree k given that it started from k_0 , which is

$$F^Z(k | k_0, z) = \frac{P^Z(k | k_0, z)}{P^Z(k | k, z)}. \quad (\text{K.1})$$

This expression shows that it is necessary to first calculate $P^Z(k | k_0, z)$ and $P^Z(k | k, z)$ in order to obtain the characteristic function of $F(k | k_0, t - 0)$. The starting point is the equation that gives the probability of a given vertex s having k edges at time $t + 1$, presented by equation (5.24), which is expressed as

$$p(k, s, t + 1) = \omega(k | k - 1) p(k - 1, s, t) + \omega(k | k + 1) p(k + 1, s, t) + \omega(k | k) p(k, s, t). \quad (\text{K.2})$$

Recall from section 5.2, equations (5.25), (5.26) and (5.27), that

$$\omega(k | k - 1) = \frac{p}{N} \left(1 - \frac{k - 1}{M} \right), \quad (\text{K.3})$$

$$\omega(k | k + 1) = \frac{k + 1}{M} p \left(1 - \frac{1}{N} \right) \quad \text{and} \quad (\text{K.4})$$

$$\omega(k | k) = \frac{kp}{MN} + \frac{k}{M}(1 - p) + \left(1 - \frac{k}{M} \right) p \left(1 - \frac{1}{N} \right) + \left(1 - \frac{k}{M} \right) (1 - p). \quad (\text{K.5})$$

Denoting $\omega(k | k - 1) = A(k - 1)$ and $\omega(k | k + 1) = B(k + 1)$, then $\omega(k | k) = 1 - A(k) - B(k)$. So equation (K.2) can be rewritten as

$$\begin{aligned} p(k, s, t + 1) = & A(k - 1) p(k - 1, s, t) - A(k) p(k, s, t) + \\ & + B(k + 1) p(k + 1, s, t) - B(k) p(k, s, t) + p(k, s, t). \end{aligned} \quad (\text{K.6})$$

The first step is to apply the characteristic function using the notation $P^K(K, t) = \sum_{k=0}^{\infty} K^k P(k, t)$. Multiplying both sides of equation (K.2) by K^k and summing over k from 0 to ∞ gives

$$\begin{aligned} P^K(K, t + 1) = & \frac{p}{N} \left[K P^K(K, t) - \frac{K^2}{M} \partial_K P^K(K, t) \right] - \frac{p}{N} \left[P^K(K, t) - \frac{K}{M} \partial_K P^K(K, t) \right] \\ & + \frac{p}{M} \left(1 - \frac{1}{N} \right) \partial_K P^K(K, t) - \frac{p}{M} \left(1 - \frac{1}{N} \right) K \partial_K P^K(K, t) + P^K(K, t). \end{aligned} \quad (\text{K.7})$$

Equation (K.7) can be simplified to

$$P^K(K, t + 1) = \frac{p}{M} (1 - K) \left(1 - \frac{1 - K}{N} \right) \partial_K P^K(K, t) + \left[1 - \frac{p}{N} (1 - K) \right] P^K(K, t). \quad (\text{K.8})$$

Next, the characteristic function is applied again, this time using the notation $P^{KZ}(K, z) = \sum_{t=0}^{\infty} z^t P^K(K, t)$. Multiplying both sides of equation (K.8) by z^t and summing over t from 0 to ∞ gives

$$\sum_{t=0}^{\infty} z^t P^K(K, t+1) = \frac{p}{M} (1-K) \left(1 - \frac{1-K}{N}\right) \partial_K P^{KZ}(K, z) + \left[1 - \frac{p}{N} (1-K)\right] P^{KZ}(K, z). \quad (\text{K.9})$$

To solve the term on the left hand side of equation (K.9), a change of variables ($t+1=l$) can be used, so that

$$\begin{aligned} \sum_{t=0}^{\infty} z^t P^K(K, t+1) &= \sum_{l=1}^{\infty} z^{l-1} P^K(K, l) \\ &= z^{-1} \left[\sum_{l=0}^{\infty} z^l P^K(K, l) - P^K(K, 0) \right] \\ &= z^{-1} [P^{KZ}(K, z) - P^K(K, 0)]. \end{aligned} \quad (\text{K.10})$$

Replacing this last result into equation (K.9) results in

$$\begin{aligned} z^{-1} P^{KZ}(K, z) - z^{-1} P^K(K, 0) &= \frac{p}{M} (1-K) \left(1 - \frac{1-K}{N}\right) \partial_K P^{KZ}(K, z) \\ &\quad + \left[1 - \frac{p}{N} (1-K)\right] P^{KZ}(K, z). \end{aligned} \quad (\text{K.11})$$

Equation (K.11) can be simplified to

$$\begin{aligned} \partial_K P^{KZ}(K, z) &= -\frac{M}{p} \left(\frac{1-z^{-1}}{1-K} + \frac{1-p-z^{-1}}{N+K-1} \right) P^{KZ}(K, z) \\ &\quad - \frac{M}{p} \left(\frac{1}{1-K} + \frac{1}{N+K-1} \right) z^{-1} P^K(K, 0), \end{aligned} \quad (\text{K.12})$$

and the differential equation (K.12) can be solved using the method of variation of parameters. In this method, the solution for the differential equation

$$\frac{d}{dt} x(t) = A(t)x(t) + B(t) \quad (\text{K.13})$$

is given by

$$x(t) = x_0 \exp \left[\int_{t_0}^t d\xi A(\xi) \right] + \int_{t_0}^t d\xi B(\xi) \exp \left[\int_{\xi}^t d\eta A(\eta) \right], \quad (\text{K.14})$$

where $x(t_0) = x_0$ is an initial (boundary) condition. Comparing equations (K.12) and (K.13), it is easy to visualize the correspondence among the terms:

$$P^{KZ}(K, z) = x(t), \quad (\text{K.15})$$

$$-\frac{M}{p} \left(\frac{1-z^{-1}}{1-K} + \frac{1-p-z^{-1}}{N+K-1} \right) = A(t) \quad \text{and} \quad (\text{K.16})$$

$$-\frac{M}{p} \left(\frac{1}{1-K} + \frac{1}{N+K-1} \right) z^{-1} P^K(K, 0) = B(t). \quad (\text{K.17})$$

The direct solution presented in equation (K.14) can be used to solve equation (K.12) term by term. The first term to be solved is the integral $\int d\xi A(\xi)$. Using the correspondence among terms, it follows that

$$-\int_{K_0}^K d\xi \frac{M}{p} \left(\frac{1-z^{-1}}{1-\xi} + \frac{1-p-z^{-1}}{N+\xi-1} \right) = -\frac{M}{p}(1-z^{-1}) \int_{K_0}^K \frac{1}{1-\xi} d\xi - \frac{M}{p}(1-p-z^{-1}) \int_{K_0}^K \frac{1}{N+\xi-1} d\xi, \quad (\text{K.18})$$

where K_0 , $|K_0| < 1$, is some fixed constant (a value will be assigned to it later). Solving equation (K.18) results in

$$-\frac{M}{p}(1-z^{-1}) [\ln(1-K_0) - \ln(1-K)] - \frac{M}{p}(1-p-z^{-1}) [\ln(N+K-1) - \ln(N+K_0-1)]. \quad (\text{K.19})$$

Equation (K.19) can be simplified using the notation

$$\alpha = \frac{M}{p}(z^{-1} - 1), \quad (\text{K.20})$$

so the solution of the integral $\int d\xi A(\xi)$ in equation (K.19) is

$$-\int_{K_0}^K d\xi \frac{M}{p} \left(\frac{1-z^{-1}}{1-\xi} + \frac{1-p-z^{-1}}{N+\xi-1} \right) = \ln \left[\left(\frac{1-K_0}{1-K} \right)^\alpha \left(\frac{N+K-1}{N+K_0-1} \right)^{\alpha+M} \right], \quad (\text{K.21})$$

where the variable K was replaced by ξ so it is not mistaken with the K from the integration interval endpoint. Next, the equation that corresponds to the term $\int d\xi B(\xi) \exp \int d\eta A(\eta)$ is

$$-\int_{K_0}^K d\xi \frac{M}{p} \left(\frac{1}{1-\xi} + \frac{1}{N+\xi-1} \right) z^{-1} P^Z(\xi, 0) \exp \left[\ln \left[\left(\frac{1-\xi}{1-K} \right)^\alpha \left(\frac{N+K-1}{N+\xi-1} \right)^{\alpha+M} \right] \right], \quad (\text{K.22})$$

where the result for the $\int A$ obtained in equation (K.21) has been directly applied. Now, equation (K.22) can be simplified to

$$-\frac{(N+K-1)^{\alpha+M}}{(1-K)^\alpha} \frac{MNz^{-1}}{p} \int_{K_0}^K d\xi \frac{(1-\xi)^{\alpha-1}}{(N+\xi-1)^{\alpha+M+1}} P^Z(\xi, 0). \quad (\text{K.23})$$

The combination of equations (K.21) and (K.23) gives that the solution for the differential equation (K.12) is

$$P^{KZ}(K, z) = P^{KZ}(K_0, z) \left(\frac{1 - K_0}{1 - K} \right)^\alpha \left(\frac{N + K - 1}{N + K_0 - 1} \right)^{\alpha+M} - \frac{MNz^{-1}}{p} \frac{(N + K - 1)^{\alpha+M}}{(1 - K)^\alpha} \int_{K_0}^K d\xi \frac{(1 - \xi)^{\alpha-1}}{(N + \xi - 1)^{\alpha+M+1}} P^Z(\xi, 0). \quad (\text{K.24})$$

Recall that $P^{KZ}(K_0, z) = \sum_{k=0}^{\infty} K_0^k \sum_{t=0}^{\infty} z^t P(k, t)$. For $K_0 = 1$, it follows that

$$P^{KZ}(1, z) = \sum_{t=0}^{\infty} z^t \sum_{k=0}^{\infty} 1^k P(k, t) = \frac{1}{1 - z}, \quad (\text{K.25})$$

and this result will be used as a boundary condition. So, if $K_0 = 1$, then equation (K.24) can be simplified to

$$P^{KZ}(K, z) = \frac{MNz^{-1}}{p} \frac{(N + K - 1)^{\alpha+M}}{(1 - K)^\alpha} \int_K^1 d\xi \frac{(1 - \xi)^{\alpha-1}}{(N + \xi - 1)^{\alpha+M+1}} P^Z(\xi, 0). \quad (\text{K.26})$$

With this solution (equation(K.26)), it is possible to calculate $P^Z(k | k_0, z)$ and $P^Z(k | k, z)$.

From the characteristic function $P^{KZ}(K, z) = \sum_{k=0}^{\infty} K^k P^Z(k, z)$, it follows that

$$P^Z(k, z) = \frac{1}{k!} \frac{\partial^k}{\partial K^k} P^{KZ}(K, z) \Big|_{K=0}. \quad (\text{K.27})$$

Recall from equation (K.1) that $F^Z(k | k_0, z) = \frac{P^Z(k | k_0, z)}{P^Z(k | k, z)}$. Hence,

$$F^Z(k | k_0, z) = \frac{\frac{1}{k!} \frac{MNz^{-1}}{p} \frac{\partial^k}{\partial K^k} \left[\frac{(N + K - 1)^{\alpha+M}}{(1 - K)^\alpha} \int_K^1 d\xi \frac{(1 - \xi)^{\alpha-1}}{(N + \xi - 1)^{\alpha+M+1}} P^Z(k | k_0, 0) \right] \Big|_{K=0}}{\frac{1}{k!} \frac{MNz^{-1}}{p} \frac{\partial^k}{\partial K^k} \left[\frac{(N + K - 1)^{\alpha+M}}{(1 - K)^\alpha} \int_K^1 d\xi \frac{(1 - \xi)^{\alpha-1}}{(N + \xi - 1)^{\alpha+M+1}} P^Z(k | k, 0) \right] \Big|_{K=0}}. \quad (\text{K.28})$$

From the initial condition is $P(k, t = 0) = \delta_{k, k_0}$, it follows that $P^Z(k | k_0, z) = K^{k_0}$ and $P^Z(k | k, z) = K^k$. Replacing these results in equation (K.28), and simplifying the equal terms that appear both in the numerator and the denominator gives

$$F^Z(k | k_0, z) = \frac{\frac{\partial^k}{\partial K^k} \left[\frac{(N + K - 1)^{\alpha+M}}{(1 - K)^\alpha} \int_K^1 d\xi \frac{(1 - \xi)^{\alpha-1}}{(N + \xi - 1)^{\alpha+M+1}} \xi^{k_0} \right] \Big|_{K=0}}{\frac{\partial^k}{\partial K^k} \left[\frac{(N + K - 1)^{\alpha+M}}{(1 - K)^\alpha} \int_K^1 d\xi \frac{(1 - \xi)^{\alpha-1}}{(N + \xi - 1)^{\alpha+M+1}} \xi^k \right] \Big|_{K=0}}. \quad (\text{K.29})$$

The term $(N + \xi - 1)^{\alpha+M+1}$ that appears in both integrals of equation (K.29) can be rewritten as

$$(N + \xi - 1)^{\alpha+M+1} = (N^{\alpha+M+1}) \left(1 + \frac{\xi - 1}{N} \right)^{k_0 N \left(\frac{z^{-1} - 1}{p} + 1 \right)} \left(1 + \frac{\xi - 1}{N} \right), \quad (\text{K.30})$$

recalling that $\alpha = \frac{M(z^{-1}-1)}{p}$ and $M = k_0 N$. If the middle term in the right hand side of equation (K.30) is rewritten as

$$\begin{aligned} \left(1 + \frac{\xi - 1}{N}\right)^{k_0 N \left(\frac{z^{-1}-1}{p} + 1\right)} &= \left[\left(1 + \frac{\xi - 1}{N}\right)^N\right]^{k_0 \left(\frac{z^{-1}-1}{p} + 1\right)} \\ &= \left[\left(1 + \frac{1}{\frac{N}{\xi-1}}\right)^{\frac{N}{\xi-1}}\right]^{(\xi-1)k_0 \left(\frac{z^{-1}-1}{p} + 1\right)}, \end{aligned} \quad (\text{K.31})$$

and $N \gg 1$, then taking the limit of equation (K.31) when $\frac{\xi-1}{N} \rightarrow 0$ gives that

$$\begin{aligned} \left[\left(1 + \frac{1}{\frac{N}{\xi-1}}\right)^{\frac{N}{\xi-1}}\right]^{(\xi-1)k_0\beta} &\simeq \left[e - \frac{e(\xi-1)}{2N}\right]^{(\xi-1)k_0\beta} \\ &\simeq e^{(\xi-1)k_0\beta} \left[1 - \frac{(\xi-1)}{2N}\right]^{(\xi-1)k_0\beta}, \end{aligned} \quad (\text{K.32})$$

where

$$\beta = \frac{z^{-1} - 1}{p} + 1. \quad (\text{K.33})$$

A Taylor expansion of the second term on the right hand side of equation (K.32) results in

$$\left[1 - \frac{(\xi-1)}{2N}\right]^{(\xi-1)k_0\beta} \simeq \left[1 - k_0\beta \frac{(\xi-1)^2}{2N}\right]. \quad (\text{K.34})$$

Since $N \gg 1$, the whole expression $k_0\beta \frac{(\xi-1)^2}{2N}$ is of the order of $\mathcal{O}(1/N)$, so equation (K.32) becomes simply

$$\left[\left(1 + \frac{1}{\frac{N}{\xi-1}}\right)^{\frac{N}{\xi-1}}\right]^{(\xi-1)k_0\beta} \simeq e^{(\xi-1)k_0\beta}. \quad (\text{K.35})$$

Therefore, the result obtained in equation (K.35) gives that

$$(N + \xi - 1)^{\alpha+M+1} \simeq (N^{\alpha+M+1}) (e^{(\xi-1)k_0\beta}). \quad (\text{K.36})$$

Similarly, the term $(N + K - 1)^{\alpha+M}$ that appears in both derivatives of equation (K.29) can be written as

$$(N + K - 1)^{\alpha+M} \simeq (N^{\alpha+M}) (e^{(K-1)k_0\beta}). \quad (\text{K.37})$$

Applying the results from (K.36) and (K.37) to equation (K.29) and simplifying some terms results that

$$F^Z(k | k_0, z) \simeq \frac{\frac{\partial^k}{\partial K^k} \left[e^{k_0\beta(K-1)} (1-K)^{-\alpha} \int_K^1 d\xi \xi^{k_0} e^{-k_0\beta(\xi-1)} (1-\xi)^{\alpha-1} \right] \Big|_{K=0}}{\frac{\partial^k}{\partial K^k} \left[e^{k_0\beta(K-1)} (1-K)^{-\alpha} \int_K^1 d\xi \xi^k e^{-k_0\beta(\xi-1)} (1-\xi)^{\alpha-1} \right] \Big|_{K=0}}. \quad (\text{K.38})$$

The multiple derivative of the product of any functions $f(x)$ and $g(x)$ is given by

$$\begin{aligned} \frac{d^k}{dx^k} [f(x) g(x)] &= \sum_{l=0}^k \binom{k}{l} f^{(k-l)}(x) g^{(l)}(x) \\ &= f^{(k)}(x) g^{(0)}(x) + \sum_{l=1}^k \binom{k}{l} f^{(k-l)}(x) g^{(l)}(x), \end{aligned} \quad (\text{K.39})$$

where $f^{(m)}$ is the m -th derivative of f (and a similar meaning for $g^{(m)}$). Applying equation (K.39) to the numerator and the denominator of equation (K.38) results in, respectively,

$$\begin{aligned} &\left(\frac{\partial^k}{\partial K^k} [e^{k_0 \beta(K-1)} (1-K)^{-\alpha}] \right) \left(\int_K^1 d\xi \xi^{k_0} e^{-k_0 \beta(\xi-1)} (1-\xi)^{\alpha-1} \right) \Big|_{K=0} \\ &+ \sum_{l=1}^k \binom{k}{l} \left(\frac{\partial^{k-l}}{\partial K^{k-l}} [e^{k_0 \beta(K-1)} (1-K)^{-\alpha}] \right) \left(\frac{\partial^l}{\partial K^l} \int_K^1 d\xi \xi^{k_0} e^{-k_0 \beta(\xi-1)} (1-\xi)^{\alpha-1} \right) \Big|_{K=0} \end{aligned} \quad (\text{K.40})$$

and

$$\begin{aligned} &\left(\frac{\partial^k}{\partial K^k} [e^{k_0 \beta(K-1)} (1-K)^{-\alpha}] \right) \left(\int_K^1 d\xi \xi^k e^{-k_0 \beta(\xi-1)} (1-\xi)^{\alpha-1} \right) \Big|_{K=0} \\ &+ \sum_{l=1}^k \binom{k}{l} \left(\frac{\partial^{k-l}}{\partial K^{k-l}} [e^{k_0 \beta(K-1)} (1-K)^{-\alpha}] \right) \left(\frac{\partial^l}{\partial K^l} \int_K^1 d\xi \xi^k e^{-k_0 \beta(\xi-1)} (1-\xi)^{\alpha-1} \right) \Big|_{K=0}. \end{aligned} \quad (\text{K.41})$$

Considering that the derivative of the integral of a function $f(x, \xi)$ is expressed as

$$\frac{d}{dx} \int_{a(x)}^{b(x)} d\xi f(x, \xi) = f(x, a(x))(-a'(x)) + f(x, b(x))b'(x) + \int_{a(x)}^{b(x)} d\xi \partial_x f(x, \xi), \quad (\text{K.42})$$

the last term on equations (K.40) and (K.41) can be derived once and hence simplified to

$$\begin{aligned} \frac{\partial^l}{\partial K^l} \int_K^1 d\xi f(K, \xi) &= \frac{\partial^{l-1}}{\partial K^{l-1}} \left[f(K, \xi)(-1) + f(K, \xi)0 + \int_K^1 d\xi \partial_K f(K, \xi) \right] \\ &= \frac{\partial^{l-1}}{\partial K^{l-1}} [-f(K, \xi)], \end{aligned} \quad (\text{K.43})$$

where $f(K, \xi) = K^{k'} e^{-k_0 \beta(K-1)} (1-K)^{\alpha-1}$, and $k' = k_0$ for the numerator and $k' = k$ for the denominator. Also, since f does not depend on K , $\partial_K f(K, \xi) = 0$. Equations (K.39) and (K.40) can be rewritten, respectively, as

$$\begin{aligned} &\left(\frac{\partial^k}{\partial K^k} [e^{k_0 \beta(K-1)} (1-K)^{-\alpha}] \right) \left(\int_K^1 d\xi \xi^{k_0} e^{-k_0 \beta(\xi-1)} (1-\xi)^{\alpha-1} \right) \Big|_{K=0} \\ &- \sum_{l=1}^k \binom{k}{l} \left(\frac{\partial^{k-l}}{\partial K^{k-l}} [e^{k_0 \beta(K-1)} (1-K)^{-\alpha}] \right) \left(\frac{\partial^{l-1}}{\partial K^{l-1}} [K^{k_0} e^{-k_0 \beta(K-1)} (1-K)^{\alpha-1}] \right) \Big|_{K=0} \end{aligned} \quad (\text{K.44})$$

and

$$\begin{aligned} & \left(\frac{\partial^k}{\partial K^k} [e^{k_0\beta(K-1)} (1-K)^{-\alpha}] \right) \left(\int_K^1 d\xi \xi^k e^{-k_0\beta(\xi-1)} (1-\xi)^{\alpha-1} \right) \Big|_{K=0} \\ & - \sum_{l=1}^k \binom{k}{l} \left(\frac{\partial^{k-l}}{\partial K^{k-l}} [e^{k_0\beta(K-1)} (1-K)^{-\alpha}] \right) \left(\frac{\partial^{l-1}}{\partial K^{l-1}} [K^k e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}] \right) \Big|_{K=0}. \end{aligned} \quad (\text{K.45})$$

Before proceeding it is convenient to analyze the multiple derivative $(\frac{\partial^l}{\partial K^{l-1}})$ of the last parenthesis of equations (K.40) and (K.41). If $k \leq k_0$, then after taking all the derivatives there will still remain, at least, the terms K^{k_0-k+1} and K^1 . These terms will take the sum on equations (K.44) and (K.45) to zero when $K = 0$. For $k > k_0$, the same happens to (K.45) only, since it does not depend on k_0 . Therefore, equations (K.44) and (K.45) become, respectively,

$$\begin{aligned} & \left(\frac{\partial^k}{\partial K^k} [e^{k_0\beta(K-1)} (1-K)^{-\alpha}] \right) \left(\int_K^1 d\xi \xi^{k_0} e^{-k_0\beta(\xi-1)} (1-\xi)^{\alpha-1} \right) \Big|_{K=0} \\ & - \sum_{l=1}^k \binom{k}{l} \left(\frac{\partial^{k-l}}{\partial K^{k-l}} [e^{k_0\beta(K-1)} (1-K)^{-\alpha}] \right) \Big|_{K=0} \\ & \quad \times \left(\frac{\partial^{l-1}}{\partial K^{l-1}} [K^{k_0} e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}] \right) \Big|_{K=0} \end{aligned} \quad (\text{K.46})$$

and

$$\left(\frac{\partial^k}{\partial K^k} [e^{k_0\beta(K-1)} (1-K)^{-\alpha}] \right) \left[\int_K^1 d\xi \xi^k e^{-k_0\beta(\xi-1)} (1-\xi)^{\alpha-1} \right] \Big|_{K=0}. \quad (\text{K.47})$$

The integrals on equations (K.46) and (K.47) can be integrated by parts, resulting in, respectively,

$$\int_K^1 d\xi \xi^{k_0} e^{-k_0\beta(\xi-1)} (1-\xi)^{\alpha-1} \Big|_{K=0} = \frac{1}{\alpha} \int_0^1 d\xi \xi^{k_0-1} e^{-k_0\beta(\xi-1)} (1-\xi)^\alpha (k_0 - k_0\beta\xi) \quad (\text{K.48})$$

and

$$\int_K^1 d\xi \xi^{k_0} e^{-k_0\beta(\xi-1)} (1-\xi)^{\alpha-1} \Big|_{K=0} = \frac{1}{\alpha} \int_0^1 d\xi \xi^{k-1} e^{-k_0\beta(\xi-1)} (1-\xi)^\alpha (k - k_0\beta\xi). \quad (\text{K.49})$$

If $z \rightarrow 1$, then $\alpha \rightarrow 0$, and since $N \gg 1$, the entire sum on equation (K.46) goes to zero. The resulting expression for $F^Z(k | k_0, 1)$ is

$$F^Z(k | k_0, 1) \simeq \frac{\frac{k_0}{\alpha} \int_0^1 d\xi \xi^{k_0-1} e^{-k_0\beta\xi} (1-\beta\xi) (1-\xi)^\alpha}{\frac{1}{\alpha} \left[k \int_0^1 d\xi \xi^{k-1} e^{-k_0\beta\xi} (1-\xi)^\alpha - k_0 \int_0^1 d\xi \beta \xi^k e^{-k_0\beta\xi} (1-\xi)^\alpha \right]} \Big|_{z=1}. \quad (\text{K.50})$$

Next, if $z = 1$, then $\beta = 1$ and $\alpha = 0$, so equation (K.50) results in

$$F^Z(k | k_0, 1) \simeq \frac{k_0 \int_0^1 d\xi \xi^{k_0-1} e^{-k_0\xi} (1 - \xi)}{k \int_0^1 d\xi \xi^{k-1} e^{-k_0\xi} - k_0 \int_0^1 d\xi \xi^k e^{-k_0\xi}}. \quad (\text{K.51})$$

Another integration by parts gives

$$\begin{aligned} F^Z(k | k_0, 1) &\simeq \frac{k_0 \left(\left[\frac{\xi^{k_0}}{k_0} e^{-k_0\xi} \right]_0^1 + \int_0^1 d\xi \xi^{k_0} e^{-k_0\xi} - \int_0^1 d\xi \xi^{k_0} e^{-k_0\xi} \right)}{k \left(\left[\frac{\xi^k}{k} e^{-k_0\xi} \right]_0^1 + \frac{k_0}{k} \int_0^1 d\xi \xi^k e^{-k_0\xi} \right) - k_0 \int_0^1 d\xi \xi^k e^{-k_0\xi}} \\ &= \frac{k_0 \left(\frac{e^{-k_0}}{k_0} + \int_0^1 d\xi \xi^{k_0} e^{-k_0\xi} - \int_0^1 d\xi \xi^{k_0} e^{-k_0\xi} \right)}{k \left(\frac{e^{-k_0\xi}}{k} + \frac{k_0}{k} \int_0^1 d\xi \xi^k e^{-k_0\xi} \right) - k_0 \int_0^1 d\xi \xi^k e^{-k_0\xi}} \\ &= 1. \end{aligned} \quad (\text{K.52})$$

Therefore, the vertex that had degree k_0 will always have degree k at some time with probability 1. Continuing the calculus to obtain $F^Z(k | k_0, z)$, the integrals in equations (K.45) and (K.46) can be expanded to simplify the notation. Both the resulting integrals in equations (K.48) and (K.49) are a function of α and β , and its expansion around $\alpha = 0$ and $\beta = 1$ can be written in a general form as

$$\begin{aligned} f(\alpha, \beta) &= f(0, 1) + \partial_\alpha^{(1)} f(0, 1) (\alpha - 0) + \partial_\beta^{(1)} f(0, 1) (\beta - 1) + \partial_\alpha^{(2)} f(0, 1) (\alpha - 0)^2 \\ &\quad + \partial_\beta^{(2)} f(0, 1) (\beta - 1)^2 + \dots \quad (\text{K.53}) \end{aligned}$$

Recall from (5.33) that $\langle t \rangle = \lim_{z \rightarrow 1} z \frac{\partial}{\partial z} F^Z(k | k_0, z)$. To calculate the mean time, $F^Z(k | k_0, z)$ will have to be derived once, so the important terms in the expansion are the linear terms — all the higher order terms will go to zero when $z = 1$, after deriving $F^Z(k | k_0, z)$ in z once. Rewriting the integrals from equations (K.48) and (K.49) in terms of the expansion in equation (K.53) gives that

$$\begin{aligned} \int_K^1 d\xi \xi^{k_0-1} e^{-k_0\beta(\xi-1)} (1 - \xi)^\alpha (k_0 - k_0\beta\xi) \Big|_{K=0} &= e^{k_0} \int_0^1 d\xi \xi^{k_0-1} e^{-k_0\xi} (k_0 - k_0\xi) \\ &\quad + \alpha \int_0^1 d\xi \xi^{k_0-1} e^{-k_0(\xi-1)} k_0 (1 - \xi) \ln(1 - \xi) \\ &\quad + (\beta - 1) \int_0^1 d\xi \xi^{k_0-1} e^{-k_0(\xi-1)} k_0 [k_0 - (2k_0 + 1)\xi + k_0\xi^2] \\ &\quad + \mathcal{O}(\alpha^2, (\beta - 1)^2, \alpha(\beta - 1)) \end{aligned} \quad (\text{K.54})$$

and

$$\begin{aligned}
\int_K^1 d\xi \xi^{k-1} e^{-k_0\beta(\xi-1)} (1-\xi)^\alpha (k-k_0\beta\xi) \Big|_{K=0} &= e^{k_0} \int_0^1 d\xi \xi^{k-1} e^{-k_0\xi} (k-k_0\xi) \\
&+ \alpha \int_0^1 d\xi \xi^{k-1} e^{-k_0(\xi-1)} (k-k_0\xi) \ln(1-\xi) \\
&+ (\beta-1) \int_0^1 d\xi \xi^{k-1} e^{-k_0(\xi-1)} k_0 [(1-\xi)(k-k_0\xi) - \xi] \\
&+ \mathcal{O}(\alpha^2, (\beta-1)^2, \alpha(\beta-1)) .
\end{aligned} \tag{K.55}$$

The integrals on the second term of the right hand side of equations (K.53) and (K.54) can be renamed as

$$\Omega(k, k_0) = \alpha \int_0^1 d\xi \xi^{k-1} e^{-k_0(\xi-1)} (k-k_0\xi) \ln(1-\xi) \tag{K.56}$$

and $\Omega(k_0, k_0)$ — which is simply $\Omega(k, k_0)$ with $k = k_0$. Then, the other integrals in equations (K.53) and (K.54) can be solved considering that

$$\int_0^1 d\xi e^{-k_0\xi} \xi^m = m! k_0^{-(m+1)} \left[1 - e^{-k_0} \sum_{l=0}^m \frac{k_0^l}{l!} \right], \tag{K.57}$$

where this relation was obtained by successive integrations by parts. Applying the result from equation (K.55) to equations (K.53) and (K.54) gives that

$$\int_K^1 d\xi \xi^{k_0} e^{-k_0\beta(\xi-1)} (1-\xi)^\alpha k_0 (1-\beta\xi) \Big|_{K=0} = 1 + \alpha [\Omega(k_0, k_0)] + \mathcal{O}(\alpha^2, (\beta-1)^2, \alpha(\beta-1)) \tag{K.58}$$

and

$$\int_K^1 d\xi \xi^{k-1} e^{-k_0\beta(\xi-1)} (1-\xi)^\alpha (k-k_0\beta\xi) \Big|_{K=0} = 1 + \alpha [\Omega(k, k_0)] + \mathcal{O}(\alpha^2, (\beta-1)^2, \alpha(\beta-1)) . \tag{K.59}$$

After these last passages, equations (K.46) and (K.47) become, respectively,

$$\begin{aligned}
&\left(\frac{\partial^k}{\partial K^k} \left[e^{k_0\beta(K-1)} (1-K)^{-\alpha} \right] \right) \left(\frac{1}{\alpha} [1 + \alpha (\Omega(k_0, k_0))] \right) \\
&- \sum_{l=1}^k \binom{k}{l} \left(\frac{\partial^{k-l}}{\partial K^{k-l}} \left[e^{k_0\beta(K-1)} (1-K)^{-\alpha} \right] \right) \left(\frac{\partial^{l-1}}{\partial K^{l-1}} \left[K^{k_0} e^{-k_0\beta(K-1)} (1-K)^{\alpha-1} \right] \right) \Theta(\Delta-1) \Big|_{K=0}
\end{aligned} \tag{K.60}$$

and

$$\left(\frac{\partial^k}{\partial K^k} \left[e^{k_0\beta(K-1)} (1-K)^{-\alpha} \right] \right) \frac{1}{\alpha} [1 + \alpha (\Omega(k, k_0))] \Big|_{K=0}, \tag{K.61}$$

recalling that equations (K.60) and (K.61) represent, respectively, the numerator and the denominator of $F^Z(k|k_0, z)$. The last multiple derivative $\frac{\partial^{l-1}}{\partial K^{l-1}}$ in equation (K.58) can be solved using the same logic presented in equation (K.38), from which it follows that

$$\begin{aligned} \frac{\partial^{l-1}}{\partial K^{l-1}} [K^{k_0} e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}] &= \sum_{n=0}^{l-1} \binom{l-1}{n} \left(\frac{\partial^n}{\partial K^n} [K^{k_0}] \right) \\ &\times \left(\frac{\partial^{l-1-n}}{\partial K^{l-1-n}} [e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}] \right). \quad (\text{K.62}) \end{aligned}$$

The only term different from zero in the sum of equation (K.62) occurs at $n = k_0$. That is because if the term K^{k_0} in equation (K.62) is derived $k_0 - 1$ times or less, then the entire term will go to zero when $K = 0$. On the other hand, the term K^{k_0} is zero if derived more than k_0 times. Therefore,

$$\begin{aligned} \frac{\partial^{l-1}}{\partial K^{l-1}} [K^{k_0} e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}] &= \binom{l-1}{k_0} \left(\frac{\partial^{k_0}}{\partial K^{k_0}} [K^{k_0}] \right) \left(\frac{\partial^{l-1-k_0}}{\partial K^{l-1-k_0}} [e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}] \right) \\ &= \frac{(l-1)!}{(l-1-k_0)!} \frac{\partial^{l-1-k_0}}{\partial K^{l-1-k_0}} [e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}] \Big|_{K=0}. \quad (\text{K.63}) \end{aligned}$$

Replacing the result from equation (K.63) to equation (K.60) and applying a change of variables ($l-1-k_0 = m$) results in

$$\begin{aligned} &\left(\frac{\partial^k}{\partial K^k} [e^{k_0\beta(K-1)} (1-K)^{-\alpha}] \right) \left(\frac{1}{\alpha} [1 + \alpha(\Omega(k_0, k_0))] \right) \\ &- \frac{k!}{(\Delta-1)!} \sum_{m=0}^{\Delta-1} \binom{\Delta-1}{m} \frac{1}{m+k_0+1} \left(\frac{\partial^{\Delta-1-m}}{\partial K^{\Delta-1-m}} [e^{k_0\beta(K-1)} (1-K)^{-\alpha}] \right) \\ &\times \left(\frac{\partial^m}{\partial K^m} [e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}] \right) \Theta(\Delta-1) \Big|_{K=0}. \quad (\text{K.64}) \end{aligned}$$

Equation (K.64) now has two types of derivatives:

$$\partial_K^\eta [e^{k_0\beta(K-1)} (1-K)^{-\alpha}] \text{ and } \partial K^\eta [e^{-k_0\beta(K-1)} (1-K)^{\alpha-1}],$$

where η is the variable that indicates the number of derivatives to be taken. These derivatives are a function of α and β , or simply $f(\alpha, \beta)$. If expanded in the form of equation (K.53), it will be seen from the calculation that it is sufficient to consider the dominant term only. Therefore, the derivatives can be written in terms of $f(0, 1)$ and solved using the same logic presented in equation (K.38), which gives

$$\frac{\partial^\eta}{\partial K^\eta} [e^{k_0\beta(K-1)}] \Big|_{K=0} = k_0^\eta e^{-k_0} \quad (\text{K.65})$$

and

$$\begin{aligned}
\left. \frac{\partial^\eta}{\partial K^\eta} [e^{-k_0(K-1)} (1-K)^{-1}] \right|_{K=0} &= \sum_{l=0}^{\eta} \binom{\eta}{l} \left(\frac{\partial^{\eta-l}}{\partial K^{\eta-l}} [e^{-k_0(K-1)}] \right) \left(\frac{\partial^l}{\partial K^l} [(1-K)^{-1}] \right) \\
&= \eta! e^{-k_0} \sum_{l=0}^{\eta} \frac{(-k_0)^{\eta-l}}{(\eta-l)!} \\
&= \eta! e^{-k_0} \sum_{n=0}^{\eta} \frac{(-k_0)^n}{n!} .
\end{aligned} \tag{K.66}$$

Applying the results obtained in equations (K.65) and (K.66) to equation (K.64) and (K.61) results in

$$\frac{k_0^k e^{-k_0}}{\alpha} (1 + \alpha [\Omega(k_0, k_0)]) - k! \sum_{m=0}^{\Delta-1} \frac{k_0^{\Delta-1-m}}{(\Delta-1-m)! (m+k_0+1)} \sum_{n=0}^m \frac{(-k_0)^n}{n!} \Theta(\Delta-1) \tag{K.67}$$

and

$$\frac{k_0^k e^{-k_0}}{\alpha} (1 + \alpha [\Omega(k, k_0)]) , \tag{K.68}$$

respectively. Next, combining equations (K.65) and (K.66) into a single equation for $F^Z(k | k_0, z)$ — equation (K.38) — gives

$$F^Z(k | k_0, z) \simeq \frac{1 + \alpha \Omega(k_0, k_0)}{1 + \alpha \Omega(k, k_0)} - \left[k! \sum_{m=0}^{\Delta-1} \frac{k_0^{-(k_0+m+1)}}{(\Delta-1-m)! (m+k_0+1)} \sum_{n=0}^m \frac{(-k_0)^n}{n!} \right] \alpha e^{k_0} \Theta(\Delta-1) . \tag{K.69}$$

APPENDIX L – Python code for the $\langle t \rangle$ - dynamical Watts-Strogatz model

The following code was implemented on Python based on the instructions provided on page 44. It generates an ensemble of graphs to estimate the mean time for the first-passage in degree space for the dynamical version of the Watts-Strogatz model.

```

N = int(input('N: '))
k0_init = int(input('k0: '))
k = int(input('k: '))
I = int(input('iterate: '))
k0 = k0_init
N_list, meantime_list, error_list = [], [], []

for N in range(1,N):
    t_list = []
    for i in range(I):
        t = 0
        k0 = k0_init
        M = k0_init*N
        while k0 != k:
            t += 1
            if random.uniform(0.0,1.0) < k0/float(M):
                if random.uniform(0.0,1.0) < 1.0 - 1/float(N):
                    k0 = 1
            else:
                if random.uniform(0.0,1.0) < 1/float(N):
                    k0 += 1
            t_list.append(t)
        mean_time = float(sum(t_list)/I)
        meantime_list.append(mean_time)
        error = np.std(t_list)/math.sqrt(I)
        error_list.append(error)
    N_list.append(N)

```

APPENDIX M – Calculating the degree variance for the dynamical Erdős-Rényi model

The degree variance is given by $var(k) = \overline{k^2} - \overline{k}^2$. Therefore, it is necessary to first obtain the mean degree and the mean degree squared for the dynamical Erdős-Rényi model.

From equation (5.1), which describes that the probability of a given vertex s having degree k at time $t + 1$ is

$$p(k, s, t + 1) = \frac{p}{N^2} p(k - 2, s, t) + \frac{2}{N} p \left(1 - \frac{1}{N} \right) p(k - 1, s, t) + \left(1 - \frac{2p}{N} + \frac{p}{N^2} \right) p(k, s, t), \quad (\text{M.1})$$

and

$$\overline{k}(s, t) = \sum_{k=0}^{\infty} k p(k, s, t), \quad (\text{M.2})$$

it is possible to obtain the equation that gives the mean degree of a vertex s at time t . Multiplying both sides of the equation (M.1) by k and summing in k from $k = 0$ to ∞ results in

$$\overline{k}(s, t + 1) = \overline{k}(s, t) + \frac{2p}{N}. \quad (\text{M.3})$$

Next, if the approximation $t \gg 1$ is used, then

$$\frac{\partial}{\partial t} \overline{k}(s, t) = \frac{2p}{N}. \quad (\text{M.4})$$

Solving equation (M.4), given that $\overline{k}(s, t = 0) = 0$ is the boundary condition, leads to

$$\overline{k}(s, t) \simeq \frac{2pt}{N}. \quad (\text{M.5})$$

Next, from equation (M.1) and

$$\overline{k^2}(s, t) = \sum_{k=0}^{\infty} k^2 p(k, s, t), \quad (\text{M.6})$$

it is possible to obtain the equation that gives the mean degree squared of a vertex s at time t . Multiplying both sides of equation (M.1) by k^2 and summing in k from $k = 0$ to ∞ leads to

$$\overline{k^2}(s, t + 1) = \overline{k^2}(s, t) + \frac{4p}{N} \overline{k}(s, t) + \frac{2p}{N^2} + \frac{2p}{N}. \quad (\text{M.7})$$

If the approximation $t \gg 1$ is used, and applying the result from equation (M.5), then

$$\frac{\partial}{\partial t} \overline{k^2}(s, t) = \frac{8p^2 t}{N^2} + \frac{2p}{N} \left(1 + \frac{1}{N} \right) . \quad (\text{M.8})$$

Solving equation (M.8), given that $\overline{k^2}(s, t = 0) = 0$, results in

$$\overline{k^2}(s, t) \simeq \left(\frac{2pt}{N} \right)^2 + \frac{2p}{N} \left(1 + \frac{1}{N} \right) t . \quad (\text{M.9})$$

With the results from equations (M.5) and (M.9), it is possible to obtain that the degree variance for the dynamical Erdős-Rényi model is given by

$$var(k) = \frac{2pt}{N} \left(1 + \frac{1}{N} \right) . \quad (\text{M.10})$$

APPENDIX N – Calculating the degree variance for the dynamical Watts-Strogatz model

The degree variance is given by $\text{var}(k) = \overline{k^2} - \overline{k}^2$. Therefore, it is necessary to first obtain the mean degree and the mean degree squared for the dynamical Watts-Strogatz model.

From equation (5.24), which describes that the probability of a given vertex s having degree k at time $t + 1$ is

$$p(k, s, t + 1) = \frac{p}{N} \left(1 - \frac{k-1}{M}\right) p(k-1, s, t) + \frac{k+1}{M} p \left(1 - \frac{1}{N}\right) p(k+1, s, t) + \left[1 - \frac{p}{N} - \frac{kp}{M} \left(1 - \frac{2}{N}\right)\right] p(k, s, t), \quad (\text{N.1})$$

and

$$\overline{k}(s, t) = \sum_{k=0}^{\infty} k p(k, s, t), \quad (\text{N.2})$$

it is possible to obtain the equation that gives the mean degree of a vertex s at time t . Multiplying both sides of the equation (N.1) by k and summing in k from $k = 0$ to ∞ results in

$$\overline{k}(s, t + 1) = \overline{k}(s, t) - \frac{p}{M} \overline{k}(s, t) + \frac{p}{N}. \quad (\text{N.3})$$

Next, if the approximation $t \gg 1$ is used, then

$$\frac{\partial}{\partial t} \overline{k}(s, t) = \frac{p}{N} - \frac{p}{M} \overline{k}(s, t). \quad (\text{N.4})$$

The differential equation can be solved through the method of the variation of parameters. In this method, the solution for the differential equation

$$\frac{d}{dt} x(t) = A(t)x(t) + B(t) \quad (\text{N.5})$$

is given by

$$x(t) = x_0 \exp \left[\int_{t_0}^t d\xi A(\xi) \right] + \int_{t_0}^t d\xi B(\xi) \exp \left[\int_{\xi}^t d\eta A(\eta) \right], \quad (\text{N.6})$$

where $x(t_0) = x_0$ is an initial (boundary) condition. Using the solution presented at equation (N.6) to solve equation (N.4), with $\overline{k}(s, t = 0) = k_0$ as the boundary condition, leads to

$$\overline{k}(s, t) \simeq \frac{M}{N}. \quad (\text{N.7})$$

Next, from equation (N.1) and

$$\overline{k^2}(s, t) = \sum_{k=0}^{\infty} k^2 p(k, s, t), \quad (\text{N.8})$$

it is possible to obtain the equation that gives the mean degree squared of a vertex s at time t . Multiplying both sides of equation (N.1) by k^2 and summing in k from $k = 0$ to ∞ leads to

$$\overline{k^2}(s, t+1) = \overline{k^2}(s, t) - \frac{2p}{M} \overline{k^2}(s, t) + \frac{2p}{N} \left(1 + k_0 - \frac{1}{N} \right). \quad (\text{N.9})$$

If the approximation $t \gg 1$ is used, then

$$\frac{\partial}{\partial t} \overline{k^2}(s, t) = -\frac{2p}{M} \overline{k^2}(s, t) + \frac{2p}{N} \left(1 + k_0 - \frac{1}{N} \right). \quad (\text{N.10})$$

Solving equation (N.10) using the method of variation of parameters results in

$$\overline{k^2}(s, t) \simeq k_0^2 + k_0 \left(1 - \frac{1}{N} \right) \left(1 - e^{-\frac{2pt}{M}} \right). \quad (\text{N.11})$$

With the results from equations (N.7) and (N.11), and recalling that $M = k_0 N$, it is possible to obtain that the degree variance for the dynamical Watts-Strogatz model is given by

$$\text{var}(k) = k_0 \left(1 - \frac{1}{N} \right) \left(1 - e^{-\frac{2pt}{M}} \right). \quad (\text{N.12})$$

Note that in the stationary limit ($t \rightarrow \infty$) and for large network ($N \rightarrow \infty$), the variance tends to the mean degree k_0 , as is expected from the stationary distribution of the dynamical Watts-Strogatz model, which is a Poisson distribution (see appendix J).